convergence. Thus, there exists $M > 0$ such that

$$\|f\|_{L^2} \leqslant 1 \quad \text{implies} \quad \sup_{x\in K_1} \Big| \int_X \varphi(x, y) f(y)\, dy \Big| \leqslant 1/M.$$

For a fixed $x \in X$,

$$\Big(\int_X |\varphi(x, y)|^2 dy \Big)^{1/2} = \sup_{\|f\|_{L^2}\leqslant 1} |R_1 \circ S \circ R_2 f(x)|$$
$$= \sup_{\|f\|_{L^2}\leqslant 1} \Big| \int_X \varphi(x, y) f(y)\, dy \Big| \leqslant 1/M.$$

Therefore

$$\int_{X\times X} |\varphi(x, y)|^2 dy\, dx \leqslant 1/M^2 \operatorname{meas}(K_1).$$

5. Proof of Theorem 2. In order to simplify the notation, U will be a fixed open set of the covering $\{U_n^a\}$ and φ will be the corresponding function in the subordinate partition of unity. Let also $\sigma_1 = \sigma^\varphi$.

It is clear that σ_1 satisfies the assumptions of Lemma 2, with $\varrho_2 = \varrho = 1$, and $\delta = 0$ if $m = 0$ or $0 < \delta < 1$ if $m > 0$.

On the other hand, $1/\sigma_1$ satisfies the assumptions of Lemma 3 with $\varrho = 1$, $m' = m$ and $\delta = 0$ if $m = 0$ or $0 < \delta < 1$ if $m > 0$. Thus, we can obtain operators A and B_1 as in Lemmas 2 and 3, respectively. Therefore $A \circ B_1$ has the properties of Theorem 3 and then its L^2-inverse $\tilde{B}$ is a pseudo-differential operator.

Therefore the operators A and $B_1 \circ \tilde{B}$ verify Theorem 2 with respect to U.

Remark. When $m = 0$, both symbols σ_1 and $1/\sigma_1$ satisfy the assumptions of Lemma 2.

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Extensions of a Fourier multiplier theorem of Paley, II*

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Abstract. Let $A(U^N)$ be the algebra of functions that are analytic in the interior of the unit polydisc U^N and continuous on the closure of U^N. Denote the positive cone in the integer lattice Z^N by Z_+^N; then, for each function f in $A(U^N)$, denote the Taylor coefficients of f by $\{\hat{f}(\alpha)\}_{\alpha\in Z_+^N}$. Call a function p on Z_+^N a *Paley multiplier* if $\sum_{\alpha\in Z_+^N} |p(\alpha)\hat{f}(\alpha)| < \infty$ for all f in $A(U^N)$. Call a region W in Z_+^N a *proper cone* if the ratios $(\min_n \alpha_n)/|\alpha|$, remain bounded away from 0 as α runs through W. Every element of $l^2(Z_+^N)$ is a Paley multiplier; it is shown in this paper that, if p is a Paley multiplier, then $\sum_{\alpha\in W} |p(\alpha)|^2 < \infty$ for every proper cone W. This is a considerable improvement on previous results, but it remains unknown, when $1 < N < \infty$, whether every Paley multiplier belong to $l^2(Z_+^N)$.

The proof is based on a simple construction that also yields partial solutions to some problems about homogenous expansions of functions in $A(U^N)$. Other applications of the constructions are also discussed.

1. Introduction. We use the notation and terminology of Rudin's book [29], except that we denote the Taylor coefficients of a function f in $A(U^N)$ by $\hat{f}(\alpha)$ rather than $c(\alpha)$. Such a function is completely determined by its restriction to the distinguished boundary T^N of U^N, and its Taylor coefficients are just the Fourier coefficients of its restriction to T^N.

Paley's theorem [26] is that, when $N = 1$, every Paley multiplier belongs to $l^2(Z^+)$. Helson [15] found a second proof of Paley's theorem, and generalized it to several variables in the following way. Choose a half-space S in Z^N, and let A be the set of continuous functions on T^N whose Fourier coefficients vanish off S; then a function p, on the set S, has the property that $\sum_{\alpha\in S} |p(\alpha)\hat{f}(\alpha)| < \infty$, for all f in A, if and only if $p \in l^2(S)$. Rudin ([28], p. 222) extended this result to the context of compact abelian groups with totally-ordered dual groups.

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Now the restriction to T^N of $A(U^N)$ is the set $A(T^N)$ of continuous functions on T^N whose Fourier coefficients vanish off Z_+^N. If $Z_+^N \subset S$, then $A(T^N)$ is included in Helson's space A, but the two spaces never coincide when $N > 1$; thus Helson's theorem does not characterize the space of Paley multipliers on $A(U^N)$ when $N > 1$. The best previous result about Paley multipliers on this case appears in [11]. Induce a partial order on Z^N by declaring that $\alpha \leqslant \beta$ if and only if $\beta - \alpha \in Z_+^N$; the every Paley multiplier p has the property that $\sum_{\alpha\in E} |p(\alpha)|^2 < \infty$, for all totally ordered subsets E of Z_+^N. The analogue of this statement for compact abelian groups with partially-ordered dual groups also holds; when the dual group is totally ordered, we recover the theorems of Paley, Helson and Rudin.

In Section 2, below, we prove a generalization of the fact that the restriction of any Paley multiplier to a totally ordered set must be square-summable. We then use this generalization to show, in Section 3, that the restriction of any Paley multiplier to any proper cone must be square-summable. As in the previous paper ([11], Sections 2 and 6), the methods used here to analyse Paley multipliers also yields results about semilacunary Fourier series, and about dual interpolation problems. In Section 4 of the present paper, we present a partial solution to an interpolation problem for homogeneous expansions of bounded analytic functions; we also extend slightly Forelli's F. and M. Riesz theorem for measures that annihilate the polydisc algebra [9]. In Section 5, we continue our study of interpolation problems for homogeneous expansions of bounded functions; the results are similar to those in Section 2, but the methods are different. Finally, in Section 6, we discuss the connection between the results in this paper and the work of other authors; we also outline briefly an alternate proof that the restriction of any Paley multiplier to any proper cone must be square-summable.

2. Paley multipliers and order-convex sets. Denote the Fourier coefficients of an integrable function f on T^N by $\hat{f}(\alpha)$. Given a subset I of Z^N, let C_I be the space of all continuous functions f on T^N such that $\hat{f}$ vanishes off I. Call a function p, on the set I, a *Paley multiplier on* C_I if

$$\sum_{\alpha\in I} |p(\alpha)\hat{f}(\alpha)| < \infty \quad \text{for all } f \text{ in } C_I.$$

Denote the space of Paley multipliers on C_I by P_I.

If p is a Paley multiplier, then, by the closed-graph theorem, the map $f \mapsto p\cdot\hat{f}$ is a bounded operator from C_I into $l^1(I)$; denote the norm of this operator by $\|p\|_I$. Observe that this norm is monotone, in the sense that if $|p_1(\alpha)| \leqslant |p_2(\alpha)|$ for all α, then $\|p_1\|_I \leqslant \|p_2\|_I$; in particular, modifying the argument of a Paley multiplier does not change its norm.

Denote by $\leqslant$ the partial order on Z^N induced by declaring that $\alpha \leqslant \beta$ if and only if $\beta - \alpha \in Z_+^N$. Call a subset I of Z^N *order-convex* if the conditions $\alpha \in I$, $\beta \in I$, and $\alpha < \gamma < \beta$ imply that $\gamma \in I$. Call a subset E of Z^N *totally-ordered* if every pair (α, β), of elements of E, satisfies $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$. Extend the partial order $\leqslant$ to pairs of subsets of Z^N by declaring that $E_1 \leqslant E_2$ if $\alpha \leqslant \beta$ for all α in E_1 and all β in E_2; that is, $E_1 \leqslant E_2$ if the set $E_2 - E_1$, of all differences $\beta - \alpha$, where $\beta \in E_2$ and $\alpha \in E_1$, is a subset of Z_+^N. Finally, call a sequence $\{E_m\}_{m=1}^M$, of subsets of Z^N, *order-increasing* if $E_1 \leqslant E_2 \leqslant E_3 \leqslant \ldots$

The results of this section hold, with the same proofs, for order-convex subsets of any partially-ordered, discrete, abelian group, but we shall concentrate on the case of Z_+^N, with the partial order defined above. It is not known whether every Paley multiplier on $A(U^N)$ belongs to $l^2(Z_+^N)$. Our first result shows, however, that this space of Paley multipliers resembles $l^2(Z_+^N)$ in the sense that we nearly have orthogonal projections.

THEOREM 1. *Let I be an order-convex subset of Z^N, and let p be a Paley multiplier on C_I. Let $\{E_m\}_{m=1}^M$ be an order-increasing sequence of disjoint subsets of I. For each m, let $\|p\|_{E_m}$ be the norm of the restriction of the function p to the set E_m, as a Paley multiplier on C_{E_m}. Then*

$$\sum_{m=1}^{M} (\|p\|_{E_m})^2 \leqslant e(\|p\|_I)^2. \tag{1}$$

Proof. It suffices to consider the case where M is finite. Moreover, it follows from the definition that $\|p\|_{E_m} \leqslant \|p\|_I$, so that the sum on the left side of inequality (1) is finite. Let us normalize p so that this sum is equal to 1; we then have to show that $\|p\|_I \geqslant e^{-1/2}$.

Fix a positive constant $K < 1$, and, for each m, choose a trigonometric polynomial g_m in C_{E_m} such that $\|g_m\|_\infty = \|p\|_{E_m}$, and

$$\sum_{\alpha\in E_m} |p(\alpha)\hat{g}_m(\alpha)| \geqslant K(\|p\|_{E_m})^2.$$

Modify the argument of the function p, on each set E_m, so that $p(\alpha)\hat{g}_m(\alpha) \geqslant 0$ for all α in E_m; this does not affect the norms $\|p\|_I$ or $\|p\|_{E_m}$.

Denote the mth Rademacher function ([39], p. 104) by r_m. Then define functions f_m and h_m, on the product space $T^N \times [0,1]$, by the following inductive procedure. Let

$$f_1(z, t) = r_1(t) g_1(z) \quad \text{and} \quad h_1(z, t) = 1,$$

for all z in T^N, and all t in $[0,1]$. Given f_m and h_m, let

$$f_{m+1}(z, t) = f_m(z, t) + r_{m+1}(t) g_{m+1}(z) h_m(z, t),$$

and

$$h_{m+1}(z, t) = h_m(z, t) - r_{m+1}(t) \overline{g_{m+1}(z)} f_m(z, t).$$

Finally, for each fixed t in the interval $[0,1]$, let f_t be the function on T^N given by letting $f_t(z) = f_M(z,t)$ for all z in T^N.

This construction is a modification of the well-known procedure due to Shapiro ([34], p. 35) and Rudin [31]. It is clear that for each value of the parameter t, the function f_t is a trigonometric polynomial. We now estimate $\|f_t\|_\infty$. As in [10], an inductive argument, based on the complex identity

$$|a+vb|^2+|b-\bar{v}a|^2 = (1+|v|^2)(|a|^2+|b|^2), \tag{2}$$

shows that

$$|f_m(z,t)|^2+|h_m(z,t)|^2 = \prod_{i=1}^{m}(1+|g_i(z)|^2),$$

for all values of m, z, and t. Now

$$\prod_{i=1}^{M}(1+\|g_i\|_\infty^2) \leqslant \exp\Big(\sum_{i=1}^{M}\|p\|_{E_i}^2\Big) = e.$$

Hence $\|f_t\|_\infty \leqslant e^{1/2}$, for all t.

We claim that $f_t \in C_I$. It is clear from the construction that f_t is a sum of products of the form

$$(-1)^{(k-1)/2} r_{m_k}(t) g_{m_k} r_{m_{k-1}}(t)\bar{g}_{m_{k-1}} \ldots r_{m_1}(t) g_{m_1},$$

for various odd integers k and sequences of indices $\{m_i\}_{i=1}^k$ with $m_1 < m_2 < \ldots < m_k$; here, we take the complex conjugate of the factor g_{m_i} above whenever i is even. It follows that, if γ belongs to the support of $\hat{f}_t$, then there exist an odd integer k, a strictly increasing sequence of indices $\{m_i\}_{i=1}^k$, and elements γ_i of E_{m_i}, such that

$$\gamma = \gamma_{m_k} - \gamma_{m_{k-1}} + \gamma_{m_{k-2}} - \ldots + \gamma_{m_1}.$$

Since the sequence of sets $\{E_m\}_{m=1}^M$ is order-increasing, the differences $\gamma_{m_i} - \gamma_{m_{i-1}}$ all belong to Z_+^N; therefore $\gamma \geqslant \gamma_{m_1}$. Similarly, $\gamma \leqslant \gamma_{m_k}$. Since I is order-convex, and γ_{m_1} and γ_{m_k} both belong to I, we also have that $\gamma \in I$. Hence $f_t \in C_I$, as claimed.

Finally, we consider how $\hat{f}_t$ depends on t. If $a \in E_m$ for some m, then

$$\hat{f}_t(a) = r_m(t)\hat{g}_m(a) + q(t,a),$$

where q is a sum of high-order terms each a product of at least three distinct Rademacher functions. If a lies in none of the sets E_m, then $\hat{f}_t(a)$ is a sum of such high-order terms only. Define a Paley multiplier p_t, on C_I, by letting $p_t(a) = r_m(t)p(a)$ if $a \in E_m$, and letting $p_t(a) = p(a)$ if a lies in none of the sets E_m. Then $\|p_t\|_I = \|p\|_I$, because $|p_t| = |p|$. Now

$$\sum_{a\in I}\int_0^1 p_t(a)\hat{f}_t(a)\,dt = \sum_{m=1}^{M}\sum_{a\in E_m} p(a)\hat{g}_m(a),$$

because the function $t \mapsto p_t(a)$ is orthogonal to all the high-order terms in the expansion of $\hat{f}_t(a)$. On the other hand,

$$\int_0^1 \sum_{a\in I}|p_t(a)\hat{f}_t(a)|\,dt \leqslant \int_0^1 \|p_t\|_I\|f_t\|_\infty\,dt \leqslant \|p\|_I e^{1/2}.$$

By the various normalizations made above,

$$1 = \sum_{m=1}^{M}(\|p\|_{E_m})^2 \leqslant K^{-1}\sum_{m=1}^{M}\sum_{a\in E_m} p(a)\hat{g}_m(a) \leqslant K^{-1}e^{1/2}\|p\|_I.$$

Letting $K \to 1$, we conclude that $\|p\|_I \geqslant e^{-1/2}$, as required. This completes the proof of the theorem.

Corollary 2 ([11], Theorem 4). *Let I be an order-convex set, let E be a totally-ordered subset of I, and let p be a Paley multiplier on C_I. Then*

$$\sum_{a\in E}|p(a)|^2 \leqslant e\|p\|_I^2.$$

Proof. Apply the theorem with each set E_m taken to be a singleton $\{a^{(m)}\}$. It is clear, in this case, that $\|p\|_{E_m} = |p(a^{(m)})|$.

Corollary 3 ([11], Theorem 4). *Let I be a totally-ordered, order-convex set, and let p be a Paley multiplier on C_I. Then*

$$\sum_{a\in I}|p(a)|^2 \leqslant e\|p\|_I^2.$$

As these corollaries show, we need good estimates for the norms $\|p\|_{E_m}$ is order to apply Theorem 1 effectively. In some cases, such estimates can be obtained by reapplying Theorem 1 separately to each set E_m, using a partial order different from the one induced by Z_N^+. For instance, let $\{i_m\}_{m=1}^\infty$ be a sequence of positive integers, with $i_{m+1} \geqslant 2i_m$ for all m, let E_1 be the set of all a in Z_+^2 such that $|a| = i_1$, and, when $m \geqslant 2$, let

$$E_m = \{a \in Z_+^2:\ |a| = i_m,\ \min(a_1, a_2) \geqslant i_{m-1}\}. \tag{3}$$

Then the sequence of sets $\{E_m\}_{m=1}^\infty$ is order-increasing relative to the partial order induced by Z_+^2. Each set E_m is in turn totally-ordered and order convex relative to the partial order on Z^2 that arises if we declare that $a < \beta$ if and only if $a_1 - \beta_1 = \beta_2 - a_2 > 0$. Thus, we can apply Theorem 1 and Corollary 2 to conclude that

$$\sum_{m=1}^{\infty}\sum_{a\in E_m}|p(a)|^2 \leqslant e^2\|p\|_I^2,$$

for all Paley multipliers p on $A(U^2)$.

The norms $\|p\|_I$ and $\|p\|_{E_m}$ do not depend on the partial order imposed on Z_+^N. For this reason, the order-theoretic hypotheses in Theorem 1 may seem unnecessary, and it is reasonable to ask if a version of inequality (1) holds for all sets I, provided merely that the subsets $\{E_m\}_{m=1}^M$ are disjoint. If $P_I = l^2(I)$, then this is indeed the case. In general, however, further hypotheses on the sets I and $\{E_m\}_{m=1}^M$ are needed to ensure that an analogue of inequality (1) holds. For instance, let I be the subset of Z given by

$$I = \{3^m:\ m = 1, 2, \ldots\}. \tag{4}$$

It is well known ([42], p. 248) that

$$\sum_{m=1}^{\infty} |\hat{f}(3^m)| \leqslant 2\|f\|_\infty,$$

for all f in C_I. It follows easily that $P_I = l^\infty(I)$, and that $\|p\|_I \leqslant 2\|p\|_\infty$, for all Paley multipliers p on C_I. Relative to the usual order on Z, the set I is totally-ordered, but not order-convex. Let E_m be the singleton $\{3^m\}$; then the sets E_m form an order-increasing sequence. Let $p(i) = 1$ for all i in I; then $\|p\|_I \leqslant 2$, but

$$\sum_{m=1}^{\infty} (\|p\|_{E_m})^2 = \infty.$$

Thus no analogue of inequality (1) can hold in this case.

As we mentioned earlier, Theorem 1 and its corollaries hold for any compact abelian group G with a partially-ordered dual group Γ. Of course, the partial order is required to commute with the group operation; that is, $\alpha \leqslant \beta$ if and only if $\alpha+\gamma \leqslant \beta+\gamma$ for all γ. Total orders with this property are discussed in [28], Section 8.1. A partial order that commutes with the group operation, exists whenever the group Γ has a subsemigroup S such that

$$\text{if } \gamma \in S, \text{ and } \gamma \neq 0, \text{ then } -\gamma \notin S; \tag{5}$$

the partial order is defined by declaring that

$$\alpha \leqslant \beta \text{ if and only if } \alpha = \beta, \text{ or } \beta - \alpha \in S. \tag{6}$$

Conversely, for any partial order that commutes with addition, the positive cone S is a subsemigroup with property (5). Now the definition (6) makes sense and gives rise to a relation $\leqslant$ that commutes with addition even if the subsemigroup S does not have property (5); the only problem is that, in the absence of property (5), there exists at least one pair (α, β) of distinct elements of Γ such that $\alpha \leqslant \beta \leqslant \alpha$. Theorem 1 and its corollaries also hold for this weakened notion of partial order. In particular, Corollary 3 can be used to show, for all compact abelian groups G with dual groups Γ, that all Paley multipliers on $C(G)$ are square summable. Just set $S = \Gamma$; then Γ is order-convex and totally-ordered relative to the "order" induced by S. This result can also be proved by other methods ([5], Theorem 2.1; [16], Theorem 36.15).

3. Paley multipliers on $A(U^N)$. In this section, we use the special properties of Z_+^N to obtain further conclusions about the space of Paley multipliers on $A(U^N)$.

We begin with an extension problem for Paley multipliers on subsets of Z^N. Let I and K be subsets of Z^N, with $I \subset K$, and let p be a function on I. By the *trivial extension* of p to K we mean the function, on K, that coincides with p on I and vanishes off I; we also denote the extended function by p. Suppose that $p \in P_I$; does it follow that $p \in P_K$? In general, the answer is no. For instance, let I be the set defined in formula (4), in Section 3, and let $K = Z$; then $P_I = l^\infty(I)$, but $P_K = l^2(Z)$, and most functions in P_I cannot be extended to functions in P_K.

We therefore consider a modification of the extension procedure. Let g be an integrable function on T^N such that $\hat{g}$ vanishes off I. Given a function p on I, let $L_g(p)$ be the product $\hat{g}\cdot p$; regard $L_g(p)$ as a function on the set K by extending it trivially to K. Suppose again that $p \in P_I$. Then, in fact,

$$L_g(p) \in P_K \quad \text{and} \quad \|L_g(p)\|_K \leqslant \|g\|_1 \|p\|_I.$$

Indeed, if $f \in C_K$, then the convolution $g*f$ belongs to C_I, and

$$\sum_{\alpha\in K} |L_g(p)(\alpha)\hat{f}(\alpha)| = \sum_{\alpha\in I} |p(\alpha)(g*f)\hat{}\,(\alpha)| \leqslant \|p\|_I \|g\|_1 \|f\|.$$

Our plan is to use this procedure is situations where we can use Theorem 1 to estimate $\sum_{\alpha\in I} |L_g(p)(\alpha)|^2$ in terms of $\|L_g(p)\|_K$. Such estimates will be useful if $\hat{g}$ is reasonably large on a large part of I, and $\|g\|_1$ is reasonably small. We now specify the sets and functions that we have in mind. Define intervals J_k, in Z_+, as follows. Let $J_0 = \{0\}$, and $J_1 = [1, 5]$; given the interval J_k, containing m_k integers, let J_{k+1} consist of the next $2m_k+3$ integers. Then let $I_0 = J_0$, and when $k > 0$, let I_k be the union of the sets J_{k-1}, J_k, and J_{k+1}. The point of these definitions is that, for each k, there is a function g_k, with $\|g_k\|_1 = 1$, such that $\hat{g}_k$ vanishes off I_k, and $\hat{g}_k \geqslant 1/2$ on J_k; to obtain such a function, simply multiply a Fejér kernel of appropriate order by an appropriate power of $e^{i\theta}$. Next, given any multiindex γ in Z_+^N, let

$$J_\gamma = J_{\gamma_1} \times J_{\gamma_2} \times \ldots \times J_{\gamma_N},$$

define I_γ similarly, and let

$$g_\gamma(z) = g_{\gamma_1}(z_1) g_{\gamma_2}(z_2) \ldots g_{\gamma_N}(z_N), \quad \text{for all } z \text{ in } T^N.$$

It is then clear that $\|g_\gamma\|_1 = 1$, that $\hat{g}_\gamma$ vanishes off I_γ, and that $\hat{g}_\gamma \geqslant 2^{-N}$ on J_γ. Also, the sets J_γ form a disjoint cover of Z_+^N.

LEMMA 4 ([11], Section 4). *Let $\gamma \in Z_+^N$, and let p be a function on L_γ. Then*

$$\sum_{\alpha \in J_\gamma} |p(\alpha)|^2 \leqslant 4^N e \|p\|_{I_\gamma}^2.$$

Proof. Apply the procedure described above with $I = I_\gamma$, $g = g_\gamma$, and $K = Z^N$. The properties of g_γ guarantee that $|p(\alpha)| \leqslant 2^N |\hat{g}_\gamma(\alpha)\, p(\alpha)|$, for all α in J_γ, and that $\|L_{g_\gamma}(p)\|_{Z^N} \leqslant \|p\|_I$. There are many ways to make Z^N a totally-ordered group ([28], p. 194); relative to any such order, the group itself is order-convex. Therefore,

$$\begin{aligned}\sum_{\alpha \in J_\gamma} |p(\alpha)|^2 &\leqslant 4^N \sum_{\alpha \in I} |\hat{g}_\gamma(\alpha)\, p(\alpha)|^2 \\ &\leqslant 4^N e \|L_{g_\gamma}(p)\|_{Z^N}^2, \quad \text{by Corollary 3}, \\ &\leqslant 4^N e \|p\|_{I_\gamma}^2.\end{aligned}$$

This completes the proof of the lemma.

We now come to the main result of this section. Recall that a *proper cone* is a subset W of Z_+^N such that $(\min_n \alpha_n)/|\alpha|$ remains bounded away from 0 as α runs through W. We adopt the convention that $0/0 = 1$, so that proper cones are allowed to contain the origin.

THEOREM 5. *Let W be a proper cone in Z_+^N. Then there is a constant K, depending on W, such that*

$$\sum_{\alpha \in W} |p(\alpha)|^2 \leqslant K(\|p\|_{Z_+^N})^2, \tag{7}$$

for all Paley multipliers p on $A(U^N)$.

Proof. Let $\bar{\gamma}$ be the multiindex $(1,1,\ldots,1)$. Given a multiindex β on one of the faces of Z_+^N, that is with $\beta_n = 0$ for some n, let V_β be the union of the sets $J_{\beta+k\bar{\gamma}}$ as k runs from 0 to ∞. Then the sets V_β form a disjoint cover of Z_+^N. The proof of the theorem consists in showing that, for any proper cone W, only finitely many of the sets V_β are required to cover W, and that

$$\sum_{\alpha \in V_\beta} |p(\alpha)|^2 \leqslant 3e^2 4^N (\|p\|_{Z_+^N})^2, \tag{8}$$

for all Paley multipliers p on $A(U^N)$, and all β.

We deal first with the covering property. Fix a proper cone W. Let β be a multiindex on one of the faces of Z_+^N, and choose integers i and j so that $\beta_i = 0$, and $\beta_j = \max_n \beta_n = l$, say. Now, if $m \in J_k$ and $m' \in J_{k+l}$,

then $m \leqslant 2^{1-l} m'$. We therefore have, for each α in V_β, that

$$\frac{\alpha_i}{|\alpha|} \leqslant \frac{\alpha_i}{\alpha_j} \leqslant 2^{1-l}.$$

On the other hand, $(\min_n \alpha_n)/|\alpha|$ is bounded away from 0 as α runs through W. It follows that, if the intersection of W with V_β is nonempty, then the number l is bounded above by a constant L depending on W. Now there are only finitely many points β on the faces of Z_+^N for which $l \leqslant L$. So, the proper cone W intersects only finitely many of the sets V_β.

Now let p be a Paley multiplier on $A(U^N)$, and let β lie on one of the faces of Z_+^N. As m runs from 0 to ∞, the sets $I_{\beta+3m\bar{\gamma}}$ form an order-increasing sequence, relative to the partial order induced by Z_+^N. We apply Lemma 4, and Theorem 1, with $I = Z_+^N$, to conclude that

$$\sum_{m=0}^{\infty} \sum_{\alpha \in J_{\beta+3m\bar{\gamma}}} |p(\alpha)|^2 \leqslant 4^N e \sum_{m=0} (\|p\|_{I_{\beta+3m\bar{\gamma}}})^2 \leqslant 4^N e^2 (\|p\|_{Z_+^N})^2.$$

Finally, we apply the same argument to the order-increasing sequences $\{I_{\beta+(3m+1)\bar{\gamma}}\}_{m=0}^{\infty}$ and $\{I_{\beta+(3m+2)\bar{\gamma}}\}_{m=0}^{\infty}$, and conclude that inequality (8) holds. This completes the proof of the theorem.

In Section 6, below, we will discuss an alternate proof of Theorem 5. The proof above is more elementary than the alternate proof, and it yields better estimates for the constant K in inequality (7).

Which sets V have the property that there exists a constant K such that

$$\sum_{\alpha \in V} |p(\alpha)|^2 \leqslant K(\|p\|_{Z_+^N})^2, \tag{9}$$

for all Paley multipliers p on $A(U^N)$? By Corollary 3, every totally ordered subset of Z_+^N has this property, with $K = e$. In the proof of Theorem 5, we showed that the sets V_β also have this property, with $K = 3e^2 4^N$. We now describe a more general class of sets for which inequality (9) also holds. Given multiindices β and γ in Z_+^N, with $\gamma \neq 0$, let $V_{\beta,\gamma}$ be the union of the sets $J_{\beta+k\gamma}$, as k runs from 0 to ∞; then the sets V_β, considered above, become $V_{\beta,\bar{\gamma}}$ in this notation. It follows by the method used in [11], Theorem 5, that, if all but one of the coordinates of γ are equal to 0, then inequality (9) holds for each of the sets $V_{\beta,\gamma}$, with $K = 4^{N-1}e$. We now state a more general result that can be proved by combining the methods used to prove Theorem 5 with those used in [11], Section 4; we omit the details of the proof. Again, we will discuss an alternate proof of a similar result in Section 6.

THEOREM 6. *Let β and γ be multiindices in Z_+^N, with $\gamma \neq 0$, and let p be a Paley multiplier on $A(U^N)$. Then*

$$\sum_{\alpha \in V_{\beta,\gamma}} |p(\alpha)|^2 \leqslant 3e^2 4^{N-1} (\|p\|_{Z_+^N})^2.$$

For any subset E of Z_+^N, let V_E be the union of the sets J_α as α runs through E. In view of Theorem 6, it is natural to ask if, for every totally-ordered set E, inequality (9) must hold for the set V_E. We do not know whether this is so, but we can show, much as in the proof of Theorem 5, that, if a subset E of Z_+^N is totally-ordered relative to the partial order induced by the semigroup $\bar{\gamma}+Z_+^N$, then inequality (9) holds for the set V_E, with $K = 3e^2 4^{N-1}$; a set is totally-ordered relative to this partial order, if it is totally-ordered relative to the usual partial order induced by Z_+^N, and if no two of its elements agree in any of their coordinates.

Finally, we apply our results to the problem of determining whether specific functions on Z_+^2 are Paley multipliers on $A(U)^2$. Let $p(\alpha) = 1/(1+|\alpha|)$, for all α in Z_+^2. This function was considered in [11], p. 425, and it was shown there that, although p is not square summable on Z_+^2, it satisfies all the restrictions placed on Paley multipliers by results of [11]. Let W be the set of points α in Z_+^2 for which the ratio α_1/α_2 lies between $1/2$ and 2. Then W is a proper cone, but p is not square-summable over W; hence, by Theorem 5, the function p is not a Paley multiplier on $A(U^2)$.

Next, for each integer k in Z_+, let

$$E_k = \{3^m\colon 1 \leqslant m \leqslant k\},$$

and let p_k be the function on Z_+^2 that vanishes off the set $E_k\times E_k$ and is equal to 1 on $E_k\times E_k$. The principal results of this paper and [11] provide lower bounds on $\|p_k\|_{Z_+^2}$; these bounds all grow no faster than $k^{1/2}$ as $k\to\infty$, whereas the norm of P_k in $l^2(Z_+^2)$ is exactly k. This means that it does not follow from the principal results of this paper and [11] that the space of Paley multipliers on $A(U^N)$ is equal to $l^2(Z_N^2)$, because, if the two spaces do in fact coincide, then their norms must be equivalent. Ironically, we can use the special properties of the functions p_k to show that $\|p_k\|_{Z_+^2} \geqslant k/e$, after all. To do this, fix a constant K in the interval $(0, 1)$, and use Paley's theorem in one variable to obtain a function g in $A(U)$ such that

$$\|g\|_\infty = 1 \quad \text{and} \quad \sum_{m=1}^{k} |\hat{g}(3^m)| \geqslant K(k/e)^{1/2};$$

then define a function f in $A(U^2)$ by setting

$$f(z) = g(z_1)\,g(z_2) \quad \text{for all } z \text{ in } T^2.$$

It follows that $\|f\|_\infty = 1$, and that

$$\sum_{\alpha\in Z_+^2} |p_k(\alpha)\hat{f}(\alpha)| = \Big(\sum_{m=1}^{k} |\hat{g}(3^m)|\Big)^2 \geqslant K^2 k/e,$$

by the choice of g. Since K is any positive number less than 1, we have that $\|p_k\|_{Z_+^2} \geqslant k/e$, as claimed.

4. Homogenous expansions and interpolation. The construction used to prove Theorem 1 appeared in [10], where it yielded an explicit solution to an interpolation problem for Fourier coefficients of functions in $A(U)$. There is an obvious way to generalize this result to several variables; we shall state this generalization later in this section, as Theorem 9. Our main goal, however, is to deal with a more subtle problem in a way that was suggested by Walter Rudin.

We first recall what is meant by the homogeneous expansion of an analytic in the open polydisc U^N ([29], p. 7). Such a function f has a power series expansion

$$f(z) = \sum_{\alpha\in Z_+^N} \hat{f}(\alpha) z^\alpha,$$

valid for all z in U^n. For each multiindex α in Z_+^N let $|\alpha| = \alpha_1+\alpha_2+\ldots$ $\ldots+\alpha_N$. Given a nonnegative integer s, let

$$F_s(z) = \sum_{|\alpha|=s} \hat{f}(\alpha) z^\alpha.$$

Then F_s is a homogeneous polynomial of degree s. The homogeneous expansion of f is the series

$$f(z) = \sum_{s=0}^{\infty} F_s(z),$$

which converges for all z in U^N.

As usual, $H^\infty(U^N)$ denotes the space of bounded, analytic functions in the open set U^N. To every function f in $H^\infty(U^N)$ there corresponds a unique element f^* of $L^\infty(T^N)$ such that $f^*(z) = \lim_{r\uparrow 1} f(rz)$, for almost all z in T^N. We identify f with f^*, and let $H^\infty(T^N)$ be the subspace of $L^\infty(T^N)$ consisting of all equivalence classes of functions that are radial limits, almost everywhere, of functions in $H^\infty(U^N)$; then $f\in H^\infty(T^N)$ if and only if $f\in L^\infty$ and $\hat{f}$ vanishes off Z_+^N. Similarly, if $1\leqslant p<\infty$, we let $H^p(T^N)$ be the subspace of all functions in $L^p(T^N)$ whose Fourier transform vanish off Z_+^N; these spaces are discussed in [29], Chapter 3.

Now the various functions F_s appearing in the homogeneous expansion of f are orthogonal elements of $H^2(T^N)$. Thus

$$\sum_{s=0}^{\infty} (\|F_s\|_2)^2 \leqslant (\|f\|_2)^2 \leqslant (\|f\|_\infty)^2.$$

In fact, it is also the case that

$$\sum_{s=0}^{\infty} |F_s(z)|^2 \leqslant (\|f\|_\infty)^2, \tag{10}$$

for all z in T^N. To see this, fix z in T^N, and define the *slice function* f_z, on U, by the rule that $f_z(w) = f(wz)$ for all w in U. Then $f_z \in H^\infty(U)$, and its power series expansion is given by

$$f_z(w) = \sum_{s=0}^{\infty} F_s(z) w^s.$$

Inequality (10) follows from Bessel's inequality, applied to the function f_z.

We now present a partial converse to inequality (10). Recall that a sequence $\{s_m\}_{m=1}^\infty$, of nonnegative integers, is called a *Hadamard set* ([32], p. 203) if there exists a constant $\lambda > 1$ such that $s_{m+1} > \lambda s_m$ for all m.

THEOREM 7. *Let W be a proper cone in Z_+^N, and let $\{s_m\}_{m=1}^\infty$ be a Hadamard set. Let $\{G_m\}_{m=1}^\infty$ be a sequence of trigonometric polynomials on T^N such that*

(i) $\hat{G}_m$ *vanishes off* W,

(ii) G_m *is homogeneous, of degree* s_m,

and

(iii) $$\sup_{z \in T^N} \sum_{m=1}^{\infty} |G_m(z)|^2 < \infty.$$

Then there exists a function f in $H^\infty(U^N)$ such that, for each m, the term of degree s_m in the homogeneous expansion of f is equal to G_m.

Proof. Let E_m be the set of elements a of the proper cone W such that $|a| = s_m$. We suppose initially that the sets E_m form an order-increasing sequence relative to the partial order induced by Z_+^N, and that $s_{m+1} > 2s_m$ for all m.

Much as in the proof of Theorem 1, we define sequences $\{f_m\}_{m=1}^\infty$ and $\{h_m\}_{m=1}^\infty$, of trigonometric polynomials, as follows. Let

$$f_1(z) = G_1(z) \quad \text{and} \quad h_1(z) = 1, \quad \text{for all } z \text{ in } T^N.$$

Then, given f_m and h_m, let

$$f_{m+1}(z) = f_m(z) + G_{m+1}(z) h_m(z) \quad \text{and} \quad h_{m+1}(z) = h_m(z) - \overline{G_{m+1}(z) f_m(z)}.$$

Now

$$|f_m(z)|^2 + |h_m(z)|^2 = \prod_{i=1}^{m} (1 + |G_i(z)|^2),$$

for all m and z. Let

$$B = \sup_z \sum_{m=1}^{\infty} |G_m(z)|^2. \tag{11}$$

Then $\|f_m\|_\infty \leqslant e^{B/2}$, for all m. Hence the sequence $\{f_m\}_{m=1}^\infty$ is bounded in $L^\infty(T^N)$, and therefore has a weak-star limit point, f say, in $L^\infty(T^N)$.

For each positive integer j, the function f_j is a sum of products of the form

$$(-1)^{(k-1)/2} G_{m_k} \bar{G}_{m_{k-1}} G_{m_{k-2}} \cdots \tag{12}$$

for various odd integers $k \leqslant j$, and sequences of indices $\{m_i\}_{i=1}^k$, with $m_1 < m_2 < \ldots < m_k$; again, we take the complex conjugate of the factor G_{m_k} above whenever k is even. Exactly as in the proof of Theorem 1, we have that, if $\hat{f}_j(\gamma) \neq 0$, then $\gamma \in Z_+^N$; hence $f_j \in H^\infty(T^N)$, for all j, and $f \in H^\infty(T^N)$.

The product (12) is homogeneous of degree

$$s_{m_k} - s_{m_{k-1}} + \ldots - s_{m_2} + s_{m_1}. \tag{13}$$

The assumption that $s_{m+1} > 2s_m$ for all m implies that each integer has at most one representation as an alternating sum (13). In particular, the only product of the form (12) that has degree s_i is the single term G_i. This means that, for all $i \leqslant j$, the term of degree s_i in the homogeneous expansion of f_j is G_i. Now f is the weak-star limit, in $L^\infty(T^N)$, of a subsequence of $\{f_j\}_{j=1}^\infty$. Therefore, for each m, the term of degree s_m in the homogeneous expansion of f is G_m.

Now we drop the assumption that $s_{m+1} > 2s_m$ for all m, and that the sequence $\{E_m\}_{m=1}^\infty$ is order-increasing. Since $\{s_m\}_{m=1}^\infty$ is a Hadamard set, and W is a proper cone, it is true for all sufficient large integers q, however, that $s_{m+q} > 2s_m$ for all m, and $E_m < E_{m+q}$ for all m; moreover, if q is sufficiently large then $s_{m+q} > s_m + s_{m+q-1}$, for all m. We fix an integer q with all these properties, split the sequence of indices m into q arithmetic progressions, each with period q, and solve the resulting interpolation problem, in the above manner, for each of these subsequences. The fact that $s_{m+q} > s_m + s_{m+q-1}$ for all m implies, as in [10], p. 405, that the H^∞-functions that we obtain for distinct subsequences have disjointly supported Fourier transforms. Therefore, we can simply add these functions to obtain a function in $H^\infty(T^N)$ whose homogenous expansion has the desired properties. This completes the proof of the theorem.

After constructing the polynomials f_m above, and showing that the sequence $\{f_m\}_{m=1}^\infty$ is bounded in $L(T^N)$, we used the weak-star compactness of closed balls in $L^\infty(T^N)$ to obtain a weak-star limit point f of the sequence $\{f_m\}_{m=1}^\infty$. In fact, this limit point is unique, because the sequence $\{f_m\}_{m=1}^\infty$ converges in the weak-star topology on $L^\infty(T^N)$. Indeed, every nonzero term in the homogeneous expansion of f_m has degree at most s_m, while the terms that we add in passing from f_m to f_{m+1} all have degree strictly greater than s_m. Thus, for each a in Z^N, the sequence $\{\hat{f}_m(a)\}_{m=1}^\infty$ is ultimately constant; the fact that the trigonometric polynomials form a dense subspace of $L^1(T^N)$, and the boundness, in $L^\infty(T^N)$, of the sequence $\{f_m\}_{m=1}^\infty$ now imply that this sequence is weak-star convergent in $L^\infty(T^N)$. If we prefer, we can work entirely with L^2-convergence rather than weak-star convergence in $L^\infty(T^N)$. The analysis above shows that $f_{m+1}-f_m$ is orthogonal to f_j for all $j \leqslant m$. The partial sums f_m of the orthogonal series

$$f_1+\sum_{m=2}^{\infty}(f_m-f_{m-1}) \tag{14}$$

form a bounded sequence in $H^\infty(T^N)$; hence this sequence is bounded in $H^2(T^N)$, and the series (14) converges in $H^2(T^N)$, to some function f. Now f is equal, almost everywhere, to the limit of some subsequence of $\{f_j\}_{j=1}^\infty$. Therefore $f\in H^\infty(T^N)$, and $\|f\|_\infty \leqslant e^{B/2}$. The convergence, in $H^2(T^N)$, of $\{f_j\}_{j=1}^\infty$ to f implies that the homogeneous expansion of f has the desired properties.

The basic construction can be modified to yield a function f, with the desired properties, such that, in addition,

$$\|f\|_\infty \leqslant q(eB)^{1/2};$$

here q is the integer used in the last paragraph of the proof, and B is the number defined in formula (11). Remember that q depends only on the set $\{s_m\}_{m=1}^\infty$ and the cone W. Suppose, for simplicity that $q=1$. If $B=1$ also, then the basic construction yields an H^∞-function f with $\|f\|_\infty \leqslant e^{B/2}=e^{1/2}$; for general B, replace the sequence $\{G_m\}_{m=1}^\infty$, by $\{G_m/B\}_{m=1}^\infty$, and apply the basic construction to get an H^∞-function f', with $\|f'\|_\infty \leqslant e^{1/2}$, such that, for all m, the term of degree m in the homogeneous expansion of f' is equal to G_m/B; then let $f=Bf'$.

When $N=1$, the set Z_+ is itself a proper cone, and Theorem 7 reduces to the known statement [10], that, if $\{s_m\}_{m=1}^\infty$ is a Hadamard set, then, for every square-summable sequence of complex numbers $\{v_m\}_{m=1}^\infty$, there exists a function f in $H^\infty(T)$ such that $\hat{f}(s_m)=v_m$ for all m. In this case, there is actually a function in $A(U)$ with these properties ([10], p. 405). When $N>1$, however, then the hypotheses of Theorem 7 do not imply the existence of a function f in $A(U^N)$ such that, for all m, the term of degree s_m in the homogeneous expansion of f is G_m. The reason for this is that, if the cardinality of the sets E_m is unbounded as $m\to\infty$, then there exists a sequence of trigonometric polynomials $\{G_m\}_{m=1}^\infty$ satisfying conditions (i), (ii), and (iii) of Theorem 1, but such that

$$\sup_z \sum_{m=M} |G_m(z)|^2 \not\to 0 \quad \text{as} \quad M\to\infty;$$

now, the terms F_s in the homogeneous expansion of any function f in $A(U^N)$ have the property that

$$\sup_z \sum_{s=S}^{\infty} |F_s(z)|^2 \to 0 \quad \text{as} \quad S\to\infty. \tag{15}$$

Indeed, assertion (15) clearly holds if f is a trigonometric polynomial, and it follows from inequality (10) that this assertion also holds for uniform limits of trigonometric polynomials.

Theorem 8. *Let W be a proper cone in Z_+^N, and let $\{s_m\}_{m=1}^\infty$ be a Hadamard set. Let $\{G_m\}_{m=1}^\infty$ **be a** sequence of trigonometric polynomials on T^N such that*

(i) $\hat{G}_m$ *vanishes off* W,

(ii) G_m *is homogeneous, of degree* s_m,

and

(iii)′ $$\sup_z \sum_{m=M}^{\infty} |G_m(z)|^2 \to 0 \quad \textit{as} \quad M\to\infty.$$

Then there exists a function f in $A(U^N)$ such that, for each m, the term of degree s_m in the homogeneous expansion of f is equal to G_m.

Proof. Let B be the number defined in formula (11), and let q be the integer introduced in the last paragraph of the proof of Theorem 7. Fix a constant R in the interval $(0,1)$. Because of hypothesis (iii)′ above, we can select a strictly increasing sequence of positive integers $\{M_k\}_{k=1}^\infty$ such that

$$\sup_z \sum_{m>M_k} |g_m(z)|^2 \leqslant R^{2k}B. \tag{16}$$

Let $M_0=0$.

For each positive integer k, construct a trigonometric polynomial g_k as follows. Adopt the temporary notation $\{G_m'\}_{m=1}^\infty$ for the sequence of functions of T^N with the property $G_m'=G_m$ if $M_{k-1}<m\leqslant M_k$, and $G_m'=0$ otherwise. Apply the modification of the basic construction proposed above to obtain an H^∞-function g_k such that, for all m, the terms of degree s_m in the homogeneous expansion of g_k is equal to G_m', and such that

$$\|g_k\|_\infty \leqslant qe^{1/2}\Big(\sup_z \sum_{M_{k-1}<m\leqslant M_k} |G_m(z)|^2\Big)^{1/2}.$$

Then $\|g_k\|_\infty \leqslant q(eB)^{1/2}R^{k-1}$, by inequality (16). Since all but finitely many of the function G_m' are trivial, the function g_k produced by the construction is actually a trigonometric polynomial.

Now the series $\sum_{k=1}^{\infty} g_k$ converges in $A(U^N)$ to a function f with

$$\|f\|_\infty \leqslant q(eB)^{1/2}\sum_{k=1}^{\infty}R^{k-1} = q(eB)^{1/2}/(1-R).$$

It is clear that the homogeneous expansion of f has the desired properties. This completes the proof of the theorem.

It is reasonable to ask if Theorem 7 and 8 remain valid with weaker hypotheses concerning the sets W and $\{s_m\}_{m=1}^{\infty}$. We shall see in the next section that some lacunarity assumption on $\{s_m\}_{m=1}^{\infty}$ is necessary. We have not been able to determine, however, whether these theorems remain valid when the proper cone W is replaced by Z_+^N. The assumption that W is a proper cone is used only in the paragraph of the proof of Theorem 7, where we observe that the sequence $\{E_m\}_{m=1}^{\infty}$ has the property that, for all sufficiently large values of q, the relation $E_m < E_{m+q}$ holds for all m. There exist sets W such that, if $s_{m+1}/s_m \to \infty$ as $m \to \infty$, then the associated sequence $\{E_m\}_{m=1}^{\infty}$ has the above property, although W is not a proper cone.

Theorems 7 and 8 are generalizations, to the case of several variables, of the main result of [10]. We now describe another generalization of this result. Call a subset $\{a^{(m)}\}_{m=1}^{\infty}$ of Z_+^N a *Hadamard set* if there exists a constant $\lambda > 1$ such that for all m, the inequality $a^{(m+1)} > \lambda a^{(m)}$ holds for the partial order on R^N induced by R_+^N. This definition implies that every Hadamard set in Z_+^N is totally-ordered relative to the partial order induced by Z_+^N.

THEOREM 9. *Let $\{a^{(m)}\}_{m=1}^{\infty}$ be a Hadamard set in Z_+^N, and let $\{v_m\}_{m=1}^{\infty}$ be a square-summable sequence of complex numbers. Then there exists a function f in $A(U^N)$ such that $\hat{f}(a^{(m)}) = v_m$ for all m.*

This is actually a special case of [11], Theorem 12. We mention it here, however, because the methods used to prove Theorems 7 and 8 provide an explicit procedure for obtaining such a function f.

We now pause to extend some of our notational conventions, and to define some useful spaces of vector-valued functions. For all multiindices a in Z^N, we denote the sum $a_1 + a_2 + \ldots + a_N$ by $|a|$; thus $|a|$ can be negative. We use the term *measure* to mean a complex, regular, Borel measure; we denote the space of measures on T^N by $M(T^N)$. It is well known ([28], p. 59) that to each integer s there corresponds a probability measure ε_s on T^N such that $\hat{\varepsilon}_s(a) = 1$ if $|a| = s$, and $\hat{\varepsilon}_s(a) = 0$ otherwise. If $f \in L^1(T^N)$, we let $F_s = \varepsilon_s * f$, and we refer to the series $\sum_{s=-\infty}^{\infty} F_s$ as the homogeneous expansion of f. Similarly, if μ is a measure on T^N, we let $\mu_s = \varepsilon_s * \mu$, and we refer to the series $\sum_{s=\infty}^{\infty} \mu_s$ as the homogeneous expansion of μ.

Let D be the closed subgroup of all elements z of T^N for which $\prod_{n=1}^{N} z_n = 1$; let π be the canonical projection of T^N onto the quotient group T^N/D. For each function f on T^N/D, let $\pi^*(f)$ be the function on T^N for which $\pi^*(f)(z) = f(\pi(z))$ for all z in T^N. Then $\pi^*(f)$ is homogeneous of

degree 0, and every function on T^N that is homogeneous of degree 0 is the image under the map π^* of a unique function on T^N/D. For each measure μ on T^N/D, let $\pi^*(\mu)$ be the measure on T^N for which

$$\int_{T^N} f(z)\, d\pi^*(\mu)(z) = \int_{T^N/D} \Big\{ \int_T f(wz)\, dw \Big\}\, d\mu(\pi(z));$$

for all continuous functions f on T^N; here the Haar measure on T is normalized so that T has mass one. If μ is absolutely continuous with Radon–Nikodym derivative f, then $\pi^*(\mu)$ is absolutely continuous, with derivative $\pi^*(f)$.

Now fix a subset S of Z, and denote the space of square-summable, complex-valued functions on S by $l^2(S)$. Then denote the standard Lebesgue spaces ([17], p. 68) of $l^2(S)$-valued, measurable functions on T^N/D by $L^p(T^N/D)(l^2(S))$. Denote the space of continuous, $l^2(S)$-valued functions on T^N/D by $C(T^N/D)(l^2(S))$, and denote the space of $l^2(S)$-valued measures on T^N/D by $M(T^N/D)(l^2(S))$. Regard any element of one of these spaces as a vector-valued function on S, with values in one of the spaces $L^p(T^N/D)$, $C(T^N/D)$, or $M(T^N/D)$.

For any such function v on S, with values in $M(T^N/D)$, say, define a function $\sigma(v)$, on S, with values in $M(T^N)$, as follows. Let γ_1 be the character on T^N for which $\gamma_1(z) = z_1$ for all z; then, for each integer s in S, let

$$\sigma(v)(s) = \gamma_1^s \cdot \pi^*[v(s)].$$

The map σ is injective, and if $v(s)$ belongs to one of the spaces $L^p(T^N/D)$, or to $C(T^N/D)$, then $\sigma(v)(s)$ belongs to the corresponding space of functions on T^N. Let X_S be the image, under the map σ, of the space $L^\infty(T^N/D)(l^2(S))$, and let X_S^0 be the image of $C(T^N/D)(l^2(S))$. Then X_S is a Banach space relative to the norm $\|\cdot\|_X$ obtained by using the map σ to transfer the norm on $L^\infty(T^N/D)(l^2(S))$ to X_S; moreover, X_S^0 is a closed subspace of X_S. Similarly, let Y_S and Y_S^0 be the images, under the map σ, of the spaces $M(T^N/D)(l^2(S))$ and $L^1(T^N/D)(l^2(S))$. Again, Y_S is a Banach space relative to the appropriate transferred norm, and Y_S^0 is a closed subspace of Y_S.

Having defined these spaces using the map σ, we now characterize them intrinsically. Let v be a function on the set S, with values in the space $M(T^N)$. Then $v \in X_S$ if and only if it satisfies the conditions

(i) $\qquad v(s) \in L^\infty(T^N), \quad$ for all s,

(ii) $\qquad v(s)\hat{\ }(a) = 0 \quad$ unless $|a| = s, \quad$ for all s,

and

(iii) $\qquad \operatorname*{ess\,sup}_{z \in T^N} \sum_{s \in S} |v(s)(z)|^2 < \infty.$

Moreover,

$$\|v\|_X = \operatorname*{ess\,sup}_{z\in T^N}\Big(\sum_{s\in S}|v(s)(z)|^2\Big)^{1/2}, \quad \text{for all } v \text{ in } X_S.$$

Such a function v belongs to X_S^0 if and only if it satisfies condition (ii) and the conditions

(i)′ $\qquad v(s)\in C(T^N), \quad$ for all s,

and

(iii)′ for each $\varepsilon>0$, there exists a finite subset S' of S such that

$$\sup_{z\in T^N}\sum_{s\in S\sim S'}|v(s)(z)|^2<\varepsilon.$$

The function v belongs to Y_S if and only if it satisfies condition (ii), and there exists a positive measure μ on T^N such that

$$\sum_{s\in S}|v(s)(B)|^2\leqslant\mu(B)$$

for all Borel sets B in T^N with the property that $wB=B$ for all w in T. Moreover, $\|v\|_Y$ is the infimum of $\mu(T)$ for measures μ with this property. Finally v belongs to Y_S^0 if and only if it satisfies condition (ii) and the conditions

(i)″ $\qquad v(s)\in L^1(T^N), \quad$ for all s,

and

(ii)″ $$\int_{T^N}\Big(\sum_{s\in S}|v(s)(z)|^2\Big)^{1/2}dz<\infty.$$

Moreover, $\|v\|_Y$ is then equal to the integral above.

For each measure μ on T^N, let $L_S(\mu)$ be the function on the set S, with values in the space $M(T^N)$, such that, for all s, the measure $L_S(\mu)(s)$ is equal to the term μ_s in the homogeneous expansion of μ. Inequality (10) implies that, if $f\in H^\infty(T^N)$, then $L_S(f)\in X_S$; inequality (15) implies that, if $f\in A(U^N)$, then $L_S(f)\in X_S^0$. Given a subset W of Z^N, let $X_{S,W}$ be the closed subspace of all elements v of X_S such that, for all s, the transform $v(s)\hat{}$ vanishes off W; define $X_{S,W}^0$ similarly. Theorem 7 states that, if W is a proper cone, and S is a Hadamard set, then $X_{S,W}$ is a subset of $L_S(H^\infty(T^N))$, and Theorem 8 states that, under the same hypotheses, $X_{S,W}^0$ is a subset of $L_S(A(U^N))$.

We now give an alternate proof of Theorem 8, using the method of [32], p. 208. For each function h in $L^1(T^N)$, and each function v in X_S, let $h*v$ be the element of X_S for which the functions $(h*v)(s)$ and $h*[v(s)]$ coincide in $L^\infty(T^N)$, for all s. Relative to product $*$, the space X_S is a Banach module over the algebra $L^1(T^N)$, and the spaces X_S^0 and $X_{S,W}^0$ are submodules of X. If $\{h_k\}_{k=1}^\infty$ is an approximate identity in $L^1(T^N)$, then $\|v-h_k*v\|_X\to 0$ as $k\to\infty$ if and only if $v\in X_S^0$. Now let W be a proper cone, let S be a Hadamard set, and let $v\in X_{S,W}^0$. By the Cohen factorization theorem ([16], Theorem 32.33) there exist functions h, in $L^1(T^N)$, and v', in $X_{S,W}^0$, such that $v=h*v'$. By Theorem 7, there exists a function f' in $H^\infty(T^N)$ such that $L_S(f')=v'$. Let $f=h*f'$. Then $f\in A(U^N)$, and $L_S(f)=v$.

Next, we give yet another proof of Theorem 8, based on a suggestion by P. Wojtaszczyk [private communication]; this proof reveals the basic idea behind our first proof of Theorem 8. Let W be a proper cone, and let S be a Hadamard set. Then let $A_{S,W}$ be the closed subspace of $A(U^N)$ consisting of all functions f in $A(U^N)$ such that $L_S(f)\in X_{S,W}^0$. To prove Theorem 8, we must show that the map L_S: $A_{S,W}\to X_{S,W}^0$ is surjective. To do this, it suffices, since $A_{S,W}$ is a Banach space, to show, for the unit ball B in $A_{S,W}$, that $L_S(B)$ has a subset that is dense in some nonempty open ball in $X_{S,W}^0$. To this end, observe, as in our first proof of Theorem 8, that, if $v\in X_{S,W}^0$, and if $v(s)=0$ except for finitely many values of s, then there exists a trigonometric polynomial f in $A(U^N)$ such that $L_S(f)=v$, and $\|f\|_\infty\leqslant qe^{1/2}\|v\|_X$. Thus $L_S(B)$ contains every such function v in $X_{S,W}^0$ for which $\|v\|_X\leqslant 1/(qe^{1/2})$. Now these functions v form a dense subset of the open ball of radius $1/(qe^{1/2})$ in $X_{S,W}^0$.

The next theorem follows from Theorem 7 and 8 by duality arguments. We omit the proof, because we plan to outline similar arguments in the proof of Theorem 13 in Section 5.

THEOREM 10. *Let W be a proper cone, and let S be a Hadamard set. Let μ be a measure on T^N with the property that $\hat\mu(\alpha)=0$ for all α in Z_+^N for which $|\alpha|\notin S$. Then there exist measures ν* and *η on T^N such that*

(a) $\mu=\nu+\eta$;

(b) $\hat\nu(\alpha)=0$ *unless* $|\alpha|\in S$,

(c) $L_S(\nu)\in Y_S$,

(d) $\hat\eta$ *vanishes on* W.

If μ is absolutely continuous, then η and ν can be chosen to be absolutely continuous also, and such that $L_S(\nu)\in Y_S^0$.

The statement of this theorem can be greatly simplified if $N=1$. It is easy to see that, in this case, Theorem 10 is equivalent to the assertion that, if S is a Hadamard set, and μ is measure on T with the property that $\hat\mu$ vanishes on the set $Z_+\sim S$, then

$$\sum_{s\in S}|\hat\mu(s)|^2<\infty.$$

This assertion is known ([11], Theorem 10) and can be proved by the method used by Rudin in [32], p. 226, to show, for a larger class of thin sets S, that every measure μ for which $\hat{\mu}$ vanishes on the complement of S in Z_+ is absolutely continuous. Further results of this kind appear in [23] and [4].

We now use Theorem 10 to obtain a slight extension of Forelli's F. and M. Riesz theorem for measures that annihilate $A(U^N)$. Let $\varphi\colon R^N \to T^N$ be the usual covering mapping. For sets V, of unit vectors in R^N, define the notion of *V-width zero* as in [29], p. 140.

THEOREM 11. *Let S be a Hadamard set. Let μ be a measure on T^N with the property that $\hat{\mu}(\alpha) = 0$ for all α in Z_+^N for which $|\alpha| \notin S$. Let V be a compact set of unit vectors in the interior of R_+^N, and let B be a Borel set, in R^N, of V-width 0. Then $|\mu|(\varphi(B)) = 0$.*

Proof. As in [32], p. 226, the idea is to split the measure μ into two pieces for which the conclusion of the theorem can be shown to hold. Fix sets V and B as above. Choose a compact set V_1 in the interior of R_+^N so that V lies in the interior of V_1. Let W' be the union of the sets tV_1 as t runs through R_+, and let $W = W' \cap Z_+^N$; then W is a proper cone.

Apply Theorem 10 to obtain measures ν and η satisfying conditions (a) to (d) of that theorem. Condition (b) implies that $\hat{\nu}(\alpha) = 0$ whenever $|\alpha| < 0$. Let $\nu' = \gamma_1 \cdot \nu$; then ν' has the property that $\hat{\nu}'(-\alpha) = 0$ whenever $\alpha \in Z_+^N$; that is, ν' annihilates the algebra $A(U^N)$, in the sense of [29], p. 140. Therefore

$$|\nu|(\varphi(B)) = |\nu'|(\varphi(B)) = 0,$$

by Forelli's theorem ([29], p. 140).

On the other hand, $\hat{\eta}(\alpha) = 0$ for all α in W. Thus, the measure $\bar{\eta}$ annihilates the algebra A_W of all continuous functions on T^N whose Fourier transforms vanish off W. Now, it is easy to see, by repeating the proof of Forelli's theorem, or by a change of variable, that, if V is a compact set of unit vectors in the interior of the cone W', and if B is a Borel set of V-width 0, then $|\eta'|(\varphi(B)) = 0$ for all measures η' that annihilate the algebra A_W. In the present case then,

$$|\eta|(\varphi(B)) = |\bar{\eta}|(\varphi(B)) = 0.$$

Hence $|\mu|(\varphi(B)) = 0$, and the proof of the theorem is complete.

5. Homogeneous expansions and $\Lambda(2)$ sets. In this section, we present some results that resemble those in Section 4, but whose proofs require different methods. We shall compare the methods and results of the two sections at the end of this section.

Recall that a subset S of Z is called a *$\Lambda(2)$ set* ([32], p. 205) if there exists a constant K such that the inequality $\|f\|_2 \leqslant K\|f\|_1$ holds for all trigonometric polynomials f on T with the property that $\hat{f}$ vanishes off S. Every Hadamard set is a $\Lambda(2)$ set, but the converse is false ([32], p. 222;

[20], Chapter 5). If I and S are subsets of Z, let $I \sim S$ be the part of I that does not lie in S. For any subset I of Z let Q_I be the operator that associates with each trigonometric polynomial f the polynomial $Q_I f$ such that $(Q_I f)\hat{}$ coincides with $\hat{f}$ on the set S, and $(Q_I f)\hat{}$ vanishes elsewhere. Call a subset of Z an *interval* if it has one of the forms $[a, b]$, $(-\infty, b]$, $[a, \infty)$, or $(-\infty, \infty)$.

We now state a slight generalization of a known results ([11], Theorem 10). The method of proof goes back to [32], p. 226.

LEMMA 12. *Let S be a $\Lambda(2)$ set in Z, and let I be an interval. Then there exists a constant K, depending only on S, such that*

$$\Big(\sum_{s \in I \cap S} |\hat{f}(s)|^2\Big)^{1/2} \leqslant K\|f\|_1,$$

for all functions f in $L^1(T^N)$ with the property that $\hat{f}$ vanishes on the set $I \sim S$.

Proof. It is easy to see that, if $I = (-\infty, \infty)$, then the desired conclusion holds, with K equal to the constant in the definition of $\Lambda(2)$ set. It is shown in [11], Theorem 10, that the conclusion also holds, for a larger value of K, if I has the form $(-\infty, b]$, or $[a, \infty)$; the key fact in the proof is that, for such intervals I, the projection Q_I is a bounded operator from $L^1(T)$ to $L^{1/2}(T)$, with a norm that is independent of I. The same proof will work for all finite intervals I, provided that the norm of Q_I, as an operator from $L^1(T^N)$ to $L^{1/2}(T^N)$, is bounded by a constant that is independent of I. In fact, such a constant exists, because $Q_{[a,b]} = Q_{(-\infty,b]} - Q_{(-\infty,a-1)}$. This completes the proof of the lemma.

The main result of this section is an extension of Lemma 12 to the context of homogeneous expansions of functions of several variables.

THEOREM 13. *Let S be a $\Lambda(2)$ set in Z, and let I be an interval. Then there exists a constant K, depending only on S, such that the following statements holds.*

(A) *If f is a function in $L^1(T^N)$ with the property that the terms F_s, in the homogeneous expansion of f, vanish for all s in $I \sim S$, then $L_{I\cap S}(f) \in Y^0_{S\cap I}$*, and

$$\|L_{I\cap S}(f)\|_Y \leqslant K\|f\|_1.$$

(B) *If μ is a measure on T^N with the property that μ_s vanishes for all s in $I \sim S$, then*

$$L_{I\cap S}(\mu) \in Y_{I\cap S} \quad \textit{and} \quad \|L_{I\cap S}(\mu)\|_Y \leqslant K\|\mu\|.$$

(C) *For each function v in $X_{I\cap S}$ there exists a function f in $L^\infty(T^N)$ such that $\|f\|_\infty \leqslant K\|v\|_X$, the terms F_s vanish for all s in $Z \sim I$, and $L_{I\cap S}(f) = v$.*

(D) *For each function v in $X^0_{I\cap S}$, and each $\varepsilon > 0$, there exists a continuous function f on T^N such that $\|f\|_\infty \leqslant (1+\varepsilon)K\|v\|_\infty$, the terms F_S vanish for all s in $Z \sim S$, and $L_{I\cap S}(f) = v$.*

Proof. We begin with assertion (A). Let $L^1_{I,S}$ be the closed subspace of all functions f in $L^1(T^N)$ with the property that F_s vanishes for all s in $I \sim S$. Let f be a trigonometric polynomial in $L^1_{I,S}$. Fix z in T^N, and let f_z be the slice function on T given by the rule that $f_z(w) = f(wz)$ for all w in T. Then $\hat{f}_z(s) = F_s(z)$ for all s; hence $\hat{f}_z$ vanishes on the set $I \sim S$. By Lemma 12,

$$\Big(\sum_{s\in I\cap S} |F_s(z)|^2_z\Big)^{1/2} \leqslant K\int_T |f(wz)|\,dw.$$

Hence,

$$\int_{T^N}\Big(\sum_{s\in I\cap S} |F_s(z)|^2\Big)^{1/2} dz \leqslant K\|f\|_1.$$

By the characterization of the space $Y^0_{I\cap S}$ given in the previous section,

$$L_{I\cap S}(f) \in Y^0_{I\cap S} \quad \text{and} \quad \|L_{I\cap S}(f)\|_Y \leqslant K\|f\|_1.$$

Since the trigonometric polynomials in $L^1_{I,S}$ form a dense subspace of $L^1_{I,S}$, the same conclusion holds for all f in $L^1_{I,S}$.

Next we show that assertion (A) implies assertion (C). In doing this we suppose, for simplicity that $I = (-\infty, \infty)$. Now it is known ([17], p. 95) that $L^\infty(T^N/D)\big(l^2(S)\big)$ can be identified with dual space of $L^1(T^N/D)\big(l^2(S)\big)$ *via* the pairing

$$(w, v) \to \sum_{s\in S}\int_{T^N/D} w(s)\big((\pi(z^{-1})\big)\, v(s)\,\big(\pi(z)\big)\,d\pi(z).$$

It follows easily that X_S can be identified with the dual space of Y^0_S *via* the pairing

$$(w, v) \mapsto \langle w, v\rangle = \sum_{s\in S}\int_{T^N} w(s)(z^{-1})v(s)(z)\,dz.$$

Fix a function v in X_s. By assertion (A), the map $g \mapsto \langle L_S(g), v\rangle$ defines a bounded linear functional, of norm at most $K\|v\|_X$, on the space $L^1_{Z,S}$. By the Hahn–Banach theorem, this functional has an extension to all of $L^1(T^N)$, with the same norm. Thus there exists a function f in $L^\infty(T^N)$, with $\|f\|_\infty \leqslant K\|v\|_X$, such that

$$\langle L_S(g), v\rangle = \int_{T^N} g(z^{-1})f(z)\,dz,$$

for all functions g in $L^1_{Z,S}$. To see that $L_S(f) = v$, use the relation above with g replaced by various characters in $L^1_{Z,S}$.

Any of the arguments used to derive Theorem 8 from Theorem 7 can be used to show that assertion (C) implies assertion (D). We omit the details.

Finally, we show that assertion (D) implies assertion (B). Again we suppose, for simplicity, that $I = (-\infty, \infty)$. An easy argument using slice functions shows that, if $f \in C(T^N)$, then $L_S(f) \in X^0_S$, and $\|L_S(f)\|_X \leqslant \|f\|_\infty$. Let $C_{S'}$ be the null space of the bounded operator L_S: $C(T^N) \to X^0_S$. Then L_S induces an injection

$$\tilde{L}_S\colon C(T^N)/C_{S'} \to X^0_S.$$

Assertion (D) states that $\tilde{L}_S$ is surjective, and that its inverse has norm at most K. It follows that the dual map

$$\tilde{L}^*_S\colon (X^0_S)^* \to [C(T^N)/C_{S'}]^*$$

also has these properties. Now $[C(T^N)/C_{S'}]^*$ is just the annihilator in $C(T^N)^*$, of $C_{S'}$. Of course, $C(T^N)^*$ can be identified with $M(T^N)$ *via* the pairing

$$(f, \mu) \mapsto \int_{T^N} f(z^{-1})\,d\mu(z);$$

the annihilator of $C_{S'}$ is then the subspace $M(T^N)_S$ consisting of all measures μ with the property that μ_s vanishes for all s in $Z \sim S$. Now $(X^0_S)^*$ can be identified with Y_S *via* the pairing

$$(w, v) \mapsto \sum_{s\in S}\int_{T^N} w(s)(z^{-1})\,dv(s)(z);$$

this follows from the fact ([3], p. 380) that $M(T^N/D)\big(l^2(S)\big)$ is the dual space of $C(T^N/D)\big(l^2(S)\big)$ relative to a similar pairing. With these conventions concerning duality, the map $(\tilde{L}^*_S)^{-1}$ becomes a bounded operator from $M(T^N)_S$ to Y_S, with norm at most K. Finally, it turns out that $(\tilde{L}^*_S)^{-1}$ is just the restriction to $M(T^N)_S$ of L_S. This completes the proof of the theorem.

Except for the setting, the arguments used to derive the remaining parts of Theorem 13 from part (A) are standard [1]; it is easy to complete the circle of implications above by showing that assertion (B) implies assertion (A). Furthermore, if a set S has the property that these assertions hold for all intervals I, then S must be a $\Lambda(2)$ set. Indeed, let f be a polynomial on T for which $\hat{f}$ vanishes off S. Let $f'(z) = f\big(\gamma_1(z)\big)$ for all z in T^N; then f' is a trigonometric polynomial on T^N, and $F'_s = \hat{f}(s)\gamma_1^s$ for all s. In particular, F'_s vanishes for all s in $Z \sim S$. By assertion (A), with $I = Z$,

$$\Big(\sum_{s\in S} |\hat{f}(s)|^2\Big)^{1/2} = \|L_S(f')\|_Y \leqslant K\|f'\|_1 = K\|f\|_1.$$

Hence S is a $\Lambda(2)$ set as claimed.

Finally, we compare the results and methods of this section with those of Section 4. In the present section, we deal with a larger class of thin sets, and we have no hypotheses involving proper cones. On the other hand, in parts (C) and (D) of Theorem 13, we obtain functions f that belong to the spaces $L^\infty(T^N)$ and $C(T^N)$ rather than to the smaller spaces $H^\infty(T^N)$ and $A(U^N)$. Thus, in the interpolation results of the present section, both the hypotheses and the conclusions are weaker than in the corresponding theorems of the previous section. When they work, the methods of Section 4 have the advantage that they yield the desired function f explicitly. If S is a Hadamard set, then we can use these methods to obtain functions f with the properties prescribed in assertion (C), and with two further properties. These are that

(a) the support of f is included in the union of the supports of the functions $v(s)$,

and

(b) F_s vanishes unless s can be expressed as an alternating sum of the form (13).

If S is merely a $\Lambda(2)$ set, then these methods fail, but we can sharpen the methods of the present section to show that there always exist functions f with the properties prescribed in assertions (C) and (D), and with property (a). We do not know, however, whether, in this case, there always exist functions f that have the desired interpolation properties, and also have property (b).

6. Related multiplier problems. In this section, we mention briefly some other applications of the basic construction; most of the results here are already known. We also discuss work by other authors on problems closely related to the Paley multiplier problem, and we outline an alternate proof of Theorem 5.

Paley's paper [26] appeared at the same time as two other papers, by Sidon [35], and Orlicz [24], that contained similar results; these three authors arrived at these conclusions independently. Sidon showed that the space P_Z, of Paley multipliers from $C(T)$ to l^1, coincides with $l^2(Z)$. Orlicz proved the corresponding statement with the trigonometric system replaced by any orthonormal system $\{\varphi_m\}_{m=1}^\infty$ in $L^2[0,1]$ with the property that $\sup_m \|\varphi_m\|_\infty < \infty$. Recently, Mahmudov [21] showed that this characterization of Paley multipliers from $C[0,1]$ to l^1 holds if and only if $\inf_m \|\varphi_m\|_1 > 0$. It is easy to use the methods of Section 2 to give new proofs of these results. In fact, these methods, or those of [21] show that if

$$\sum_{m=1}^{\infty} |p(m)\hat{f}(m)| < \infty \quad \text{for all } f \text{ in } C[0,1],$$

then

$$\sum_{m=1}^{\infty} (|p(m)|\, \|\varphi_m\|_1)^2 < \infty.$$

This leads to a new proof of the fact ([18], p. 236) that if the orthonormal system $\{\varphi_m\}_{m=1}^\infty$ is complete, then there exist functions, on the positive integers, that tend to 0 at ∞, but are not Paley multipliers from $C[0,1]$ to l^1. Indeed, by a theorem of Orlicz ([24], p. 101), every complete orthonormal system $\{\varphi_m\}_{m=1}^\infty$, in $L^2[0,1]$ has the property that

$$\sum_{m=1}^{\infty} \Big(\int_E |\varphi_m(x)|\,dx\Big)^2 = \infty,$$

for all sets E of positive measure. In particular, $\sum_{m=1}^{\infty} \|\varphi_m\|_1^2 = \infty$ for any such system $\{\varphi_m\}_{m=1}^\infty$, and there exist functions p, on the positive integers, such that $p(m) \to 0$ as $m \to \infty$, and $\sum_{m=1} (|p(m)|\, \|\varphi_m\|_1)^2 = \infty$.

We now give an alternate proof of the theorem of Orlicz cited in the previous paragraph. Suppose that $\sum_{m=1}^{\infty} \big(\int_E |\varphi_m(x)|\,dx\big)^2 < \infty$ for some set E of positive measure; denote the measure of E by $|E|$. By splitting the set E into two pieces of measure $|E|/2$, and splitting each of these pieces, etc. ([8], p. 180), we can imitate the usual construction of the Rademacher system, and obtain an orthornormal system $\{\psi_k\}_{k=1}^\infty$ such that, for all k, the function ψ_k vanishes off E, and $\|\psi_k\|_\infty = |E|^{-1/2}$. Denote the inner product in $L^2[0,1]$ by $\langle\cdot,\cdot\rangle$. For each fixed value of k, the sequence $\{\langle\varphi_m, \psi_k\rangle\}_{m=1}^\infty$ is dominated by the square-summable sequence $\{|E|^{-1/2}\int_E |\varphi_m|\}_{m=1}^\infty$; moreover, for each fixed value of m, the sequence $\{\langle\varphi_m, \psi_k\rangle\}_{k=1}^\infty$ tends to 0 as k tends to ∞. Therefore

$$\lim_{k\to\infty} \sum_{m=1}^{\infty} |\langle\varphi_m, \psi_k\rangle|^2 = 0.$$

In particular, for all sufficiently large values of k,

$$\sum_{m=1}^{\infty} |\langle\varphi_m, \psi_k\rangle|^2 < 1 = \|\psi_k\|_2^2,$$

and the system $\{\varphi_m\}_{m=1}^\infty$ cannot be complete.

Let q be an index in the interval $(0, \infty]$, let I be a subset of Z, and let $M(C_I, l^q)$ be the space of Paley multipliers from C_I to l^q, that is the space of all functions p on the set I such that the product $p\cdot\hat{f}$ belongs to $l^q(I)$ whenever $f \in C_I$. It is trivial that if $q \geqslant 2$, then $M(C_I, l^q) = l^\infty(I)$. If $q < 2$, let $r = 2q/(2-q)$; an easy application of Holder's inequality shows that $l^r \subset M(C_I, l^q)$. Edwards ([5], p. 468) has shown that, if $1 < q < 2$, then $M(C_Z, l^q) = l^r$; his argument uses Sidon's theorem [35]. Stechkin ([37], Theorem 1) used Paley's theorem to show that, if $0 < q < 2$, then there is a positive constant, C_q such that, for all intervals I, and all functions p in $l^r(I)$, the norm of p in $M(C_I, l^q)$ is at least $C_q\|p\|_r$; in particular, $M(A(U), l^q) = l^r$, for all such q. It is easy to modify the basic construction of Section 2 to give simple, direct proofs of these results. Similarly, we can show that, if W is a proper cone in Z_+^N, and if $0 < q < 2$, then the restriction to W of every element of $M(A(U^N), l^q)$ belongs to $l^r(W)$. Finally, if in fact $M(A(U^N), l^1) = l^2(Z_+^N)$, then, by Stechkin's methods, $M(A(U^N), l^q) = l^r(Z_+^N)$, for all q in the interval $(0, 2)$; indeed, the methods of [37] show that if the equality $M(A(U^N), l^q) = l^r(Z_+^N)$ holds for one value of q in the interval $(0, 2)$, then this equality holds for all q in the interval $(0, 2)$.

The following generalization of Paley's theorem was suggested by A. Pełczyński (private communication). Let $\{\varphi_m\}_{m=1}^{\infty}$ be an orthonormal system in $H^2(T)$ with the property that $\inf_m \|\varphi_m\|_1 > 0$; then every function p on Z_+ with the property that

$$\sum_{m=1}^{\infty} |p(m)\langle f, \varphi_m\rangle| < \infty \quad \text{for all } f \text{ in } A(U)$$

must be square-summable. This assertion can be proved by Paley's original method [26], or by the methods of [15] and [28], p. 222, but not by the methods of the present paper or those of [11].

For each function w on Z_+^N, let Q_w be the operator, on the space of trigonometric polynomials, such that $Q_w(f)\hat{\ }(\alpha) = w(\alpha)\hat{f}(\alpha)$ for all f and all α. The key fact in the proofs of Paley's theorem in [15] and [28], p. 222, is that, if w is the characteristic function of Z_+, then the associated operator Q_w is of weak type (1,1), and hence is bounded from $L^1(T)$ to $L^q(T)$ when $q < 1$. If $N > 1$, and w is the characteristic function of Z_+^N, then there is no positive index q for which the operator Q_w is bounded from $L^1(T^N)$ to $L^q(T^N)$. Indeed, if such an index q existed, then by a theorem of Sawyer ([33]; [12], p. 7), the operator Q_w would be of weak type $(1, 1)$; it is easy ([38], p. 158) however, to verify directly that Q_w is not weak type $(1, 1)$.

We now outline an alternate proof of Theorem 5 using the methods of [15] or [28], p. 222. These methods will work provided that we can associate with each proper cone W in Z_+^N a function w on Z^N such that

(17) $\quad w$ vanishes off Z_+^N,

(18) $\quad |w| \geq 1$ on W,

(19) $\quad Q_w$ is of weak type $(1, 1)$.

In fact, it is easy to specify such a function. First observe that each point α in Z_+^N belongs to at most 3^N of the sets I_γ defined in Section 3; thus, for each α, the series $\sum_{\gamma \in Z_+^N} \hat{g}_\gamma(\alpha)$ has at most 3^N nonzero terms. Given a proper cone W, and a point α in Z_+^N, let

$$w(\alpha) = 2^N \sum_{W \cap J_\gamma \neq \varnothing} \hat{g}_\gamma(\alpha).$$

It is easy to verify that the function w has properties (17) and (18). To prove that the operator Q_w is of weak type $(1, 1)$, use the Calderon–Zygmund theorem as in [39], p. 106, or [6], Section 6.2. A similar argument yields a proof of Theorem 6, with a larger constant in place of $3e^2 4^{N-1}$. It does not seem possible to obtain the results of Section 2 by the methods we have just outlined.

Various Soviet authors ([14]; [41], Theorem 3.17) have shown that, in Paley's theorem, the space $A(U)$ can be replaced by a class of functions analytic on a much larger domain that the open unit disc. For instance, S. A. Vinogradov has shown that, if $\sum_{m=0}^{\infty} |p(m)|^2 = \infty$, then there exists a meromorphic function f on the Riemann sphere such that

(i) the only singularity of f occurs at $z = 1$,

(ii) $\lim_{\substack{z \to 1 \\ |z| \leq 1}} f(z)$ exists,

(iii) $\sum_{m=0}^{\infty} |p(m)\hat{f}(m)| = \infty$.

The methods of the present paper do not seem to apply in this setting.

It is well known that spaces of multipliers can usually be represented as dual spaces ([7], [27]). We now describe the predual of $M(A(U^N), l^1)$. Let V_N be the space of all functions v on Z_+^N for which there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of trigonometric polynomials in $A(U^N)$ such that $\sum_{k=1}^{\infty} \|f_k\|_\infty < \infty$, and

$$|v(\alpha)| \leq \sum_{k=1}^{\infty} |\hat{f}_k(\alpha)|, \quad \text{for all } \alpha; \tag{20}$$

for each function v in V_N, let

$$\|v\|_V = \inf\Big\{\sum_{k=1}^{\infty} \|f_k\|_\infty : \text{relation (20) holds}\Big\}.$$

It follows easily from these definitions that V_N is a Banach space relative to the norm $\|\cdot\|_V$, and that, if $v \in V_N$, and $p \in M(A(U^N), l^1)$, then

$\sum_{m=1}^{\infty} |v(m)\, p(m)| < \infty$. It turns out that the pairing

$$(v, p) \mapsto \sum_{m=1}^{\infty} v(m)\, p(m)$$

represents $M(A(U^N), l^1)$ isometrically as the dual space of V_N; we omit the proof of this fact since it is similar to that of [2] Theorem 3.6. Now $V_N \subset l^2(Z_+^N)$ in any case, and, when $N = 1$, the spaces $M(A(U^N), l^1)$ and $l^2(Z_+^N)$ coincide, so that $V_1 = l^2(Z_+)$; this equality is also stated in [19]. In general, $M(A(U^N), l^1) = l^2(Z_+^N)$ if and only if $V_N = l^2(Z_+^N)$ but we have been unable to determine whether either of these equalities hold when $N > 1$. We can use the duality between $M(A(U^N), l^1)$ and V_N, however, to restate Theorem 5 in the dual form asserting that, if $v \in l^2(Z_+^N)$, and v vanishes outside a proper cone, then $v \in V_N$.

The fact that $V_1 = l^2(Z_+)$ suggests an obvious conjecture. Is it true that for each function v in $l^2(Z_+)$ there exists a function f in $A(U)$ such that $|v(m)| \leqslant |\hat{f}(m)|$ for all m? This question is also mentioned in [2], Section 5. The corresponding question with Z in place of Z_+, and $C(T)$ in place of $A(U)$ was posed by Sidon ([36,] p. 479), and it is still unanswered.

Further conjectures concerning the space $A(U)$ and its dual are suggested by various proofs of Sidon's theorem. The simplest proof [35] depends on the fact that $M(T)$, the dual space of $C(T)$, has the property that every unconditionally convergent series in $M(T)$ is absolutely square-summable; does the dual space of $A(U)$ also have this property? Other proofs [13] can be based on factorizations of bounded operators from $C(T)$ to l^1. For instance, every such operator a can be factored

$$a\colon C(T) \xrightarrow{b} l^2 \xrightarrow{d} l^1,$$

where d is a diagonal operator [22]; is a similar factorization always possible if $C(T)$ is replaced by $A(U)$? Also, every such operator a can be factored

$$a\colon C(T) \xrightarrow{i} L^2(T, \sigma) \xrightarrow{c} l^1,$$

where σ is bounded, regular, Borel measure on T, and i is the canonical injection; is such a factorization always possible if $C(T)$ is replaced by $A(U)$?

The basic construction of Section 2 works in any C^*-algebra with identity. Indeed, suppose that $\{A_m\}_{m=1}^M$ is a sequence of operators on a Hilbert space H. Define sequences $\{B_m\}_{m=1}^M$ and $\{C_m\}_{m=1}^M$, of operators on H, as follows. Let $B_1 = A_1$, and $C_1 = I$; given B_m and C_m, let

$$B_{m+1} = B_m + A_{m+1} C_m \quad \text{and} \quad C_{m+1} = C_m - A_{m+1}^* B_m.$$

It is easy to verify for each element h of H that

$$\|B_m h\|^2 + \|C_m h\|_2^2 \leqslant \|h\|^2 \prod_{i=1}^{m} (1 + \|A_i\|^2),$$

for all m. Thus,

$$\|B_m\|^2 \leqslant \prod_{i=1}^{m} (1 + \|A_i\|^2),$$

for all m.

We mention two applications of this construction in this setting. First, we formulate a generalization of a key lemma in [11]. Given a sequence $\{A_m\}_{m=1}^M$ of bounded operators on a Hilbert space, and a set S consisting of an odd number of integers from the interval $[1, M]$, define an operator A_S as follows. Enumerate the elements of S in increasing order as $\{m_i\}_{i=1}^k$, and let

$$A_S = A_{m_k} A_{m_{k-1}}^* \ldots A_{m_2}^* A_{m_1};$$

here, as in Section 2, we insert the adjoint operator $A_{m_i}^*$ whenever i is even.

LEMMA 14. *Let H be a Hilbert space, let g and h be elements of H, and let $\{A_m\}_{m=1}^M$ be a sequence of bounded operators on H, all of norm at most* 1. *Suppose that $\langle g, A_S h\rangle = 0$ whenever S is a subset of $\{1, 2, \ldots, M\}$, of odd cardinality greater than* 1. *Then*

$$\sum_{m=1}^{M} |\langle g, A_m h\rangle|^2 \leqslant e\|g\|^2 \|h\|^2.$$

We omit the proof. The special case of this result when the operators A_m are all unitary, and the constant e is replaced by 4, is essentially the same as Lemma 1 of [11].

We can also use the above construction to improve slightly on some results due to Nicole Tomczak-Jaegermann [40]. A Banach space X is said to be of *cotype* 2 if there exists a constant C such that, for all finite sequences $\{x_m\}_{m=1}^M$ in X,

$$\Big(\sum_{m=1}^{M} \|x_m\|^2\Big)^{1/2} \leqslant C \int_0^1 \Big\| \sum_{m=1}^{M} x_m r_m(t) \Big\|\, dt.$$

Tomczak-Jaegermann showed that, if X is a subspace of the dual space of a C^*-algebra with identity, then X is of cotype 2, with $C \leqslant 2e^{1/2}$; she also showed that, if $1 \leqslant p \leqslant 2$, then the Schatten class S_p is of cotype 2, again with $C \leqslant 2e^{1/2}$. In the proof of these facts, she used a generalization of a Riesz-product construction due to Salem and Zygmund; by using the above construction instead, we can show that these conclusions hold with $C \leqslant e^{1/2}$.

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