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On the problem of convergence of a bounded-below sequence of symmetric forms for the Schrödinger operator

by

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Abstract. In the paper the problem of smooth approximation of a Hamiltonian with singular positive potentials for which – roughly speaking – Friedrich's extension does not have to exist has been solved. An application to the theory of potential scattering is also given.

1. Introduction. In Robinson's paper [6] the approximation procedure for a self-adjoint two-particle system Hamiltonian with strongly singular potentials was presented in the case where two well-known definitions of a Hamiltonian, i.e. Friedrichs' extension and the form-sum, coincide.

This allowed him to extend Lavine's results on the theory of potential scattering with positive decreasing H_0-bounded potentials to positive decreasing potentials with arbitrary singularity at point zero.

The intention of the present paper is to establish some approximation theorems for singular potentials under much wider conditions than those given by Robinson and to present their application to the theory of potential scattering. The main results are included in Theorems 2 and 4.

2. Formulation of the problem and results. Let us consider the (complex) Hilbert space $\mathscr{H} = L^2(\boldsymbol{R}^l, d^l x)$, $l \geqslant 2$. Let us define self-adjoint operators H_0 and V acting in $\mathscr{H}$ by:

$$(T_0 u)(x) = -\Delta u(x), \quad \mathscr{D}(T_0) = C_0^\infty(\boldsymbol{R}^l), \quad H_0 = T_0^\sim;$$

$$(Vu)(x) = v(x)u(x),$$

$$\mathscr{D}(V) = \Big\{u \in L^2;\ \|Vu\|_2 = \int_{\boldsymbol{R}^l} |v(x)u(x)|^2 d^l x < \infty\Big\},$$

where $\Delta = \sum_{k=1}^{l} (\partial^2/\partial x_k^2)$ and $v(x)$ is a real-valued measurable function such that

$$0 \leqslant v(x) \in L^1_{\mathrm{loc}}(\boldsymbol{R}^l \setminus S), \quad S = \{a_0, a_1, \ldots, a_\nu\}.$$

Let $\{V_n\}_{n\geqslant 1}$ be a sequence of self-adjoint operators associated with such bounded functions that

1. $0 \leqslant v_1(x) \leqslant \ldots \leqslant v_n(x) \leqslant \ldots \leqslant v(x)$,
2. a.e.-$\lim\limits_{n\to\infty} v_n(x) = v(x)$.

The following truncation of operator V will by denoted by $V_{(n)}$:

$$(V_{(n)}u)(x) = v_{(n)}(x)u(x),$$

$$v_{(n)}(x) = \begin{cases} v(x) & \text{for} \quad v(x) < n, \\ 0 & \text{for} \quad v(x) > n. \end{cases}$$

Any positive self-adjoint operator A defines the following form:

$$\mathcal{I}_A[u, v] = \langle A^{1/2}u, A^{1/2}v\rangle, \qquad \mathcal{D}(\mathcal{I}_A) = \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}).$$

Now, let us consider the sequence of forms $\mathcal{I}_n$:

$$0 \leqslant \mathcal{I}_1 \leqslant \mathcal{I}_2 \leqslant \ldots \leqslant \mathcal{I}_n \leqslant \ldots \leqslant \mathcal{I}_H,$$

where

$$\mathcal{I}_n = \mathcal{I}_{H_0} + \mathcal{I}_{V_n}, \qquad \mathcal{D}(\mathcal{I}_n) = \mathcal{D}(\mathcal{I}_{H_0});$$

$$\mathcal{I}_H = \mathcal{I}_{H_0} + \mathcal{I}_V, \qquad \mathcal{D}(\mathcal{I}_H) = \mathcal{D}(\mathcal{I}_{H_0}) \cap \mathcal{D}(\mathcal{I}_V).$$

Let $H_n = H_0 + V_n$ with $\mathcal{D}(H_n) = \mathcal{D}(H_0)$ and let $H = H_0 + V$ (form-sum).

Then ([5], p. 569) there exists such a self-adjoint operator $H_\infty \geqslant 0$ that

$$\mathcal{I}_n[u, v] \to \mathcal{I}_{H_\infty}[u, v], \qquad u, v \in \mathcal{D}(\mathcal{I}_{H_\infty})$$

and

$$(H_n + \lambda)^{-1} \to (H_\infty + \lambda)^{-1}, \qquad \operatorname{Re}\lambda > 0.$$

It is well known that $\mathcal{I}_{H_\infty} \subset \mathcal{I}_H$. Moreover, if V_n $(= V_{(n)})$ is a truncation of operator V, then $\mathcal{I}_{H_\infty} = \mathcal{I}_H$ ([2], [3]). Generally, it is not clear whether $\mathcal{I}_{H_\infty} = \mathcal{I}_H$.

If we assume that $0 \leqslant v(x) \in L^2_{\text{loc}}(\boldsymbol{R}^l \setminus S)$, then Friedrich's extension H_F of the operator $H_0 + V$ on $\mathcal{D}(H_0 + V) = \mathcal{D}(H_0) \cap \mathcal{D}(V)$ is well defined. Since $\mathcal{D}(\mathcal{I}_{H_\infty}) \supset \mathcal{D}(\mathcal{I}_{H_F})$, the equality $H = H_F$ implies the equality $H = H_F = H_\infty$.

Robinson achieved the following result:

THEOREM R. *Let $\mathcal{H} = L^2(\boldsymbol{R}^l, d^l x)$ with $l \neq 2$. Let us assume that*

$$0 \leqslant v(x) \in L^p_{\text{loc}}(\boldsymbol{R}^l \setminus \{0\}),$$

where

$$p = \begin{cases} 2 & \text{when} \quad l = 1 \text{ or } l \geqslant 4, \\ \infty & \text{when} \quad l = 3. \end{cases}$$

Then $H_\infty = H$.

We shall now prove the theorem as follows:

THEOREM 1. *Let $\mathcal{H} = L^2(\boldsymbol{R}^l, d^l x)$, $l \geqslant 3$. Let us assume that*

$$0 \leqslant v(x) \in L^2_{\text{loc}}(\boldsymbol{R}^l \setminus S), \qquad S = \{a_0, a_1, \ldots, a_\nu\}.$$

Then $H_F = H$.

Remark. Since $H_0 + V$ is essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^l \setminus S)$ where $l \geqslant 4$ (see, for instance, [7]), then useful statements of Theorem R and Theorem 1 are suitable only for cases where

$$(p, l) = \{(2, 1), (\infty, 3), (2, 3)\}.$$

If $l = 2$, then the following theorem holds true:

THEOREM 1′. *Let $\mathcal{H} = L^2(\boldsymbol{R}^2, d^2 x)$. Let us assume that*

1. $v(x) \in L^2_{\text{loc}}(\boldsymbol{R}^2 \setminus S)$, $S = \{a_0, a_1, \ldots, a_\nu\}$,
2. $v(x) \geqslant \alpha_i |x - a_i|^{-2}$, $\alpha_i > 0$, $i = 0, 1, \ldots, \nu$.

Then we have $H_F = H$.

Now, let us consider the case where

$$0 \leqslant v(x) \in L^q_{\text{loc}}(\boldsymbol{R}^l \setminus S), \qquad q < 2.$$

Clearly, H_F does not now have to exist. Nevertheless, the following theorems are true:

THEOREM 2. *Let $\mathcal{H} = L^2(\boldsymbol{R}^l, d^l x)$ with $l \geqslant 3$. Let us assume that*

$$0 \leqslant v(x) \in L^1_{\text{loc}}(\boldsymbol{R}^l \setminus S).$$

Then $H_\infty = H$.

THEOREM 2′. *Let $\mathcal{H} = L^2(\boldsymbol{R}^2, d^2 x)$. Let us assume that*

1. $v(x) \in L^1_{\text{loc}}(\boldsymbol{R}^2 \setminus S)$.
2. $v(x) \geqslant \alpha_i |x - a_i|^{-2}$, $\alpha_i > 0$, $i = 0, 1, \ldots, \nu$.

Then $H_\infty = H$.

3. Proof of Theorem 1. The idea of the proof is the same as in Robinson's work [6], but here the information about the core of the form $\mathcal{I}_H$ is taken into consideration. So, let all the assumption of Theorem 1 be satisfied. Let $\mathcal{D}_H$ be a certain core of the form $\mathcal{I}_H$. To prove the theorem it is sufficient to construct such a sequence $\{\Psi_N\}$ for any $\Psi \in \mathcal{D}_H$ that:

1. $\Psi_N \in \mathcal{D}(H_0) \cap \mathcal{D}(V)$,
2. $\|\Psi_N - \Psi\|_2 \to 0$,
3. $\mathcal{I}_H[\Psi_N - \Psi] \to 0$.

Let us take $\mathcal{D}_H = \exp(-H)\mathcal{E}$, where $\mathcal{E} = \bigcap\limits_{1\leqslant p\leqslant\infty} L^p$. Then $\mathcal{D}_H$ is a core of H and hence a core of $\mathcal{I}_H$. Moreover, $\mathcal{D}_H \subset \mathcal{E}$ ([2], [3]).

Let $\Psi \in \mathscr{D}_H$. We shall define $\Psi_{n,k}$ assuming that $\Psi_{n,k} = \Psi_n * \varrho_k = (\Psi w_n) * \varrho_k$, where ϱ_k is a smooth approximation of the identity and where

$$w_n = \prod_{i=1}^{\nu} w_n^{(i)}, \quad w_n^{(i)} \in C_0^\infty(\mathbf{R}^l \setminus \{a_i\}), \quad 0 \leqslant w_n^{(i)} \leqslant 1,$$

$$w_n^{(i)}(x) = \begin{cases} 1 & \text{when} \quad 1/n < |x-a_i| < n, \\ 0 & \text{when} \quad |x-a_i| < 1/n \text{ or } |x-a_i| > 2n, \end{cases}$$

$$|\nabla w_n^{(i)}(x)| \leqslant \begin{cases} d_i |x-a_i|^{-1} & \text{when} \quad |x-a_i| < 1/n, \\ 1 & \text{when} \quad |x-a_i| > n. \end{cases}$$

Lemma 1. *The sequence $\{\Psi_{n,k}\}$ with any fixed n satisfies the following conditions:*

(a) $\Psi_{n,k} \in C_0^\infty(\mathbf{R}^l \setminus S)$,

(b) $\|\Psi_{n,k} - \Psi_n\|_2 \to 0$,

(c) $\mathscr{I}_{H_0}[\Psi_{n,k} - \Psi_n] \to 0$,

(d) $\mathscr{I}_V[\Psi_{n,k} - \Psi] \to 0$.

Proof. Obviously, only point (d) needs proving. We have

$$I = \int_{R^l} v(x) |\Psi_{n,k}(x) - \Psi_n(x)|^2 d^l x \leqslant \|\Psi_{n,k} - \Psi_n\|_\infty \cdot \|\Psi_{n,k} - \Psi_n\|_2 \cdot \int_{K'} v^2(x) d^l x,$$

where $K' = K + \varepsilon_n$, $\forall \varepsilon_n > 0$, $K = \operatorname{supp}(w_n)$.

By the inequality $\|\Psi_{n,k}\|_\infty \leqslant \|\Psi_n\|_\infty \leqslant \|\Psi\|_\infty < \infty$ and by condition (b) we eventually obtain:

$$I \leqslant \|v\|_{L^2(K')} \cdot 2\|\Psi\|_\infty \cdot \|\Psi_{n,k} - \Psi_n\|_2 \to 0, \quad \text{when} \quad k \to \infty.$$

Lemma 2. *Let $\Psi_n = \Psi w_n$, $\Psi \in \mathscr{D}_H$. Then*

(a) $\|\Psi_n - \Psi\|_2 \to 0$,

(b) $\mathscr{I}_V[\Psi_n - \Psi] \to 0$,

(c) $\mathscr{I}_{H_0}[\Psi_n - \Psi] \to 0$.

Proof. Points (a) and (b) are obvious. To prove point (c) we mention that

$$\|\nabla w_n^{(i)} \Psi\|_2^2 \leqslant d_i^2 \int_{|x-a_i|<1/n} \left|\Psi(x)|x-a_i|^{-1}\right|^2 d^l x + \int_{|x-a_i|>n} |\Psi(x)|^2 d^l x = I_1^{(i)} + I_2^{(i)},$$

$$I_2^{(i)} \to 0 \quad \text{as} \quad n \to \infty.$$

Since $l \geqslant 3$, we have

$$I_1^{(i)} = d_i^2 \|\Psi\|_\infty \int_{|t|<1/n} t^{l-3} dt \to 0, \quad \text{when} \quad n \to \infty.$$

Hence,

$$\mathscr{I}_{H_0}[\Psi_n - \Psi] = \|\nabla(\omega_n - 1)\Psi\|_2^2 = \|(\nabla w_n)\Psi - (1-w_n)\nabla\Psi\|_2^2$$
$$\leqslant 2\left(\|(\nabla w_n)\Psi\|_2^2 + \|(1-w_n)\nabla\Psi\|_2^2\right) \leqslant 2\|(1-w_n)\nabla\Psi\|_2^2 + 2\sum_{i=1}^{\nu}(I_1^{(i)} + I_2^{(i)}),$$

i.e. $\mathscr{I}_{H_0}[\Psi_n - \Psi] \to 0$ as $n \to \infty$.

Thus the required sequence has been constructed. This completes our proof.

Remarks. 1. Applying condition 2 of Theorem 1′ we have

$$a_i \int_{|x-a_i|<1/n} |x-a_i|^{-2} \Psi(x) \overline{\Psi(x)} \leqslant \int_{|x-a_i|<1/n} \Psi(x) v(x) \overline{\Psi(x)} \to 0, \quad \text{when} \quad n \to \infty,$$

since $\Psi(x) v(x) \overline{\Psi(x)} \in L^1(\mathbf{R}^l)$.

2. Proving point (d), one may proceed as follows: Choose $\{V_m\}_{m=0}^\infty$ such that $V_m \xrightarrow{L^2(K')} V$ and $V_m \in L^\infty$. Then

$$I \leqslant \int_{R^l} |v_m(x) - v(x)| \cdot |\Psi_{n,k} - \Psi_n|^2 d^l x + $$
$$+ \|\Psi_{n,k} - \Psi_n\|_\infty \cdot \|\Psi_{n,k} - \Psi_n\|_2 \cdot \int_{K'} v_m^2(x) d^l x < \frac{\varepsilon(m)}{2} + \frac{\varepsilon(k)}{2} < \varepsilon.$$

Thus a stronger result than Theorem 1 has been proved here (see [1]).

Theorem 0. *Let $0 \leqslant v(x) \in L^1_{\rm loc}(\mathbf{R}^l \setminus S)$, $S = \{a_0, a_1, \ldots, a_\nu\}$. Then the form $\mathscr{I}_H[f, g] \upharpoonright C_0^\infty(\mathbf{R}^l \setminus S)$ is closable.*

Corollary. *Let $v = v_+ - v_-$, let v_+ satisfy the conditions of Theorem 0, and let $|V_-|^{1/2}$ be $H_0^{1/2}$-bounded with $a < 1$, i.e. $\mathscr{D}(|V_-|^{1/2}) \supset \mathscr{D}(H_0^{1/2})$, $\||V_-|^{1/2} f\|_2 \leqslant a\|H_0^{1/2} f\|_2 + b\|f\|_2$, $a \in [0, 1)$, $b \geqslant 0$, $f \in \mathscr{D}(H_0^{1/2})$.*

Then $\mathscr{I}_H[f, g] = \langle H_0 f, g\rangle + \langle Vf, g\rangle$ is closable on $C_0^\infty(\mathbf{R}^l \setminus S)$, $S = \{a_0, a_1, \ldots, a_\nu\}$.

4. Proof of Theorem 2. First of all let us mention that the statement of Theorem 2 under the restrictive assumptions $0 \leqslant v(x) \in L^q_{\rm loc}(\mathbf{R}^l \setminus S)$ with a certain $q > 1$ is an easy consequence of [5], Proposition 2.5 and Proposition 3.4.

Now, let $q = 1$. Let

$$Y(x) = \begin{cases} -C^2 & \text{when} \quad |x| \leqslant 1, \\ 0 & \text{when} \quad |x| > 1. \end{cases}$$

Let us define an operator $Y\colon \mathscr{H} \to \mathscr{H}$ by $(Y\Psi)(x) = Y(x)\Psi(x)$. Then $H_0 + Y$ with $\mathscr{D}(H_0 + Y) = \mathscr{D}(H_0)$ possesses a strongly negative eigenvalue $-\mu^2$ and a strongly positive eigenvector Ψ_μ with $\|\Psi_\mu\|_2 = 1$.

Let $\mathscr{H}' = L^2(\boldsymbol{R}^l, d\gamma)$, $d\gamma(x) = \Psi_\mu^2(x)\, d^l x$. Let us define the mapping $W\colon \mathscr{H}' \to \mathscr{H}$ by the equality $(Wu)(x) = \Psi_\mu(x)\, u(x)$.

Let $A = H_0 + Y + \mu^2$ and $A' = W^{-1} A W$. Then $(L^1(\boldsymbol{R}^l, d\gamma) \equiv \mathscr{B}_1)$ $\|\exp(-tA') f\|_{\mathscr{B}_1} \leqslant \|f\|_{\mathscr{B}_1}$ (see, for instance, [1]).

Lemma 3. *Let* $F = H_\infty$ *or* H. *Then* $\|\exp(-tF') f\|_{\mathscr{B}_1} \leqslant M^t \|f\|_{\mathscr{B}_1}$, $f \in \mathscr{H}'$.

Proof. Using the Trotter formula, we obtain

$$\|\exp(-tF_n') f\|_{\mathscr{B}_1} \leqslant \|\exp(-tH_0') f\|_{\mathscr{B}_1},$$

where $F_n' = W^{-1} F_n W$, $(F_n = H_0 + V_n$ or $H_0 + V_{(n)})$.

Now,

$$\|\exp(-tH_0') f\|_{\mathscr{B}_1} = \|\exp(-t[A' - Y - \mu^2) f\|_{\mathscr{B}_1}$$
$$\leqslant \|\exp(-t[Y+\mu]^2]) f\|_{\mathscr{B}_1} \leqslant M^t \|f\|_{\mathscr{B}_1}.$$

Hence

$$\|\exp(-tF') f\|_{\mathscr{B}_1} \leqslant \varlimsup_{n\to\infty} \|\exp(-tF_n') f\|_{\mathscr{B}_1} \leqslant M^t \|f\|_{\mathscr{B}_1}.$$

Defining $\exp(-tF_1')$ to be the unique bounded extension of the mapping $\exp(-tF') \restriction \mathscr{B}_1 \cap \mathscr{H}'$ on $\mathscr{B}_1$, one may see that $\exp(-tF_1')$ is a C_0-semigroup of quasicontractions on $\mathscr{B}_1$ and $-F_1'$ is its infinitesimal operator.

Lemma 4. *For any* $\varphi \in \mathscr{D}(H_{0,1}') \cap \mathscr{D}(V_1') \cap \mathscr{H}'$ *the following equality holds true:*

$$\exp(-tF_1')(H_{0,1}' + V_1')\varphi = F_1' \exp(-tF_1')\varphi,$$

where $-H_{0,1}'$ *and* $-V_1'$ *are infinitesimal operators of the semigroups* $(\exp(-tH_0') \restriction \mathscr{B}_1 \cap \mathscr{H}')^\sim$ *and* $(\exp(-tV_1') \restriction \mathscr{B}_1 \cap \mathscr{H}')^\sim$, *respectively.*

Proof. For any $\varphi \in \mathscr{H}'$

$$\|F' \exp(-tF')\varphi - F_n' \exp(-tF_n')\varphi\|_1$$
$$\leqslant \|F' \exp(-tF')\varphi - F_n' \exp(-tF_n')\varphi\|_2 \to 0, \quad \text{as} \quad n \to \infty,$$
$$\|\exp(-tF_1')(H_{0,1}' + V_1')\varphi - F_n' \exp(-tF_n')\varphi\|_1$$
$$\leqslant \varepsilon/2 + M^t \|V_1'\varphi - W_n \varphi\|_1 \leqslant \varepsilon, \quad \forall \varphi \in \mathscr{D}(H_0') \cap \mathscr{D}(V_1').$$

Here $w_n = V_n$ or $V_{(n)}$.

Let us notice that $W^{-1} C_0^\infty(\boldsymbol{R}^l \setminus S) \subset \mathscr{D}(H_{0,1}') \cap \mathscr{D}(V_1') \cap \mathscr{H}'$. Hence, as a consequence of Lemma 4 and of the fact that F_1' is closed we obtain

$$W^{-1} C_0^\infty(\boldsymbol{R}^l \setminus S) \subset \mathscr{D}(F_1')$$

and

$$F_1'\varphi = H_{0,1}'\varphi + V_1'\varphi, \qquad \varphi \in W^{-1} C_0^\infty(\boldsymbol{R}^l \setminus S).$$

Thus, operators $H_{\infty,1}'$ and H_1' are both restrictions of the operator N_0, where

$$N_0 = [H_{0,1}' + V_1'] \restriction W^{-1} C_0^\infty(\boldsymbol{R}^l \setminus S).$$

Lemma 5. $R(N_0 + 1)$ *is dense in* $\mathscr{B}_1 \equiv L^1(\boldsymbol{R}^l, d\gamma)$.

In fact, the proof of this lemma is given in [7], Proposition 3.4 and will not be presented here.

It follows from Lemmas 4 and 5 that $H_{\infty,1}'$ and H_1' coincide. So the semigroups $\exp(-tH_{\infty,1}')$ and $\exp(-tH_1')$, and hence the semigroup $\exp(-tH_\infty)$ and $\exp(-tH)$, are equal. Thus $H_\infty = H$.

Theorem 2 is now proven.

Remark. The following theorem strictly follows from the considerations given above (compare [7], Theorem 1.3).

Theorem 3. *Let* $\mathscr{H} = L^2(\boldsymbol{R}^l, d^l x)$, $l \geqslant 3$. *We assume that* $0 \leqslant v(x) \in L^1_{\mathrm{loc}}(\boldsymbol{R}^l \setminus S)$, $S = \{a_0, a_1, \ldots, a_\nu\}$. *Then for any* $t > 0$

$$S - \mathscr{L})(\mathscr{H}) - \lim_{n\to\infty} (\exp(-H_0 t/n) \exp(-Vt/n))^n = \exp(-tH).$$

5. Application in potential scattering. Let $\mathscr{H} = L^2(\boldsymbol{R}^3)$. Let $q\colon \boldsymbol{R}^3 \to q(x) \in \boldsymbol{R}^1$ be a measurable function. We assume that

1. $q(x) = v(x) + w(x)$, $x \in \boldsymbol{R}^3$,
2. $0 \leqslant v(x) \in L^1_{\mathrm{loc}}(\boldsymbol{R}^3 \setminus S)$, $S = \{a_0, a_1, \ldots, a_\nu\}$,
3. $w(x) \in R$ (R is the Rollnic class; see [8]).

Lemma 6. *For any* $N \geqslant 1$ *and* $\lambda > 0$ *the operators*

$$A = |W|^{1/2}(H_0 + V + \lambda)^{-N},$$
$$B = |W|^{1/2}(H_0 + V + \lambda)^{-N}|W|^{1/2},$$
$$C = (H_0 + V + \lambda)^{-N/2}|W|\,(H_0 + V + \lambda)^{-N/2}$$

are the Hilbert–Schmidt operators.

The following estimations hold true:

$$\|A\|_{\mathscr{H}.\mathscr{S}.} \leqslant \|A_0\|_{\mathscr{H}.\mathscr{S}.},$$
$$(*) \qquad \|B\|_{\mathscr{H}.\mathscr{S}.} \leqslant \|B_0\|_{\mathscr{H}.\mathscr{S}.},$$
$$\|C\|_{\mathscr{H}.\mathscr{S}.} \leqslant \|C_0\|_{\mathscr{H}.\mathscr{S}.},$$

where, for instance, $A_0 = |W|^{1/2}(H_0 + \lambda)^{-N}$.

Proof. Let K be a positive cone in $L^2(\boldsymbol{R}^3)$. Then for any $f \in K$ and $\lambda > 0$

$$(H_0 + \lambda)^{-N} f - (H_0 + V + \lambda)^{-N} f \in K.$$

This inclusion is a consequence of Theorem 3 and of the representation of a resolvent by its semigroup.

Since $|W|^{1/2}$ is the Kato-perturbation of the operator $H_0^{1/2}$, the operators A_0 and A are bounded mappings in $L^2(\boldsymbol{R}^3)$ and obviously $A_0 f - Af \in K$ $\forall f \in K$. In particular $\|A_0 f\| \leqslant \|Af\|$, $\forall f \in K$.

Now, let us take an increasing sequence of finite-dimensional subspaces $\{E_n\}_{n\geqslant 1}$ from $L^2(\boldsymbol{R}^3)$ such that:

1. basis vectors for $E_n - \{\Phi_k\}_{k=1}^n$ belong to K,

2. $P_{E_n}\colon L^2 \to E_n$ strongly converges to 1, where P_{E_n} is an orthogonal projection on E_n.

Then

$$\|P_{E_n} A_0\|^2_{\mathscr{H}.\mathscr{S}.} = \sum_{k=1}^{n} \|A_0 \Phi_k\|^2 \geqslant \sum_{k=1}^{n} \|A\Phi_k\|^2 = P_{E_n} A\|^2_{\mathscr{H}.\mathscr{S}.}.$$

Hence we have the required inequality

$$\|A\|_{\mathscr{H}.\mathscr{S}.} \leqslant \|A_0\|_{\mathscr{H}.\mathscr{S}.}.$$

The remaining two inequalities may be proved the same way (see also [6]).

THEOREM 4. *Let $\mathscr{H} = L^2(\boldsymbol{R}^3)$. Let $q = v + w$. We assume that*

1. $0 \leqslant v(x) \in L^1(\boldsymbol{R}^3 \setminus \{0\})$,
2. $v(\lambda x) \leqslant v(x)$, $\forall \lambda \geqslant 1$, $x \in \boldsymbol{R}^3$,
3. $\lim_{|x|\to\infty} |x|^3 v(x) = 0$,
4. $w(x) \in L^1 \cap R$.

Let us define $H = H_0 + Q \equiv H_0 + V + W$.

Then the wave operators $\Omega_\pm(H_0, H)$, $\Omega_\pm(H, H_0)$ do exist as unitary transformations between $\mathscr{H}$ and $\mathscr{H}^H_{a.c.}$.

Proof. By Theorem 2 and the appropriate results of Robinson [6] it follows that the wave operators $\Omega_\pm(H_0, H_0+V)$ exist and are complete.

Now, using Lemma 6 and repeating the proof of Simon's Theorem IV. 1.1, [8], one may see that the wave operators $\Omega_\pm(H_0+V, H_0+Q)$ exist and are complete. Hence, according to the transitive property of wave operators, we infer that $\Omega_\pm(H_0, H_0+Q)$ exist and are complete.

Remark. To prove inequality (*) Robinson [1] employed the following estimation, restrictive in this case, for the potential V

$$(**) \qquad (H_0 + V + \lambda)^2 \geqslant H_0^2 \quad \text{(in the sense of forms).}$$

Estimation (**) means that $H_0 + V$ is self-adjoint on $\mathscr{D}(H_0) \cap \mathscr{D}(V)$.

The proof presented here shows that the inclusion $\exp(-t[H_0+V])K \subset K$ (i.e. lemma Davis–Faris [4]) is essential for our considerations.

References

[1] A. G. Beliy and Yu. A. Semenov, *Kato's inequality and semigroup product formulae*, Funktional. Anal. i Priložen. 9, 4 (1975), pp. 59–60 (Russian).

[2] — *One criterion of semigroup product convergence*, Letters in Math. Phys. (to appear).

[3] W. G. Faris, *Quadratic forms and essential self-adjointness*, Helv. Phys. Acta 45 (1972), pp. 1074–1088.

[4] — *Essential self-adjointness of operators in ordered Hilbert space*, Commun. Math. Phys. 30 (1973), pp. 23–27.

[5] T. Kato, *Perturbation theory with singular potentials*, Mir, Moskva 1972 (Russian).

[6] D. W. Robinson, *Scattering theory with singular potentials*, preprint, Marseille, France 1973.

[7] Yu. Semenov, *Schrödinger operators with L^p_{loc}-potentials*, Commun. Math. Phys. (to appear).

[8] B. Simon, *Dissertation*, Princeton University Press, Princeton 1970.

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