If ν is a normal Radon probability measure concentrated on G, then we have

$$\langle S\nu, f\rangle = \langle T'\nu, f\rangle = \sum_j |\beta_{1j}| = \|T'\| = \|S\|,$$

which concludes the proof of the proposition.

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Smooth and $\boldsymbol{R}$-analytic negligibility of subsets and extension of homeomorphisms in Banach spaces

by

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Abstract. It is proved that, if A is a compact set in the space $E = E' \times l_p(A)$, where E' is an infinite-dimensional, separable Banach space, then $E\setminus A$ and E are $\boldsymbol{R}$-analytically isomorphic. It is also established that, if A is a closed subspace of infinite codimension in a separable Banach space or in an arbitrary Hilbert space E, then $E\setminus A$ and E are $\boldsymbol{R}$-analytically isomorphic. In the smooth category analogous facts hold true if E is any infinite-dimensional weakly compactly generated (WCG-) Banach space. It is shown that any embedding of a compact subset A of a Banach space E admits an extension to an autohomeomorphism of E which is $\boldsymbol{R}$-analytic off A provided that either A is finite-dimensional and $E = E' \times l_p(A)$ for a separable infinite-dimensional Banach space E' or $E = E' \times l_p(A)$, where E' has an unconditional Schauder basis. Other results of this type are proved.

Introduction. Let us say that a closed subset A of a manifold M is *smoothly* or *$\boldsymbol{R}$-analytically negligible in M* if $M\setminus A$ and M are smoothly or $\boldsymbol{R}$-analytically isomorphic. Negligibility of subsets was investigated by Renz [17], Moulis [14], West [21], Burghelea and Kuiper [5], Szigeti [18]. The most general theorem known in this field so far was established by Renz; it stated (in its weaker form) that compact sets are smoothly negligible in smooth Banach spaces with unconditional Schauder bases. The main result of the first part of this paper is the theorem stating that compact subsets are $\boldsymbol{R}$-analytically negligible in any infinite-dimensional separable Banach space. This is a strengthening of the result of Renz concerning smooth negligibility. (The only fact concerning $\boldsymbol{R}$-analytic negligibility known earlier was obtained by Burghelea and Kuiper [5]; it stated that $l_2\setminus\{0\}$ and l_2 are $\boldsymbol{R}$-analytically isomorphic.) We observe also that our theorem does not extend to all infinite-dimensional Banach spaces; e.g. one-point sets are not $\boldsymbol{R}$-analytically negligible in the space $c_0(A)$ with uncountable A.

In the second part of this paper we deal with questions concerning the extension of embeddings of compact subsets K of a Banach space E into the space E. Let us recall that Renz proved in [17] that if $E = l_a$

or c_0, then any such embedding can be extended to an autohomeomorphism of E which is of class C^∞ off K. Here we extend this result to the $\boldsymbol{R}$-analytic category by proving that, under the assumption that E is a Banach space with an unconditional Schauder basis, any embedding of K has an extension to an autohomeomorphism of E which is $\boldsymbol{R}$-analytic off K. We also show that an analogous theorem is true if $2 \leqslant \dim E < \infty$ and K is countable and compact. On the other hand, we observe that facts of this type are false if $E = c_0(\Lambda)$ with uncountable Λ.

The methods we employ are based on non-complete norm techniques of Bessaga and Klee [2] and Bessaga [1] and on Klee's [12] extending homeomorphisms by "pushing graphs around". However, to obtain $\boldsymbol{R}$-analytic "neglecting isomorphisms" we have to construct certain auxiliary $\boldsymbol{R}$-analytic paths and real functions in Banach spaces. This is achieved by using Whitney's approximation technique [22]. One of the main lemmas that we prove in this way states that if K is any compact subset of the Banach space $l_p(\Lambda)$, then there is a lipschitzian function which vanishes precisely on K and is $\boldsymbol{R}$-analytic off K (we call such functions "Whitney functions").

The results of this paper were announced without proofs in [8].

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Preliminaries. The following letters denote sets: $\boldsymbol{N}$ = positive integers; $\boldsymbol{R}$ = reals; $\boldsymbol{C}$ = complex numbers. We put $\overline{\boldsymbol{N}} = \boldsymbol{N} \cup \{\infty\}$ and $\boldsymbol{R}^+ = [0, \infty)$.

If M_1 and M_2 are C^p manifolds and $p \in \boldsymbol{N}$, then $C^p(M_1, M_2)$ denotes the set of p-times continuously differentiable maps from M_1 to M_2. If M_1 and M_2 are $\boldsymbol{R}$-analytic manifolds, then by $C^\omega(M_1, M_2)$ we denote the subset of $\boldsymbol{R}$-analytic maps of the set $C^\infty(M_1, M_2) = \bigcap_{p \in \boldsymbol{N}} C^p(M_1, M_2)$. $D^{(p)}\alpha$ denotes the pth derivative of a C^p map α; we put $D^{(1)}\alpha = D\alpha$. Whenever α is a continuous map we put $D^{(0)}\alpha = \alpha$. Isomorphisms in the C^r category for $r \in \overline{\boldsymbol{N}} \cup \{\omega\}$ are called C^r isomorphisms.

Let $E = (E, \|\cdot\|)$ be a normed linear space and let w be a pseudonorm on E. If $f\colon X \to E$ is a map and $Z \subset X$, we put

$$w(f)_Z = \sup\{w(f(z))\colon z \in Z\}.$$

For every $A \subset E$ we denote by $\mathrm{dist}_w(x, A)$ the distance between x and A in the pseudometric induced by w. We put

$$B_w(A, \varepsilon) = \{x \in E\colon \mathrm{dist}_w(x, A) < \varepsilon\}.$$

If $w = \|\cdot\|$, we also shall use the corresponding symbols: $\|f\|_Z$, $\mathrm{dist}_{\|\cdot\|}(x, A)$ and $B_{\|\cdot\|}(A, \varepsilon)$. A sequence $\{x_n^*\}$ of the elements of the dual space E^* is said to be *total* if the functionals x_n^* separate the points of E. Whenever $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space, we denote by $|\cdot|$ the norm induced by the scalar product $\langle\cdot,\cdot\rangle$. For the spaces $\boldsymbol{R}^n$ and $l_2(\Lambda)$ the symbols $\langle\cdot,\cdot\rangle$ (and $|\cdot|$) denote the standard scalar products $\langle\cdot,\cdot\rangle$ (and the norm defined by it).

A pseudonorm w on E is called *of class* C^r for $r \in \overline{\boldsymbol{N}} \cup \{\omega\}$ if it is of class C^r on the set $E \setminus w^{-1}(\{0\})$. The space E is said to be C^p *smooth* for $p \in \overline{\boldsymbol{N}}$ if it admits a non-trivial function $\varphi \in C^p(E, [0,1])$ whose support $\mathrm{supp}\,\varphi = \{x\colon \varphi(x) > 0\}$ is bounded. We say that E admits C^p *partitions of unity* ($p \in \overline{\boldsymbol{N}}$) if for any open cover $\mathcal{U}$ of E there exists a family $\{\varphi_a\} \subset C^p(E, [0,1])$ such that $\sum \varphi_a(x) = 1$ for every $x \in E$ and $\{\mathrm{supp}\,\varphi_a\}$ is a locally finite cover of E subordinate to $\mathcal{U}$.

We use the usual symbols for the topological operations: cl = closure and int = interior.

1. Whitney functions. The constructions in the smooth case. Suppose that $(X, \|\cdot\|)$ is a normed linear space and $w\colon X \to \boldsymbol{R}^+$ is a continuous pseudonorm. Let $A \subset X$ be a w-closed set and let U be a w-neighbourhood of A. A continuous function $\varphi\colon X \to \boldsymbol{R}^+$ will be called a C^r *Whitney function for* $(A, U; w)$, $r \in \overline{\boldsymbol{N}} \cup \{0, \omega\}$, if it satisfies the following conditions:

(1) $$\varphi^{-1}(\{0\}) = A,$$

(2) $$\varphi\colon (X \setminus A, \|\cdot\|) \to \boldsymbol{R}^+ \text{ is of class } C^r,$$

(3) $$|\varphi(x_1) - \varphi(x_2)| \leqslant w(x_1 - x_2) \text{ for all } x_1, x_2 \in X,$$

(4) $$\varphi(x) = \text{const} \quad \text{for} \quad x \notin U.$$

If $w = \|\cdot\|$, we shall call a C^r Whitney function for $(A, U; w)$ a C^r *Whitney function for* (A, U).

Let $A, B \subset X$ be disjoint closed sets. A C^p function ($p \in \overline{\boldsymbol{N}}$) $\psi\colon X \to [0,1]$ will be called a C^p *Urysohn function for* (A, B) if $\psi^{-1}(\{0\}) = A$ and $\psi^{-1}(\{1\}) = B$.

1.1. Lemma. *Suppose that $w\colon (X, \|\cdot\|) \to \boldsymbol{R}^+$ is a norm of class C^p with $p \in \overline{\boldsymbol{N}}$. If $A \subset X$ is a compact set and U is a w-neighbourhood of A, then there exists a C^p Whitney function for $(A, U; w)$. If X is w-separable and $A \subset X$ is an arbitrary w-closed set, then there exists a C^p Whitney function for $(A, X; w)$.*

Proof. A is compact. We fix $n \in \boldsymbol{N}$. For any $x \in A$ there exists a w-lipschitzian C^p function $\varphi_x\colon X \to [0,1]$ such that

$$\varphi_x | X \setminus B_w(A, 1/n) = 1 \quad \text{and} \quad \varphi_x | U_x = 0,$$

where U_x is some w-neighbourhood of x. Let $\{U_{x_i}\}_{i=1}^k$ be a cover of A. The C^p function $\varphi_n = \varphi_{x_1} \cdot \ldots \cdot \varphi_{x_n}$ satisfies $\varphi_n | A = 0$,

$$\varphi_n | X \setminus B_w(A, 1/n) = 1 \quad \text{and} \quad |\varphi_n(x) - \varphi_n(y)| \leqslant M_n \cdot w(x - y),$$

for some $1 \leqslant M_n < \infty$ and any $x, y \in X$.

Pick n_0 such that $B_w(A, 1/n_0) \subset U$. Then the function $\varphi = \sum_{n=n_0}^{\infty} 2^{-n-1}(M_n)^{-1}\varphi_n$ is the required C^p Whitney function for $(A, U; w)$.

(X, w) is separable. If $P, Q \subset X$ are disjoint w-closed sets, then there exists a C^p partition of unity $\{a_n\}_{n=1}^{\infty}$ subordinated to $\{X \setminus Q, X \setminus P\}$ with

$$|a_n(x_1) - a_n(x_2)| \leqslant M_n w(x_1 - x_2)$$

for some $1 \leqslant M_n < \infty$ and any $x_1, x_2 \in X$ (see [19], p. 44). Then, letting

$$\psi = \sum_{\operatorname{supp} a_n \subset X \setminus P} 2^{-n-1}(M_n)^{-1} a_n,$$

we have

$$\psi \in C^p(X, [0,1]), \quad \psi | P = 0, \quad \psi | Q > 0$$
$$\text{and} \quad |\psi(x_1) - \psi(x_2)| \leqslant w(x_1 - x_2)$$

for all $x_1, x_2 \in X$.

So, for any $n \in \boldsymbol{N}$, we can construct C^p functions $0 \leqslant \varphi_n \leqslant 1$ such that

$$\varphi_1 | \mathrm{cl}_w B_w(A, 1/3) = 0, \quad \varphi_1 | X \setminus B_w(A, 1/2) > 0,$$
$$\varphi_n | \mathrm{cl}_w B_w\big(A, 1/(n+2)\big) \cup X \setminus B_w\big(A, 1/(n-1)\big) = 0,$$
$$\varphi_n | \mathrm{cl}_w B_w(A, 1/n) \setminus B_w\big(A, 1/(n+1)\big) > 0 \quad \text{for} \quad n \geqslant 2$$

and

$$|\varphi_n(x_1) - \varphi_n(x_2)| \leqslant w(x_1 - x_2) \quad \text{for all } x_1, x_2 \in X \text{ and } n \in \boldsymbol{N}.$$

Finally, the function $\varphi = \sum_{n=1}^{\infty} 2^{-n}\varphi_n$ is the required C^p Whitney function for $(A, X; w)$.

The next lemma concerns Urysohn functions.

1.2. Lemma. *For every pair (K, L) of disjoint compact sets of a Hilbert space H there exists a $|\cdot|$-lipschitzian C^∞-Urysohn function.*

Proof. 1. Assume that $L = \emptyset$. Denote $H_0 = \mathrm{cl\,span}\,K$. We have $H = H_0 \oplus H_0^{\perp}$, where $H_0^{\perp}$ is the orthogonal complement of the separable subspace H_0. Let $x_n \in H_0$, $\varrho_n > 0$ for $n \in \boldsymbol{N}$, be such that $\bigcup_{n \in \boldsymbol{N}} B_{|\cdot|}(x_n, 2\varrho_n) = H_0 \setminus K$. For each n there is a function $\gamma_n \in C^\infty(\boldsymbol{R}, [0,1])$ with $\gamma_n | [0, \varrho_n] = 1$ and $\gamma_n | [4\varrho_n^2, \infty) = 0$. Then the C^∞ function φ_n given by $\varphi_n(x) = \gamma_n(|x - x_n|^2)$ for $x \in H$ satisfies $\varphi_n | B_{|\cdot|}(x_n, \varrho_n) = 1$ and $\varphi_n | H \setminus B_{|\cdot|}(x_n, 2\varrho_n) = 0$. It is evident that the C^∞ function $\psi_0 = \sum_{n=1}^{\infty} 2^{-n}(c_n)^{-1}\varphi_n$, where $c_n = \sup\{\|D^{(k)}\varphi_n\|_{H_0}: 0 \leqslant k \leqslant n\}$, is $|\cdot|$-lipschitzian and $\psi_0^{-1}(\{0\}) = K$. Take a $|\cdot|$-lipschitzian function $\gamma \in C^\infty(\boldsymbol{R}, [0,1])$ with $\gamma^{-1}(\{0\}) = 0$. The function $\psi(x, y) = \psi_0(x) + \gamma(|y|^2)$ for $(x, y) \in H_0 \times H_0^{\perp}$ has the required property.

2. The general case. Let ψ_1 and ψ_2 be C^∞ Urysohn functions for the pairs $(K, \emptyset)$ and $(L, \emptyset)$, respectively. We define $g: H \to \boldsymbol{R}^2$ by $g(x) = \big(\psi_1(x), \psi_2(x)\big)$ for $x \in H$. There is a $|\cdot|$-lipschitzian C^∞ Urysohn function $\lambda: \boldsymbol{R}^2 \to [0,1]$ for $\big(g(K), g(L)\big)$. We see that $\lambda \circ g$ is the required function.

1.3. Remark. For every pair (K, L) of disjoint compact sets of a separable normed linear space $(X, \|\cdot\|)$ there exists a $\|\cdot\|$-lipschitzian C^∞ Urysohn function. (This follows from the fact that X admits a continuous linear injection into l_2.) For every pair (A, B) of disjoint closed subsets of the space l_2 there exists a C^∞ Urysohn function. (The proof is analogous to that of 1.2.)

2. Auxiliary constructions in the $\boldsymbol{R}$-analytic case. The construction of $\boldsymbol{R}$-analytic Whitney functions is not so straighforward as it is in the smooth case. We shall perform this construction basing ourselves on Whitney's finite-dimensional technique of approximating smooth functions by $\boldsymbol{R}$-analytic ones. Our argument starts with a technically complicated lemma (Lemma 2.1), whose proof includes an adaptation of Whitney's technique. This lemma will be proved in such generality in order to allow us to construct C^ω Whitney functions (Lemma 2.2), a certain $\boldsymbol{R}$-analytic approximation of smooth homotopies (Lemma 2.3) and $\boldsymbol{R}$-analytic paths (Lemma 2.7).

To begin with, assuming that X is a Banach space, we introduce the following notation:

$$C_0^\infty(\boldsymbol{R}^n, X) = \{\alpha \in C^\infty(\boldsymbol{R}^n, X): \operatorname{span} \alpha(\boldsymbol{R}^n) \text{ is finite-dimensional}\}.$$

2.1. Technical lemma. *Suppose that H is a Hilbert space, X and E are Banach spaces, $\{x_n^*: n \in \boldsymbol{N}\}$ is a total set in E^* such that $\|x_n^*\| \leqslant 2^{-n}$ for $n \geqslant 1$ and $\{m_n: n \in \boldsymbol{N}\} \subset \boldsymbol{N}$ an increasing sequence. Write $P_n = (x_1^*, \ldots, x_{m_n}^*): E \to \boldsymbol{R}^{m_n}$ and let $\eta(z) = z^2/(z^2+1)$ for $z \in \boldsymbol{C} \setminus \{-i, i\}$. Let Q_n: $E \times H \times \boldsymbol{R} \to \boldsymbol{R}^{m_n} \times \boldsymbol{R} \times \boldsymbol{R}$ be defined by*

$$Q_n(x, h, t) = \big(P_n(x), \eta(|h|^2), t\big).$$

Moreover, let K be a compact subset of E and consider the sequence of closed neighbourhoods of K

$$A_n = Q_n^{-1}\big(\mathrm{cl}\, B_{|\cdot|}\big(P_n(K), 1/n\big) \times [-2^{-n}, 2^{-n}] \times [-2^{-n}, 2^{-n}]\big) \subset E \times H \times \boldsymbol{R} \quad \textit{for } n \geqslant 1,$$

$$A_0 = E \times H \times \boldsymbol{R},$$

which is nested in K. Write

$$B_n = A_n \setminus A_{n+1} \quad \textit{for} \quad n \geqslant 0.$$

Then there exist real numbers $0 < k_0 < k_1 < k_2 \dots$ such that for every continuous pseudonorm $\|\cdot\|'$ on X, for every sequence $\{\overline{F}_n\}_{n\geqslant 0}$ of maps such that $F_0 = \mathrm{const}\colon E\times H\times \boldsymbol{R}\to X$, $F_n = \overline{F}_n\circ Q_n \in C_0^\infty(E\times H\times \boldsymbol{R}, X)$ $(n\geqslant 1)$, and for arbitrary positive numbers δ_n $(n\geqslant 0)$, there exists a map

$$g\colon E\times H\times \boldsymbol{R}\setminus K\times\{0\}\times\{0\}\to X$$

of class C^∞ satisfying for $n\geqslant 0$ the following conditions:

(i) $\|g-F_n\|'_{B_n}\leqslant \|F_{n+1}-F_n\|'_{B_n}+\delta_n$,

(ii) $\|Dg-DF_n\|'_{B_n}\leqslant \|DF_{n+1}-DF_n\|'_{B_n}+k_n(\|F_{n+1}-F_n\|'_{B_n}+\delta_n)+\delta_n$,

(iii) $\left\|\frac{\partial}{\partial t}g-\frac{\partial}{\partial t}F_n\right\|'_{B_n}\leqslant \left\|\frac{\partial}{\partial t}F_{n+1}-\frac{\partial}{\partial t}F_n\right\|'_{B_n}+2^{n+3}(\|F_{n+1}-F_n\|'_{B_n}+ +\delta_n)+\delta_n.$

Proof. Fix positive numbers δ'_n $(n\geqslant 0)$ with

(1) $$\sum_{i\geqslant n}\delta'_n\leqslant \delta_n/2 \quad\text{and}\quad \sum_{i\geqslant 0}\delta'_n<\infty.$$

Write $K_n = \mathrm{cl}\,B_{|\cdot|}(P_n(K), 1/n)$ and observe that

(2) $$K_{n+1}\subset \mathrm{int}\,K_n\times \boldsymbol{R}^{m_{n+1}-m_n}\quad\text{for}\quad n\in\boldsymbol{N}$$

and

(3) $$K\times\{0\}\times\{0\}\subset\dots\subset Q_n^{-1}(K_n\times[-2^{-n},2^{-n}]\times[-2^{-n},2^{-n}])\subset A_{n-1}\subset\dots\subset A_1\subset A_0,$$

$$\bigcap_{n\in\boldsymbol{N}}A_n = K\times\{0\}\times\{0\}.$$

Using (2) we can find for each $n\in\boldsymbol{N}$ a function $\overline{\varphi}_n\in C^\infty(\boldsymbol{R}^{m_n},[0,1])$ such that

(4) $$\mathrm{supp}\,\overline{\varphi}_n \text{ is compact},$$

(5) $$\overline{\varphi}_n(u) = 1 \quad\text{if}\quad u \text{ belongs to some neighbourhood of } K_n,$$

(6) $$\overline{\varphi}_n(u) = 0\quad\text{if}\quad u\notin K_{n-1}\times\boldsymbol{R}^{m_n-m_{n-1}}\quad\text{for } n\geqslant 2.$$

Moreover, for each $n\in\boldsymbol{N}$ let the function $\lambda_n\in C^\infty(\boldsymbol{R},[0,1])$ be such that

(7) $$\lambda_n(t) = 1\quad\text{if}\quad t \text{ belongs to some neighbourhood of } [-2^{-n},2^{-n}],$$

(8) $$\lambda_n(t) = 0\quad\text{if}\quad |t|>2^{-n+1},$$

(9) $$|D\lambda_n|_{\boldsymbol{R}}\leqslant 4\cdot 2^n.$$

Next, for $(u,s,t)\in\boldsymbol{R}^{m_n}\times\boldsymbol{R}\times\boldsymbol{R}$, we put

$$\psi_n(u,s,t) = \overline{\varphi}_n(u)\cdot\lambda_n(s)\cdot\lambda_n(t).$$

Define $\varphi_n = \overline{\varphi}_n\circ Q_n\colon E\times H\times\boldsymbol{R}\to[0,1]$, i.e.

$$\varphi_n(x,h,t) = \overline{\varphi}_n(P_n(x))\cdot\lambda_n(\eta(|h|^2))\cdot\lambda_n(t)$$

for $(x,h,t)\in E\times H\times\boldsymbol{R}$. Conditions (5)–(8) imply the following:

(10) $$\varphi_n(x,h,t) = 1 \quad\text{if}\quad (x,h,t) \text{ belongs to some neighbourhood of } A_n,$$

(11) $$\varphi_n(x,h,t) = 0\quad\text{if}\quad (x,h,t)\notin A_{n-1}$$

for $n\in\boldsymbol{N}$.

From the fact that $\|P_n\|\leqslant 1$ and $|D_\eta(|\cdot|^2)|_H\leqslant 1$ we conclude that

(12) $$\|DQ_n\|_{E\times H\times\boldsymbol{R}}\leqslant 1\quad\text{for}\quad n\in\boldsymbol{N}.$$

Using (4), (8), (9) and (12), we infer that

(13) $$\left\|\frac{\partial}{\partial t}\varphi_n\right\|_{E\times H\times\boldsymbol{R}}\leqslant 4\cdot 2^n\quad\text{for}\quad n\in\boldsymbol{N}$$

and that there exist $0\leqslant k_0\leqslant k_1\leqslant\dots$ such that

(14) $$\|D\varphi_{n+1}\|_{E\times H\times\boldsymbol{R}}\leqslant k_n\quad\text{for}\quad n\geqslant 0.$$

We shall inductively construct $\overline{g}_n\in C_0^\infty(\boldsymbol{R}^{m_n}\times\boldsymbol{R}\times\boldsymbol{R},X)$ for $n\geqslant 0$. We put $\overline{g}_0 = F_0$. Assume that we have already constructed the maps $\overline{g}_0,\dots,\overline{g}_{n-1}$ for $n\geqslant 1$. We consider, for $a>0$, the map

$$g_a\colon \boldsymbol{R}^{m_n}\times\boldsymbol{R}\times\boldsymbol{R}\to X$$

given by

(15) $$g_a(u,s,t) = \left(\frac{a}{\pi}\right)^{n/2}\int_{\boldsymbol{R}^{m_n}\times\boldsymbol{R}\times\boldsymbol{R}}\overline{\varphi}_n(v,r_1,r_2)\times \times[\overline{F}_n(v,r_1,r_2)-(\overline{g}_0(v,r_1,r_2)+\overline{g}_1(v_1,\dots,v_{m_1},r_1,r_2)+\dots \dots+\overline{g}_{n-1}(v_1,\dots,v_{m_{n-1}},r_1,r_2))]\times \times\exp[-a(|u-v|^2+(s-r_1)^2+(t-r_2)^2)]\,dv\,dr_1\,dr_2,$$

where $v = (v_1,\dots,v_{m_n})\in\boldsymbol{R}^{m_n}$ and $r_1,r_2\in\boldsymbol{R}$. One can choose a_n so large that

(16) $$\left(\frac{a_n}{\pi}\right)^{n/2}\|\overline{\varphi}_n(\overline{F}_n-(\overline{g}_0+\dots+\overline{g}_{n-1}))\|_{\boldsymbol{R}^{m_n}\times\boldsymbol{R}\times\boldsymbol{R}}\exp(-a_n)\leqslant 2^{-n}$$

and

(17) $$\|D^{(i)}g_{a_n}-D^{(i)}[\overline{\varphi}_n(\overline{F}_n-(\overline{g}_0+\dots+\overline{g}_{n-1}))]\|'_{\boldsymbol{R}^{m_n}\times\boldsymbol{R}\times\boldsymbol{R}}\leqslant\delta'_n$$

for $i = 0,1$ and $n\in\boldsymbol{N}$; see [22] and [15], Theorem 1.6.7. (We identify the function g_k defined on $\boldsymbol{R}^{m_k}\times\boldsymbol{R}\times\boldsymbol{R}$ with the function defined on $\boldsymbol{R}^{m_n}\times\boldsymbol{R}\times\boldsymbol{R}$, which g_k induces by composition with obvious projection.) We put $\overline{g}_n = \overline{g}_{a_n}$.

Let

$$g_n = \bar{g}_n \circ Q_n \colon E \times H \times \boldsymbol{R} \to X.$$

Combining (17) with (12), we get the following basic estimates:

$$\|D^{(i)} g_n - D^{(i)}[\psi_n(F_n - (g_0 + \ldots + g_{n-1}))]\|_{E\times H\times \boldsymbol{R}} \leqslant \delta_n' \tag{18}$$

for $i = 0, 1$ and $n \in \boldsymbol{N}$. By (18), (10), and (11),

$$\|D^{(i)} g_n - D^{(i)}(F_n - (g_0 + \ldots + g_{n-1}))\|_{A_n} \leqslant \delta_n', \tag{19}$$

$$\|D^{(i)} g_n\|_{E\times H\times \boldsymbol{R}\setminus A_{n-1}} \leqslant \delta_n' \tag{20}$$

for $i = 0, 1$ and $n \in \boldsymbol{N}$. Without loss of generality we can assume that $\|\cdot\|' \leqslant \|\cdot\|$. Thus, it follows from (20), (1) and (3) that the formula

$$g = \sum_{n=0}^{\infty} g_n | E \times H \times \boldsymbol{R} \setminus K \times \{0\} \times \{0\}$$

gives a C^1 map.

Now, we shall show that g satisfies condition (ii). Taking into consideration conditions (18)–(20), (14) and (4), we can estimate as follows:

$$\begin{aligned}
\|Dg - DF_n\|_{B_n}' &\leqslant \|D(g_0 + \ldots + g_n) - DF_n\|_{B_n}' + \|Dg_{n+1}\|_{B_n}' + \sum_{i\geqslant n+2} \|Dg_i\|_{B_n}' \\
&\leqslant \delta_n' + \|Dg_{n+1} - D[\psi_{n+1}(F_{n+1} - (g_0 + \ldots + g_n))]\|_{B_n}' + \\
&\quad + \|D[\psi_{n+1}(F_n - (g_0 + \ldots + g_n))]\|_{B_n}' + \\
&\quad + \|D(\psi_{n+1}(F_{n+1} - F_n))\|_{B_n}' + \sum_{i\geqslant n+2} \delta_i' \\
&\leqslant \delta_n' + \delta_{n+1}' + \|g_0 + \ldots + g_n - F_n\|_{B_n}' \|D\psi_{n+1}\|_{B_n} + \\
&\quad + |\psi_{n+1}|_{B_n} \|D(g_0 + \ldots + g_n - DF_n)\|_{B_n}' + \\
+ \|F_{n+1} - F_n\|_{B_n}' \|D\psi_{n+1}\|_{B_n} &+ |\psi_{n+1}|_{B_n} \|DF_{n+1} - DF_n\|_{B_n}' + \sum_{i\geqslant n+2} \delta_i' \\
&\leqslant \|DF_{n+1} - DF_n\|_{B_n}' + k_n \|F_{n+1} - F_n\|_{B_n}' + k_n \delta_n + \delta_n.
\end{aligned}$$

Using (18)–(20), (13) and (4) and estimating in the same way as above, we check (i) and (ii). It remains to establish that g is an $\boldsymbol{R}$-analytic map.

To this end, together with each Banach space $(Y, \|\cdot\|)$ we shall consider its complexification $\tilde{Y}$, i.e. a complex Banach space $(Y, |||\cdot|||)$ which contains Y as a real linear subspace such that $Y \oplus iY = \tilde{Y}$.[1] Let $\tilde{l}_2$ be the complex Hilbert space of square summable sequences. If $\tilde{x} = x + iy \in \tilde{Y} = Y \oplus iY$, where $x, y \in Y$, we put $\tilde{x}_* = x - iy \in \tilde{Y}$.

Consider the continuous linear operator $P\colon E \to l_2$ given by

$$P(x) = (x_j^*(x)) \quad \text{for} \quad x \in E,$$

[1] For instance, one can take $\tilde{Y} = Y \times Y$ with the norm $|||(x, y)||| = \sup\{\|x\cos\alpha - y\sin\alpha\|\colon 0 < \alpha < 2\pi\}$ and multiplication $(a + ib)(x, y) = (ax - by, bx + ay)$.

and its natural extension $\tilde{P}\colon \tilde{E} \to \tilde{l}_2$ given by

$$\tilde{P}(x + iy) = P(x) + iP(y) \quad \text{for} \quad x, y \in E.$$

Fix a sequence of positive numbers $\{\varrho_n\}_{n\geqslant 1}$ so that

$$(\varrho_n)^{1/2} < \min\big(2^{-n-1}, (1/2)\operatorname{dist}_{|\cdot|}(\boldsymbol{R}^{m_{n+1}} \setminus K_n \times \boldsymbol{R}^{m_{n+1}-m_n}, K_{n+1})\big). \tag{22}$$

For each $n \in \boldsymbol{N}$ put

$$W_n = \Big\{(z_j) \in \tilde{l}_2\colon z_j = a_j + ib_j, \\ \inf_{(u_1,\ldots,u_{m_{n+1}})\in K_{n+1}} \operatorname{Re}\big((z_1 - u_1)^2 + \ldots + (z_{m_{n+1}} - u_{m_{n+1}})^2\big) \leqslant \varrho_n,\ \sum_{j=1}^{\infty} b_j^2 \leqslant \varrho_n/4\Big\}$$

and

$$V_n = \{a + ib \in \boldsymbol{C}\colon \inf_{-2^{-n-1}<s<2^{-n-1}} \operatorname{Re}(a + ib - s)^2 \leqslant 2^{-n-1},\ b^2 \leqslant \varrho_n/4\}.$$

For $((z_j), \zeta_1, \zeta_2) \in \tilde{l}_2 \times \boldsymbol{C} \times \boldsymbol{C} \setminus W_n \times V_n \times V_n$ and for $((u_i), s, t) \in l_2 \times \boldsymbol{R} \times \boldsymbol{R}$ with $(u_1, \ldots, u_{m_{n+1}}, s, t) \in K_{n+1} \times [-2^{-n}, 2^{-n}] \times [-2^{-n}, 2^{-n}]$ we have

$$\operatorname{Re}[(z_1 - u_1)^2 + \ldots + (z_k - u_k)^2 + (\zeta_1 - s)^2 + (\zeta_2 - t)^2] \geqslant \varrho_n/2 \tag{23}$$

for $k \geqslant m_{n+1}$ and $n \in \boldsymbol{N}$.

Note that the map $\bar{G}_k\colon \boldsymbol{C}^{m_k} \times \boldsymbol{C} \times \boldsymbol{C} \to \tilde{X}$ given by

$$\begin{aligned}
\bar{G}_k(z, \zeta_1, \zeta_2) = \Big(\frac{a_k}{\pi}\Big)^{k/2} \int_{\operatorname{supp}\bar{\psi}_k} &\bar{\psi}_k(u, r_1, r_2)\big[\bar{F}_k(u, r_1, r_2) - \\
&- (g_0(u, r_1, r_2) + \ldots + g_{k-1}(u_1, \ldots, u_{m_{k-1}}, r_1, r_2))\big] \times \\
\times \exp\big[-a_k\big((z_1 - u_1)^2 + \ldots &+ (z_{m_k} - u_{m_k})^2 + (\zeta_1 - r_1)^2 + (\zeta_2 - r_2)^2\big)\big]\,du\,dr_1\,dr_2
\end{aligned} \tag{24}$$

where $z = (z_1, \ldots, z_{m_k}) \in \boldsymbol{C}^{m_k}$, $u = (u_1, \ldots, u_{m_k}) \in \boldsymbol{R}^{m_k}$, $r_1, r_2 \in \boldsymbol{R}$, is holomorphic. Moreover, observe that the map g_k is the restriction of the holomorphic map

$$G_k\colon \tilde{E} \times \{\tilde{h} \in \tilde{H}\colon \eta(\langle \tilde{h}, \tilde{h}_* \rangle)^2 \neq -1\} \times \boldsymbol{C} \to \tilde{X}$$

given by

$$G_k(\tilde{x}, \tilde{h}, \zeta) = \bar{G}_k(\tilde{P}_k(\tilde{x}), \eta(\langle \tilde{h}, \tilde{h}_* \rangle), \zeta)$$

for $(\tilde{x}, \tilde{h}, \zeta) \in \tilde{E} \times \tilde{H} \times \boldsymbol{C}$. Let

$$U_n = \{(\tilde{x}, \tilde{h}, \zeta) \in \tilde{E} \times \tilde{H} \times \boldsymbol{C}\colon (\tilde{P}(\tilde{x}), \eta(\langle \tilde{h}, \tilde{h}_* \rangle), \zeta) \notin W_n \times V_n \times V_n\}.$$

Then, according to (22) we have $E \times H \times \boldsymbol{R} \setminus A_n \subset U_n$ for any $n \in \boldsymbol{N}$. Moreover, using (24), (23), and (16), we can write for $k > n$, $n \geqslant 1$,

$$\begin{aligned}
|||G_k|||_{U_n} &\leqslant \Big(\frac{a_k}{\pi}\Big)^{k/2} \cdot \|\psi_k(F_k - (g_0 + \ldots + g_{k-1}))\|_{E\times H\times \boldsymbol{R}} \exp(-a_k \varrho_k/2) \\
&\leqslant \Big(\frac{a_k}{\pi}\Big)^{k/2} \cdot \|\psi_k(F_k - (g_0 + \ldots + g_{k-1}))\|_{E\times H\times \boldsymbol{R}} \exp(-a_k) \leqslant 2^{-k}.
\end{aligned} \tag{25}$$

Hence $\sum_{k=1}^{\infty} G_k$ is uniformly convergent on the set U_n. From the Weierstrass Theorem ([10], Theorem 4.6.2) the map

$$G = \sum_{k=1}^{\infty} G_k\colon\ U = \bigcup_{n=1}^{\infty} U_n \to X$$

is holomorphic and therefore

$$g = G \mid U \cap (E \times H \times \boldsymbol{R} \setminus K \times \{0\} \times \{0\})$$

is $\boldsymbol{R}$-analytic.

Let us derive from 2.1 the following fact about C^∞ Whitney functions.

2.2. Lemma. *Suppose that E_0 is a separable Banach space, H_0 is any Hilbert space and $K \subset E_0 \times H_0$ is a compact set. There exists a C^∞ Whitney function for $(K, E_0 \times H_0)$.*

Proof. We may assume that E_0 is infinite-dimensional. Let $H' \subset H_0$ be a closed separable linear subspace such that $K \subset E_0 \times H' \subset E_0 \times H_0$. Denote by H the orthogonal complement of H' in H_0, write $E = E_0 \times H'$ and identify K with the projection K onto E. Since E is separable, there is a total sequence $\{x_n^*\}_{n \in \boldsymbol{N}} \subset E^*$ with $\|x_n^*\| \leqslant 2^{-n}$.

Let $0 < k_0 < k_1 < \ldots$ be the sequence of Lemma 2.2 applied for the given $E, H, \{x_n^*\}$, $m_n = n$ and $X = \boldsymbol{R}$. Now, let us specify

$$\delta_n = \frac{1}{2k_{n+1}} \quad \text{and} \quad F_n = \frac{1}{k_n} \quad \text{for } n \geqslant 0 \text{ and } \|\cdot\|' = \|\cdot\|.$$

Continuing the use of 2.1, we determine an $\boldsymbol{R}$-analytic function

$$g\colon E \times H \times \boldsymbol{R} \setminus K \times \{0\} \times \{0\} \to \boldsymbol{R}$$

such that

$$\|Dg\|_{B_n} \leqslant 3 \quad \text{and} \quad \left| g - \frac{1}{k_n} \right|_{B_n} < \frac{1}{k_n} \quad \text{for } n \geqslant 0$$

(for the definition of B_n see 2.1).

The function $\psi\colon E \times H \to \boldsymbol{R}$ given by $\psi(x, h) = (1/3)g(x, h, 0)$ for $(x, h) \notin K$ and $\psi(x, h) = 0$ for $(x, h) \in K$ is the required C^∞ Whitney function.

2.3. Lemma. *Suppose that H is a Hilbert space, E is a separable Banach space, $K \subset E$ is a compact set and $f\colon K \to X$ is a continuous map from K into a Banch space X. Then there exists a continuous map*

$$F\colon E \times H \times \boldsymbol{R} \to X$$

such that

(i) $F(x, 0, 0) = f(x)$ *for* $x \in K$,

(ii) $F | E \times H \times \boldsymbol{R} \setminus K \times \{0\} \times \{0\}$ *is of class* C^∞,

(iii) $\sup \left\{ \left\| \frac{\partial}{\partial t} F(x, h, t) \right\| : (x, h, t) \notin K \times \{0\} \times \{0\} \right\} < \infty.$

The proof of 2.3 is preceded by the following

2.4. Sublemma (cf. Renz [17]). *Suppose that E is a Banach space and assume that a sequence $\{x_n^*\}_{n \in \boldsymbol{N}} \subset E^*$ separates the points of K. Then there exists an increasing sequence $m_1 < m_2 < \ldots$ of positive numbers and a sequence $\{\gamma_n\}_{n \in \boldsymbol{N}}$ with $\gamma_n \in C_0^\infty(\boldsymbol{R}^{m_n}, X)$ such that, if we write $P_n = (x_1^*, \ldots, x_{m_n}^*)\colon E \to \boldsymbol{R}^{m_n}$ and $f_n = \gamma_n \circ P_n\colon E \to X$, the following conditions are satisfied:*

$(\mathrm{a})_n$ $$\Big\| f - \sum_{i=1}^{n} f_i \Big\|_K \leqslant 2^{-2n-4} \quad \textit{for} \quad n \in \boldsymbol{N},$$

$(\mathrm{b})_n$ $$\|f_n\|_E \leqslant 2^{-2n-2} \quad \textit{for} \quad n \geqslant 2.$$

Proof of 2.4. We endow K with the metric

$$\varrho(x, y) = \sum_{i=1}^{\infty} \min\left(2^{-i}, |x_i^*(x) - x_i^*(y)|\right).$$

Since K is compact and the embedding $(K, \|\cdot\|) \to (K, \varrho)$ is continuous, the metric induces the topology of K.

1. Let $n = 1$. It follows from the uniform continuity of f that there exists a $\delta_1 > 0$ such that $x, y \in K$, $\varrho(x, y) < \delta_1$ implies $\|f(x) - f(y)\| \leqslant \varepsilon_1 = 2^{-2-4}$. Fix $m_1 \in \boldsymbol{N}$ so that $2^{-m_1-1} < \delta_1$ and write $P_1 = (x_1^*, \ldots, x_{m_1}^*)$. Using the compactness of K, one can construct a finite open cover $\{U_a\}$ of the set $P_1(K) \subset \boldsymbol{R}^{m_1}$ such that $\operatorname{diam}_\varrho\left(K \cap P_1^{-1}(U_a)\right) < \delta_1$ for all a. Let $\{\varphi_a\} \cup \{\varphi\}$ be a C^∞ partition of unity of the space $\boldsymbol{R}^{m_1}$ with the property $\operatorname{supp} \varphi \subset \boldsymbol{R}^{m_1} \setminus P_1(K)$ and $\operatorname{supp} \varphi_a \subset U_a$ for all a. For each a we take $p_a \in P_1^{-1}(U_a) \cap K$ and we put $\gamma_1 = \sum_a \varphi_a f(p_a)$. Then for any $x \in K$ we have

$$\begin{aligned} \|f(x) - f_1(x)\| &= \Big\| f(x) - \sum_a \varphi_a\big(P_1(x)\big) f(p_a) \Big\| \\ &= \Big\| \sum_a \varphi_a\big(P_1(x)\big) f(x) - \sum_a \varphi_a\big(P_1(x)\big) f(p_a) \Big\| \\ &\leqslant \sum_a \varphi_a\big(P_1(x)\big) \|f(x) - f(p_a)\| \leqslant \varepsilon_1 = 2^{-2-4}. \end{aligned}$$

We observe, moreover, that $\|f_1\|_E \leqslant \|f\|_K$.

2. Assume that we already know $m_1 < \ldots < m_{n-1}$ and $f_1, \ldots, f_{n-1}$ for $n \geqslant 2$, so that $(a)_{n-1}$ and $(b)_{n-1}$ are satisfied. Then employing in 1 the map $f - \sum_{k=1}^{n-1} f_k$ instead of f, the number $\varepsilon_n = 2^{-2n-4}$ instead of ε_1 and the operator $P_n = (x_1^*, \ldots, x_{m_n}^*)$ instead of P_1 one can construct f_n (and simultaneously m_n) satisfying $(a)_n$. Moreover, then

$$\|f_n\|_E \leqslant \Big\|f - \sum_{k=1}^{n-1} f_k\Big\|_K \leqslant \varepsilon_{n-1} = 2^{-2n-2}.$$

Proof of 2.3. We may assume that E is infinite-dimensional. Since E is separable, there exists a total sequence $\{x_n^*\}_{n\in N} \subset E^*$ with $\|x_n^*\| \leqslant 2^{-n}$. Take the sequences $\{m_n\}_{n\in N}$ and $\{\gamma_n\}_{n\in N}$ of Sublemma 2.4 applied for the data E, X, $\{x_n^*\}$. Now, specify sequences $\{\overline{F}_n\}_{n\geqslant 0}$, $\overline{F}_n\colon \boldsymbol{R}^{m_n} \times \boldsymbol{R} \times \boldsymbol{R} \to X$ by $F_0 = 0$ and

$$\overline{F}_n(u_1, \ldots, u_{m_n}, s, t) = \gamma_1(u_1, \ldots, u_{m_1}) + \sum_{k=2}^{n} \lambda_k(t)\gamma_k(u_1, \ldots, u_{m_k})$$

for $(u_1, \ldots, u_{m_n}, s, t) \in \boldsymbol{R}^{m_n} \times \boldsymbol{R} \times \boldsymbol{R}$, where the functions $\lambda_k \in C^\infty(\boldsymbol{R}, [0, 1])$ satisfy conditions (7)–(9) of the proof of 2.1. Let Q_n be the maps of 2.1 and write $F_n = \overline{F}_n \circ Q_n$. We see that

$$\|F_{n+1} - F_n\|_{E\times H\times \boldsymbol{R}} \leqslant 2^{-2n-2} \quad \text{for} \quad n \in \boldsymbol{N}$$

and

$$\Big\|\frac{\partial}{\partial t} F_{n+1} - \frac{\partial}{\partial t} F_n\Big\|_{E\times H\times \boldsymbol{R}} \leqslant 4\cdot 2^n \cdot 2^{-2n-2} = 2^{-n} \quad \text{for} \quad n \geqslant 0.$$

Applying 2.1, to the sequence $\{F_n\}_{n\geqslant 0}$ and $\delta_n = 2^{-n-2}$ for $n \geqslant 0$ and $\|\cdot\|' = \|\cdot\|$, we conclude that there exists an $\boldsymbol{R}$-analytic map

$$g\colon E \times H \times \boldsymbol{R} \setminus K \times \{0\} \times \{0\} \to X$$

such that

$$\|g - F_n\|_{B_n} \leqslant 2^{-2n-2} + 2^{-n-2} \quad \text{for} \quad n \geqslant 1$$

and for $n \geqslant 0$

$$\Big\|\frac{\partial}{\partial t} g - \frac{\partial}{\partial t} F_n\Big\|_{B_n} \leqslant 2^{-n} + 2^{n+3}\cdot 2^{-2n-2} + 2^{n+3}\cdot 2^{-n-2} + 2^{-n-2} \leqslant 6$$

(use (i) and (iii) of 2.1; for the definition of B_n's see 2.1).

It follows from $(a)_n$ and $(b)_n$ of 2.4 that the sequence $\{\overline{F}_n\}_{n\geqslant 0}$ is uniformly convergent and that the map $F\colon E \times H \times \boldsymbol{R} \to X$ given by

$$F(x, h, t) = \begin{cases} g(x, h, t), & (x, h, t) \notin K \times \{0\} \times \{0\}, \\ f(x) & x \in K,\ h = 0,\ t = 0 \end{cases}$$

is continuous. Since $\Big\|\frac{\partial}{\partial t} F_n\Big\|_{E\times H\times \boldsymbol{R}} \leqslant 1$, we get

$$\Big\|\frac{\partial}{\partial t} F\Big\|_{E\times H\times \boldsymbol{R} \setminus K\times\{0\}\times\{0\}} \leqslant 8.$$

2.5. Remark. The assertions of 2.1 and consequently of 2.2 and 2.3 hold true if H is replaced by the space $l_p(\Lambda)$ with $p \in [1, \infty)$. (This is a consequence of the fact that every such $l_p(\Lambda)$-space admits a linear continuous injection into a certain $l_{2n}(\Lambda)$ and the $2n$th power of the standard norm of $l_{2n}(\Lambda)$ is a polynomial.)

2.6. Corollary. *Let E be a separable Banach space and let $K \subset E$ be a weakly compact set. Then there exists a C^∞ Whitney function for $\big(K \times \{0\}, E \times l_p(\Lambda)\big)$.*

Proof. There exists a total sequence of linear functionals $\{x_n^*\}_{n\in N} \subset E^*$ with $\|x_n^*\| \leqslant 2^{-n-1}$, $|x_n^*|_K \leqslant 2^{-n}$. We define an operator $S\colon E \to l_2$ by $S(x) = \big(x_n^*(x)\big)$ for $x \in E$. Then S is an injection, $S(K) \subset l_2$ is compact and $\|S\| \leqslant 1$. By 2.2 and 2.5 there exists a C^∞ Whitney function $\psi\colon l_2 \times l_p(\Lambda) \to \boldsymbol{R}^+$ for $\big(S(K) \times \{0\}, l_2 \times l_p(\Lambda)\big)$. Hence $\varphi = \psi \circ (S \times \mathrm{id}_{l_p(\Lambda)})$ is a C^∞ Whitney function for $\big(K \times \{0\}, E \times l_p(\Lambda)\big)$.

For the arguments of the next chapter the following "reduced form" of Lemma 2.1 is sufficient.

2.7. Lemma. *Let X be a Banach space and let $\|\cdot\|'$ be a continuous pseudonorm on X. Then, for every sequences $\{F_n\}_{n\geqslant 0} \in C_0^\infty(\boldsymbol{R}, X)$ and $\{\delta_n > 0\}_{n\geqslant 0}$ there exists a map $g \in C^\infty(\boldsymbol{R} \setminus \{0\}, X)$ such that, writing $B_0 = \{t \in \boldsymbol{R}\colon |t| > 1\}$ and $B_n = \{t \in \boldsymbol{R}\colon 2^{-n-1} < |t| \leqslant 2^{-n}\}$ for $n \in \boldsymbol{N}$, we have*

(i) $\|g - F_n\|'_{B_n} \leqslant \|F_{n+1} - F_n\|'_{B_n} + \delta_n$,

(ii) $\|Dg - DF_n\|'_{B_n} \leqslant \|DF_{n+1} - DF_n\|'_{B_n} + 2^{n+3}(\|F_{n+1} - F_n\|'_{B_n} + \delta_n) + \delta_n$

for $n \geqslant 0$.

The proof of 2.7 is analogous to the proof of 2.1.

3. The negligibility scheme. We are going to extend Bessaga's negligibility scheme (see [3], Chapter III, § 5) to the smooth and the $\boldsymbol{R}$-analytic categories.

We shall empoly the following notation and terminology. Given a continuous pseudonorm $w\colon X \to \boldsymbol{R}^+$ on a normed linear space X. We consider $X_w = X/w^{-1}(\{0\})$, the normed linear space equipped with the norm w (induced by the pseudonorm w on X) and we put the coset map $i_w\colon X \to X_w$. A subset $A \subset X$ is said to be *w-complete* if $i_w(A) \subset X_w$ is complete. The pseudonorm w is said to be *non-complete* if the whole space X is not w-complete. We denote the completion X_w by $\hat{X}_w$. A subset $Z \subset X$ will be called *cylindrical* if $i_w^{-1} \circ i_w(Z) = Z$.

The following proposition summarizes the arguments of [3], Chapter III, Proposition 5.1, with the distance functions replaced by Whitney functions.

3.1. PROPOSITION. *Let E be a Banach space and let $w: E \to \boldsymbol{R}^+$ be a continuous non-complete pseudonorm. Further, let $A \subset E$ be a w-complete set and let $U \subset E$ be a neighbourhood of A such that there exists a C^r Whitney function for $(A, U; w)$, where $r \in \bar{\boldsymbol{N}} \cup \{\omega\}$. Then there exists a C^r isomorphism $H: E \setminus A \xrightarrow[\text{onto}]{} E$ such that $H(x) = x$ for $x \notin U$.*

In the proof of 3.1 we shall need the following

3.2. SUBLEMMA. *Given $\varepsilon > 0$ and $r \in \{\infty, \omega\}$, there is a path $q \in C^r\big((0, \infty), E\big)$ satisfying the following conditions:*

(1) $$w\big(q(t)-q(s)\big) \leqslant \varepsilon|t-s| \quad \text{for} \quad t, s > 0,$$

(2) $$\lim_{t\to 0} i_w\big(q(t)\big) \in \hat{E}_w \setminus E_w.$$

In the case where $r = \infty$ we can require that

(3) $$q(t) = 0 \quad \text{iff} \quad t \geqslant 1.$$

Proof of 3.2. The case $r = \infty$ (cf. Renz [17]). There exists a linearly independent sequence $\{y_k\}_{k=2}^{\infty}$ in E_w such that $\sum_{k=1}^{\infty} 2^{k+6} w(y_{k+1}-y_k) < \varepsilon$, (where $y_1 = 0$) and $\lim y_k \in \hat{E}_w \setminus E_w$. Take: $x_k \in E$ with $x_1 = 0$ and $i_w(x_k) = y_k$ and a function $\gamma \in C^\infty(\boldsymbol{R}, [0, 1])$ with $\gamma|(-\infty, 1/2] = 1$, $\gamma^{-1}(\{0\}) = [1, \infty)$ and $\|D\gamma\|_{\boldsymbol{R}} \leqslant 4$. Define the required C^∞ path by

$$q(t) = x_1 + \sum_{k=1}^{\infty} \gamma(2^{k-1}t)(x_{k+1}-x_k) \quad \text{for} \quad t > 0.$$

The fact that $q(t) = 0$ iff $t \geqslant 1$ is a consequence of the linear independence of the vectors $y_2, y_3, \ldots$ and the property $\gamma^{-1}(\{0\}) = [1, \infty)$.

The case $r = \omega$. We shall use Lemma 2.7. We specify $\{F_n\}_{n\geqslant 0} \subset C_0^\infty(\boldsymbol{R}, E)$ by letting $F_0 = 0$ and for $n \geqslant 1$

$$F_n(t) = x_1 + \sum_{k=1}^{n} \gamma(2^{k-1}t)(x_{k+1}-x_k) \quad \text{for} \quad t \in \boldsymbol{R},$$

where γ and $\{x_n\}$ are those of the previous case. Applying 2.7 to the space $X = E$, the pseudonorm $\|\cdot\|' = w$, the sequence $\{F_n\}_{n\geqslant 0}$ and $\delta_n = 2^{-n-6}\varepsilon$ for $n \geqslant 0$, we infer that there exists an $\boldsymbol{R}$-analytic function $g: \boldsymbol{R}\setminus\{0\} \to X$ such that, writing $B_0 = \{t \in \boldsymbol{R}: |t| > 1\}$, $B_n = \{t \in \boldsymbol{R}: 2^{-n-1} < |t| \leqslant 2^{-n}\}$ for $n \geqslant 1$, we have

(i) $$w(g-F_n)_{B_n} \leqslant w(F_{n+1}-F_n) + 2^{-n-6}\varepsilon \leqslant 2^{-n-6}\varepsilon + 2^{-n-6}\varepsilon = 2^{-n-5}\varepsilon,$$

(ii) $$w(Dg-DF_n)_{B_n} \leqslant w(DF_{n+1}-DF_n) + 2^{n+3}w(F_{n+1}-F_n)_{B_n} + \\ + 2^{n+3}2^{-n-6}\varepsilon + 2^{-n-6}\varepsilon \leqslant (2^{-5}+2^{n+3}2^{-n-6}+2^{-3}+2^{-n-6})\varepsilon \leqslant \varepsilon/2$$

for $n \geqslant 0$. We define

$$q = g|(0, \infty).$$

Since $w(DF_n)_{\boldsymbol{R}} \leqslant 2^{-5}\varepsilon$, we get by (ii)

$$w\big(Dq(t)\big) < \varepsilon \quad \text{for} \quad t > 0,$$

which is equivalent to (1). Finally, (2) is a consequence of (i) and the fact that $\lim_{t\to 0}\lim_n i_w\big(F_n(t)\big) = \lim y_n \in \hat{E}_w \setminus E_w$.

Proof of 3.1. Let φ be a C^r Whitney function for $(A, U; w)$. Since φ is w-lipschitzian, there exists a w-lipschitzian function $\varphi_0: E_w \to \boldsymbol{R}^+$ such that $\varphi = \varphi_0 \circ i_w$. The function φ_0 has a w-lipschitzian extension $\hat{\varphi}_0: \hat{E}_w \to \boldsymbol{R}$. We may assume that

(4) $$|\hat{\varphi}_0(y_1)-\hat{\varphi}_0(y_2)| \leqslant M\cdot w(y_1-y_2)$$
$$\text{for a certain } M < \infty \text{ and every } y_1, y_2 \in \hat{E}_w,$$

(5) $$\varphi(x) = 1 \quad \text{for} \quad x \notin U.$$

By 3.2 there exists a C^r path $q: (0, \infty) \to E$ satisfying (1) and (2) for $\varepsilon = M/2$. Using these conditions and (4), one can show that the formula

(6) $$H(x) = x - q\cdot\varphi(x) \quad \text{for} \quad x \in E \setminus A$$

defines a 1-1 transformation $E\setminus A$ onto E (cf. [3], Chapter III, Theorem 5.1). Note that $H \in C^r(E\setminus A, E)$. For each $x \in E\setminus A$ we can write $DH(x) = I - v_0\cdot D\varphi(x)$, where $w(v_0) \leqslant 1/2$. Since $\sup\{w\big(D\varphi(x)(v)\big): w(v) \leqslant 1/2\} \leqslant 1/2$, the operator $DH(x)$ is invertible. Thus, applying the Inverse Function Theorem (cf. [7], 10.2.5 and [23]), we infer that $H^{-1} \in C^r(E, E\setminus A)$.

Now we deduce the following fact, which was conjectured by Bessaga (see [3], p. 109).

3.3. THEOREM. *Let E be a Banach space and let $w: E \to \boldsymbol{R}^+$ be a non-complete C^p pseudonorm with $p \in \bar{\boldsymbol{N}}$. If $A \subset E$ is a cylindrical set such that $i_w(A)$ is compact and $U \subset E$ is a cylindrical neighbourhood of A, then there exists a C^p isomorphism $H: E\setminus A \xrightarrow[\text{onto}]{} E$ with $H(x) = x$ for $x \notin U$. I, (E, w) is separable and $A \subset E$ is an arbitrary w-complete, cylindrical setf then $E\setminus A$ and E are C^p isomorphic.*

Proof. Using the trivial fact that $w: E \to \boldsymbol{R}$ is a w-lipschitzian function, one can construct, as in 1.1, a C^0 Whitney function $\varphi_0: E_w \to \boldsymbol{R}^+$ for $\big(i_w(A), i_w(U); w\big)$ [resp. for $\big(i_w(A), E_w; w\big)$] such that $\varphi = \varphi_0 \circ i_w$ is of class C^p on $E\setminus A$. Now, the assertion of 3.3 follows from 3.1.

4. $\boldsymbol{R}$-analytical negligibility. We shall need the following fact concerning the existence of non-complete C^ω norms.

4.1. PROPOSITION. *If a Banach space E is infinite-dimensional and admits a total sequence of continuous functionals and $K \subset E$ is a weakly* *compact set* [resp. *If* $E = Y^*$, *where* Y *is an infinite-dimensional and separable Banach space and* $K \subset E$ *is a weakly-star compact set*], *then there exists a non-complete* C^∞ *norm* w *on* E *such that* E *is* w-*separable and* K *is* w-*compact.*

Proof. Suppose that $\{x_n^*: n \in N\} \subset E^*$ is a total set; one can additionally assume that $\|x^*\| \leqslant 2^{-n}$ and $|x_n^*|_K \leqslant 2^{-n}$. We define the operator $S: E \to (l_2, |\cdot|)$ by $S(x) = (x_n^*(x)) \in l_2$ for $x \in E$. If $S(E) \subset l_2$ is not closed, then we put $w = |S(\cdot)|$. Otherwise, $S(E)$ is a subspace of l_2 spanned by an orthonormal system (e_i). We let $w(x) = \left(\sum_{i=1}^{\infty} |\langle S(x), e_i\rangle|^3\right)^{1/3}$.

The proof of the second part of 4.1 is analogous to the above one.

The main results concerning C^∞ negligibility are:

4.2. Theorem. *Let* E *be an infinite-dimensional, separable Banach space and let* K *be a weakly compact subset of* E. *Then* $E\setminus K$ *and* E *are* C^∞ *isomorphic.*

4.3. Theorem. *If either* E *is a separable Banach space or* E *is an arbitrary Hilbert space and* F *is a closed linear subspace of* E *with* $\dim(E/F) = \infty$, *then* $E\setminus F$ *and* E *are* C^∞ *isomorphic.*

Proofs. Theorem 4.2 follows from 3.1, 4.1 and 2.2. Theorem 4.3 is a consequence of 3.1, 4.1 and the fact that any C^∞ norm w: $E/F \to \boldsymbol{R}^+$ induces a C^∞ Whitney function for $(w^{-1}(\{0\}), E; w)$.

Theorem 4.2 admits the following generalization:

4.4. Theorem. *If* E *is an infinite-dimensional Banach space admitting a total sequence of linear functionals and* $K \subset E$ *is a weakly compact set* [*resp. If* $E = Y^*$, *where* Y *is infinite-dimensional and separable Banach space and* $K \subset E$ *is a weakly-star compact set*], *then* $E \times l_p(\Lambda)\setminus K \times \{0\}$ *is* C^∞ *isomorphic with* $E \times l_p(\Lambda)$ *for every* $p \in [1, \infty)$ *and every* Λ.

Proof. This follows from 3.1, 4.1 and 2.6.

4.5. Corollary. *If* $K \subset l_p(\Lambda)$ *is a compact set where* $p \in [1, \infty)$ *and* Λ *infinite, then* $l_p(\Lambda)\setminus K$ *and* $l_p(\Lambda)$ *are* C^∞ *isomorphic.*

Let us notice the following consequence of 4.2:

4.6. Corollary. *If* K *is a convex, closed and bounded subset of an infinite-dimensional, separable, reflexive Banach space* E, *then* $E\setminus K$ *and* E *are* C^∞ *isomorphic.*

Hirschowitz [11] has shown that 4.2 cannot be extended to the non-separable case. Namely, he gave an example of a non-separable Banach space E such that $E\setminus\{0\}$ and E are not C^∞ isomorphic. The following shows that $E = c_0(\Lambda)$, with Λ uncountable, may also serve as an example.

4.7. Proposition. *Let* Λ *be uncountable and let* φ: $c_0(\Lambda)\setminus\{0\} \to c_0(\Lambda)$ *be a closed homeomorphic embedding. Then* $\varphi \notin C^\infty(c_0(\Lambda)\setminus\{0\}, c_0(\Lambda))$.

Proof. Denote $E = c_0(\Lambda)$ and $U = c_0(\Lambda)\setminus\{0\}$. We first show that

(i) Given any $f \in C^\infty(U, \boldsymbol{R})$, there is an extension $\tilde{f} \in C^\infty(E, \boldsymbol{R})$ of f.

To this end write $f|V = \sum_{n=1}^{\infty} P_n|V$, where $0 \notin V$ is an open set and $P_1, P_2, \ldots$ are polynomials on E. It follows from the result of Pełczyński [16] that there is a countable set $T \subset \Lambda$ such that $P_n(x+z) = P_n(x)$ for every $x \in E$, $z \in E' = c_0(\Lambda\setminus T)$, and every $n \in \boldsymbol{N}$ (we identify E and $E'\oplus c_0(T)$). We define a C^∞ function g on the set $G = \{(x, z) \in E \times E'$: $x+z \neq 0$, $x \neq 0\}$ by $g(x, z) = f(x+z) - f(x)$. Since g vanishes on some open subset of the connected set G, it follows that $g \equiv 0$. Hence an extension $\tilde{f}$ can be defined by $\tilde{f}(x) = f(x)$ for $x \neq 0$ and $\tilde{f}(0) = f(x+z)$ for some $x+z \neq 0$ with $z \in E'$; this completes the proof of (i).

Now, supposing a contrario that $\varphi \in C^\infty(U, E)$, φ has an extension $\tilde{\varphi} \in C^\infty(E, E)$ according to [11]. Then $\varphi^{-1}\circ\varphi(x) = x$ for all $x \in E$, which is impossible.

5. C^∞ negligibility. We shall deal with Banach spaces E satisfying the following condition:

(L) *there exists a continuous, linear injection* T: $E \to c_0(\Lambda)$ *for some set* Λ.

For a list of spaces satisfying (L) we refer the reader to [20]. We only mention that the weakly compactly generated (= WCG) Banach spaces, hence in particular all separable and all reflexive Banach spaces, satisfy (L), cf. [13], Theorem 2.4. The Banach space $l_\infty(\Lambda)$, with uncountable Λ, may serve as an example of a space which does not satisfy (L).

We shall employ the following fact concerning the existence of non-complete C^∞ norms.

5.1. Proposition. *Every infinite-dimensional Banach space satisfying* (L) *admits a non-complete* C^∞ *norm.*

Proof. It is known (see [4]) that $c_0(\Gamma)$ admits an equivalent C^∞ norm. Thus it is sufficient to show that there exists a non-closed continuous linear injection T_0: $E \to c_0(\Gamma)$ for some Γ. By the assumption (L), there is a continuous linear-injection T: $E \to c_0(\Lambda)$. If $T(E)$ is not closed, the proof is complete. Otherwise we may assume (the inverse mapping theorem) that E is a closed linear subspace of $c_0(\Lambda)$. By Theorem 2.1 of [13], there are Banach spaces E_1, E_2 such that $E = E_1\oplus E_2$ and E_1 is infinite-dimensional and separable. One can check that there exists a non-closed continuous linear injection S: $E_1 \to c_0(\boldsymbol{N})$. Finally, we put T_0: $E \to c_0(\Lambda) \times c_0(\boldsymbol{N}) = c_0(\Lambda \cup \boldsymbol{N})$ by $T_0(x) = (S(x_1), T(x_2))$ for $x = (x_1, x_2) \in E_1 \times E_2$ (assume that $\Lambda \cap \boldsymbol{N} = \emptyset$).

From 3.3 and 5.1 we obtain our main Theorems 5.2 and 5.3, about C^∞ negligibility.

5.2. Theorem. *Let* E *be an infinite-dimensional Banach space satis-*

fying (L) *and let* $K \subset E$ *be a compact set. Then there exists a* C^∞ *isomorphism* H *of* $E \setminus K$ *onto* E. *Moreover, if* w *is an arbitrary, non-complete* C^∞ *norm on* E *and* U *is a* w*-neighbourhood of* K, *then we may additionally require that* $H(x) = x$ *for* $x \notin U$.

5.3. THEOREM. *Let* E *be a* WCG-*space and let* $F \subset E$ *be a closed, linear subspace with* $\dim(E/F) = \infty$. *Then there exists a* C^∞ *isomorphism* H *of* $E \setminus F$ *onto* E. *Moreover, if* w *is an arbitrary, non-complete* C^∞ *pseudonorm on* E *with* $w^{-1}(\{0\}) = F$, *then we may additionally require that* $H(x) = x$ *for* $w(x) \geqslant 1$.

We also have

5.4. THEOREM. *Let* E *be an infinite-dimensional, separable Banach space and let* K, L *be disjoint, weakly compact subsets of* E. *Then there exists a* C^∞ *isomorphism* H *of* $E \setminus K$ *onto* E *with* $H(x) = x$ *iff* $x \in L$.

Proof. By 4.1 there is a non-complete C^∞ norm w on E such that $K \cup L$ is w-compact. Applying 1.2, we take a C^∞ Urysohn function φ for (K, L) with

$$|\varphi(x_1) - \varphi(x_2)| \leqslant M \cdot w(x_1 - x_2)$$

for a certain $M < \infty$ and every $x_1, x_2 \in E$.

By 3.2, for $\varepsilon = M/2$ there exists a path $q \in C^\infty((0, \infty), E)$ satisfying (1)–(3) of 3.2.

The formula

$$H(x) = x - q(\varphi(x)) \quad \text{for} \quad x \in E \setminus K$$

gives the required C^∞ isomorphism.

5.5. Remark. The assertion of 5.4 is also true if $E = l_p(\Lambda)$, where $p \in [1, \infty)$ and Λ is infinite, and K, L are compact sets. (The proof may be obtained, as in 5.4, by using the fact that for each pair of compact sets of the space $l_p(\Lambda)$ there exists a lipschitzian Uryshn function.)

Let us notic that the $\boldsymbol{R}$-analytic version of 5.4 is false.

5.6. EXAMPLE. Let $\mathcal{Э} = \{(x_n) \in l_2 \colon \sum_{n=1}^{\infty} n^2 x_n^2 \leqslant 1\}$ be an ellipsoid in the space l_2. Then $\mathcal{Э}$ is compact and has the following property: if U is an open and connected subset l_2 containing $\mathcal{Э}$, then for any Banach space F and any $f_1, f_2 \in C^\omega(U, F)$ with $f_1|\mathcal{Э} = f_2|\mathcal{Э}$ we have $f_1 \equiv f_2$.

6. Negligibility in manifolds. It is clear that the Negligibility Scheme does not generally work in the case of manifolds. However, the Scheme can be extended to the spaces $Z = E \times M$, where E is an infinite-dimensional Banach space and M is a manifold. Since many concrete infinite-dimensional manifolds admits such a product structure (cf. [5], [14]), the study of negligibility in the spaces Z seems to be reasonable.

We adopt the following notation. Given a continuous, non-complete pseudonorm w on a Banach space E and a metrizable C^r manifold ($r \in \overline{\boldsymbol{N}} \cup \{\omega\}$) M, possibly with boundary, modelled on a normed linear space, we fix a metric d on M and we put

$$(*) \qquad \overline{w}((x_1, x_2), (y_1, y_2)) = \max(w(x_1 - x_2), d(y_1, y_2)) \quad \text{for} \quad (x_i, y_i) \in E \times M \ (i = 1, 2);$$

$\overline{w}$ is a pseudometric on $E \times M$. We denote

$$i = i_w \times \mathrm{id} \colon E \times M \to E_w \times M.$$

Suppose we are given a $\overline{w}$-closed set $A \subset E \times M$ and $\overline{w}$-neighbourhood U of A. A continuous function $\varphi \colon E \times M \to \boldsymbol{R}^+$ will be called a C^r *Whitney function for* $(A, U; \overline{w})$ if $\varphi^{-1}(\{0\}) = A$, φ is of class C^r on $E \times M \setminus A$,

$$|\varphi(x_1, y) - \varphi(x_2, y)| \leqslant w(x_1 - x_2) \quad \text{for} \quad (x_i, y) \in E \times M \ (i = 1, 2)$$

and $\varphi(z) = \text{const}$ for $z \notin U$.

We start with the following generalization of 3.1.

6.1. PROPOSITION. *Assume that there exists a* C^r *Whitney function for* $(A, U; \overline{w})$ *with* $r \in \overline{\boldsymbol{N}} \cup \{\omega\}$ *and* A *is* $\overline{w}$*-complete* (*i.e.* $i(A)$ *is complete in* $E_w \times M$). *Then there exists a* C^r *isomorphism* H *of* $E \times M \setminus A$ *onto* $E \times M$ *with* $H(z) = z$ *for* $z \in E \times M \setminus U$.

Proof. It is easy to check that under our assumption there exists a C^r Whitney function φ for $(A, U; \overline{w})$ of the form $\varphi = \varphi_0 \circ i$ such that

$$|\varphi_0(x_1, y) - \varphi_0(x_2, y)| \leqslant M w(x_1 - x_2) \quad \text{for a certain} \quad M < \infty \quad \text{and every} \quad x_1, x_2 \in E_w$$

and $\varphi(z) = 1$ for $z \notin U$. By 3.2, there is a path $q \in C^r((0, \infty), E)$ satisfying (1) and (2) of 3.2 for $\varepsilon = M/2$. Then the formula

$$H(x, y) = (x - q \circ \varphi(x, y), y) \quad \text{for} \quad (x, y) \in E \times M \setminus A$$

gives the required C^r isomorphism.

6.2. THEOREM. *Assume that, under our notation,* w *is of class* C^p *and then model of* M *is* C^p *smooth with* $p \in \overline{\boldsymbol{N}}$. *If* $A = i^{-1} \circ i(A)$ *is a set of* $E \times M$ *such that* $i(A)$ *is compact and* $U = i^{-1} \circ i(U)$ *is a neighbourhood of* A, *then there exists a* C^p *isomorphism* $H \colon E \times M \setminus A \xrightarrow[\text{onto}]{} E \times M$ *with* $H(z) = z$ *for* $z \notin U$. *If* $(E \times M, \overline{w})$ *is separable and* $A = i^{-1} \circ i(A)$ *is an arbitrary* $\overline{w}$*-complete set of* $E \times M$, *then* $E \times M \setminus A$ *and* $E \times M$ *are* C^p *isomorphic.*

Proof. It is clear that for any $z \in E \times M$ and for any $\overline{w}$-neighbourhood U of z there is a function $f \in C^p(E \times M, [0, 1])$ with $f(z) > 0$ and $\operatorname{supp}(f) \subset U$ which is w-lipschitzian in the first coordinate. Therefore, arguing as in 1.1, one can construct a C^p Whitney function φ_0 for $(i(A), i(U); \overline{w})$ [resp. for $(i(A), E_w \times M; \overline{w})$] such that $\varphi_0 \circ i$ is of class C^p on $E \setminus A$. Now, the assertion follows from 6.1.

6.3. Remark. The first part of the assertion of 6.2 holds true under the additional assumption that the model of M admits C^p partition of unity (e.g. if M is separable and C^p smooth, cf. [4]), for an abitrary closed set A and its neighbourhood U such that $A = i_w^{-1}(A_1)\times A_2 \subset i^{-1}(U_1)\times U_2 = U \subset E\times M$ for some compact set $A_1 \subset E_w$. (This follows from the fact that in this case a C^p Whitney function for $(A, U; \overline{w})$ can easily be constructed.)

6.4. Corollary. *Suppose that E is an infinite-dimensional Banach space satisfying* (L) *and the model of M admits C^p partitions of unity* ($p \in \overline{N}$). *If $A \subset \{0\}\times M \subset E\times M$ is a closed set, then there exists a C^∞ isomorphism H of $E\times M\setminus A$ onto $E\times M$. Moreover, if w is an arbitrary, non-complete C^p norm on E and $U\subset M$ is a neighbourhood of the projection A onto M, then we may additionally require that $H(x, y) = (x, y)$ for $(x, y)\notin B_w(0, 1)\times U$.*

We note that this corollary is a generalization of the main lemma of Szigeti [18].

Theorem 6.2 gives some information about the negligibility of one-point sets in non-separable C^p smooth Banach spaces E, e.g. if E admits a continuous linear projection on an infinite-dimensional, separable linear subspace or, more generally, on a subspace satisfying (L), then $E\setminus\{0\}$ and E are C^p isomorphic.

The study of C^ω negligibility in non-trivial C^ω manifolds is much more difficult than in the smooth case. We derive from 6.1 the following fact about it:

6.5. Corollary. *Assume that M admits a C^ω embedding into a separable Banach space F, E is either an infinite-dimensional separable Banach space or $E = l_p(\Lambda)$ with $p\in[1, \infty)$ and Λ infinite, and $K \subset E\times M$ is a compact set. Then $E\times M\setminus K$ and $E\times M$ are C^ω isomorphic. In particular, the assertion holds if M is finite-dimensional.*

Proof. Let w be a non-complete C^ω norm on E, see 4.1. By 2.5 there is a C^ω Whitney function for $(K, E\times F; \overline{w})$, where $\overline{w}$ is given by $(*)$ with the metric d defined by the norm on F. Now, the assertion follows from 6.1.

Finally, let us notice that 6.2 and 6.5 make it possible to formulate isotopic versions (cf. [5], p. 381, for the definitions) of all our statements concerning negligibility.

7. R-analytic and smooth extensions of homeomorphisms by the technique of Klee. The primary purpose of this chapter is to prove the following two theorems:

7.1. Theorem. *Let E be a Banach space of the form $E = E'\times l_p(\Lambda)$ and let $h\colon K_1 \xrightarrow[\text{onto}]{} K_2$ be a homeomorphism between compact subsets of E. If E' has an unconditional Schauder basis, then h admits an extension to an autohomeomorphism H of E such that $H\colon E\setminus K_1 \to E\setminus K_2$ is a C^∞ isomorphism.*

7.2. Theorem. *Let E be a separable Banach space with $\dim E \geqslant 2$ and let $h\colon K_1 \xrightarrow[\text{onto}]{} K_2$ be a homeomorphism between finite-dimensional compact subsets of E. If either E is infinite-dimensional or K_1 is countable, then h admits an extension to an autohomeomorphism H of E such that $H\colon E\setminus K_1 \to E\setminus K_2$ is a C^ω isomorphism.*

A proof of the C^∞ version of 7.1 in the special case where $E = l_p$ or c_0 has been given by Renz [17], Corollary 5. The idea we use in establishing 7.1 and 7.2 is motivated by Renz's paper. The basic role is played by the following "flattening lemma".

7.3. Lemma. *If $E_i = E_i'\times l_{p_i}(\Lambda_i)$, where E_i' is a separable Banach space and $p_i\in[1, \infty)$ for $i = 1, 2$, $K\subset E_1$ is a compact set and $f\colon K\to E_2$ is a continuous map, then there exists an autohomeomorphism Φ of the space $E_1\times E_2$ such that*

$$\text{(1)}\qquad \Phi(x_1, f(x_1)) = (x_1, 0) \quad \text{for} \quad x_1\in K,$$

$$\text{(2)}\qquad \Phi \text{ is } C^\omega \text{ isomorphism off the graph } G(f) = \{(x_1, f(x_1))\colon x_1\in K\}.$$

Proof. There exists a splitting $l_{p_1}(\Lambda_1) = l_{p_1}(\Lambda_0)\oplus l_{p_1}(\Lambda)$ for some countable set Λ_0 ($\Lambda = \Lambda_1\setminus\Lambda_0$) such that $K\subset E_1'\times l_{p_1}(\Lambda_0)\subset E_1'\times l_{p_1}(\Lambda_1)$. Write $E = E_1'\times l_{p_1}(\Lambda_0)$ and identify K with the projection onto E.

Let $F\colon E_1\times \boldsymbol{R}\to E_2$ be the map of 2.3 (see also 2.5) applied to the triple $\big(E\times l_{p_1}(\Lambda)\times \boldsymbol{R} = E_1\times\boldsymbol{R}, f, X = E_2\big)$. So, there exists an $M<\infty$ such that $\left\|\frac{\partial}{\partial t}F(x, t)\right\|\leqslant M$ for $(x, t)\notin K\times\{0\}\subset E_1\times\boldsymbol{R}$. Since $G(f)\subset \subset E_1\times E_2$ is compact, there exists, according to 2.2 and 2.5, a C^ω Whitney function φ for $\big(G(f), E_1\times E_2\big)$.

The required homeomorphism can be defined by

$$\Phi(x_1, x_2) = \Big(x_1, x_2 - F\big(x_1, (2M)^{-1}\varphi(x_1, x_2)\big)\Big) \quad \text{for} \quad (x_1, x_2)\in E_1\times E_2.$$

7.4. Lemma. *If E_1, E_2 and $K\subset E_1$ are such as in 7.3 and $f\colon K\to E_2$ is an embedding, then there exists an autohomeomorphism $\tilde{\Phi}$ of the space $E_1\times E_2$ such that*

$$\text{(3)}\qquad \tilde{\Phi}(x_1, 0) = \big(0, f(x_2)\big) \quad \text{for} \quad x_1\in K,$$

$$\text{(4)}\qquad \tilde{\Phi}\,|E_1\times E_2\setminus K\times\{0\} \text{ is a } C^\omega \text{ isomorphism}.$$

Proof. Let Φ_1 and Φ_2 be the homeomorphisms of 7.3 applied to the triples (E_1, f, E_2) and (E_2, f^{-1}, E_1), respectively. It is sufficient to put $\tilde{\Phi} = \Phi_2\circ\Phi_1^{-1}$.

7.5. LEMMA. *Let E be a Banach space of the form $E = E' \times l_p(\Lambda)$ where E' is separable and $p \in [1, \infty)$ and let $h\colon K_1 \to K_2$ be a homeomorphism between compact subsets of E. If there exists a closed complemented subspace $E_2 \subset E$ which contains a set homeomorphic to K_2 and any translation of which intersects $K_1 \cup K_2$ at most at one point, then h admits an extension to an autohomeomorphism H of the space E such that $H\colon E \setminus K_1 \to E \setminus K_2$ is a C^∞ isomorphism.*

Proof. Let $i\colon K_2 \to E_2$ be any homeomorphic embedding. By E_1 denote a closed complement of E_2. Under our assumption there are continuous maps $f_i\colon \pi_{E_i}(K_i) \to E_2$ such that $K_i = G(f_i)$ for $i = 1, 2$ (π_{E_i} denotes the projection of $E_1 \times E_2$ onto E_i). Denote by Φ_i the homeomorphism obtained by 7.3 and applied to the triple (E_1, f_i, E_2) for $i = 1, 2$. Let $\tilde{\Phi}_1, \tilde{\Phi}_2$ be the homeomorphisms of 7.4 applied to the triples

$$\big(E_1, i \circ h \circ (\pi_{E_1} \circ \Phi_1)^{-1} | \pi_{E_1} \circ \Phi_1(K_1), E_2\big) \quad \text{and} \quad (E_2, \pi_{E_1} \circ \Phi_2 \circ i^{-1}, E_1),$$

respectively. Finally, put $H = \Phi_2^{-1} \circ \tilde{\Phi}_2 \circ \tilde{\Phi}_1 \circ \Phi_1$.

Proof of 7.1. Since E' has an unconditional Schauder basis, it follows from a theorem of Corson ([17], Corollary 8) that there exists an infinite-dimensional complemented subspaces $E_2 \subset E$ any translation of which intersects $K_1 \cup K_2$ at most at one point. We conclude the proof applying 7.5.

Proof of 7.2. Assume that E is infinite-dimensional. Since the space $\operatorname{span}(K_1 \cup K_2)$ is sigma-compact, E admits a complemented subspace E_2 such that $E_2 \cap \operatorname{span}(K_1 \cup K_2) = \{0\}$ and $\dim E_2 \geqslant 2 \dim K_1 + 1$. Now, the assertion follows from 7.5 because K_1 admits an embedding into E_2.

Assume that E is finite-dimensional. Write $K_1 \cup K_2 = \{a_i\}_{i \in \boldsymbol{N}}$. Take a vector a in E not parallel to any $a_i - a_j$ ($i \neq j$). Then $\boldsymbol{R} \cdot a = E_2$ is as required in 7.5; thus the assertion follows.

Let us observe the following facts concerning extensions of homeomorphisms in the C^∞ category:

7.6. Remarks. A. Assume that E is a non-separable WCG-space, $K_1, K_2 \subset E$ are compact sets [resp. E is a non-separable space $c_0(\Lambda)$ and $K_1, K_2 \subset E$ are closed separable sets] and $h\colon K_1 \to K_2$ is a homeomorphism. Then h admits an extension to an autohomeomorphism H of the space E such that $H\colon E \setminus K_1 \to E \setminus K_2$ is a C^∞ isomorphism.

B. Assume that M is a metrizable and connected C^p manifold of dimension $\geqslant 2$ ($p \in \overline{\boldsymbol{N}}$) modelled on a C^p smooth Banach space satisfying (L), $K_1, K_2 \subset M$ are countable compact sets and $h\colon K_1 \to K_2$ is a homeomorphism. Then h admits an extension to an autohomeomorphism H of the space M such that $H\colon M \setminus K_1 \to M \setminus K_2$ is a C^p isomorphism.

The proofs employ the following C^∞ version of 7.3.

7.7. LEMMA. *Suppose that E_1, E_2 are Banach spaces satisfying* (L), *$K \subset E_1$ is a compact set, $f\colon K \to E_2$ is a continuous map and w is a C^∞ norm on $E_1 \times E_2$; then there exists an autohomeomorphism Φ of the space $E_1 \times E_2$ satisfying*

(1′) $\quad \Phi(x_1, f(x_1)) = (x_1, 0) \quad$ *for* $\quad x_1 \in K,$

(2′) $\quad \Phi$ *is C^∞ isomorphism off the graph $G(f)$,*

(3′) $\quad$ *if $w(x) \geqslant \eta$, then $\Phi(x) = x \quad$ (for some $\eta > 0$).*

Questions and remarks.

A. Let us say that a normed linear space X is *C_L^p smooth* ($p \in \overline{\boldsymbol{N}}$) if X admits a non-constant lipschitzian function $\varphi \in C^p(X, [0, 1])$ whose support is bounded. It is obvious that if X admits an equivalent C^p norm, then X is C_L^p smooth. Let us note that the assertion of Lemma 1.1 remains valid under the assumption that X is C_L^p smooth.

Q1. Does any C_L^p smooth space admit an equivalent C^p norm? Is a C^p smooth space C_L^p smooth?

B. The following problem is still unsolved.

Q2. Are one-point sets C^1 negligible in any non-separable Banach space E?

To the author's best knowledge, the answer is unknown even if $E = l_\infty(\Lambda)$, where Λ is uncountable. (Let us recall that $l_\infty(\Lambda)$ does not possess property (L) and therefore Theorem 5.2 is not applicable.) Also the following problems concerning $l_\infty(\Lambda)$-spaces are open:

Q3. Let Λ be uncountable and let $f \in C^p\big(l_\infty(\Lambda) \setminus \{0\}, l_\infty(\Lambda)\big)$. Does f admit a C^p extension to the whole $l_\infty(\Lambda)$?

Q4. Let Λ be uncountable and let $f \in C^1\big(l_\infty(\Lambda), \boldsymbol{R}\big)$. Does f depend on countably many coordinates?

Q5. Let Λ be uncountable. Is there an $f \in C^p\big(l_\infty(\Lambda), \boldsymbol{R}^+\big)$ with $f^{-1}(\{0\}) = \{0\}$?

As mentioned at the end of the proof of 4.7, a positive answer to Q3 would imply a negative answer to Q2, whereas it follows from Theorem 6.2 that a positive answer to Q5 would imply the positive solution of Q2 in the case where $E = l_\infty(\Lambda)$ with Λ uncountable. Let us note, moreover, that a positive answer to Q4 would give a positive answer to Q3.

C. It was observed by Szigeti [18] that, if E is a Banach space such that there is a C^p isomorphism H of $E \setminus \{0\}$ onto E with $H(x) = x$ for $\|x\| \geqslant 1$, then E is C^p smooth. On the other hand, if E is infinite-dimensional and separable, then the converse holds true.

Q6. Assume that E is a non-separable C^p smooth Banach space [or E is C_L^p smooth, or E admits an equivalent C^p norm], where $p \in \overline{\boldsymbol{N}}$. Does there exist a C^p isomorphism H of $E \setminus \{0\}$ onto E with $H(x) = x$ for $\|x\| \geqslant 1$?

It follows from 5.2 and from Lemma 3 of [1] that the answer to Q6 is affirmative if we additionally assume that the space E satisfies (L). Hence it is natural to ask:

Q7. Must a C^p smooth Banach space satisfy condition (L)?

Recently, Lindenstrauss and Johnson discovered (see the Israel Journal of Mathematics 17 (1974), p. 222) a Banach space E which admits an equivalent Fréchet differentiable norm and E is not a WCG-space but E still satisfies condition (L).

The corresponding question to Q6 in the $\boldsymbol{R}$-analytic category is the following:

Q8. Assume that every continuous function on an infinite-dimensional Banach space E can be approximated by a function $\boldsymbol{R}$-analytic in the C^0 Whitney topology [resp. E is infinite-dimensional and admits an equivalent C^ω norm]. Is then $E\setminus\{0\}$ C^ω isomorphic with E?

The proof of Theorem 6.2 suggests the following question, which is more general than Q8.

Q9. Assume that E is an infinite-dimensional Banach space and there is a C^r function on $E\setminus\{0\}$, where $r \in \bar{N}\cup\{\omega\}$, which admits no C^r extension to the whole E. Is the one-point set C^r negligible in E?

D. The assertion of Theorem 7.1 does not hold for the non-separable space $E = c_0(A)$. This follows from the fact that if $K \subset E$ is a compact set and $H \in C^\omega(E\setminus K, E)$, then there exists an extension $\tilde{H} \in C^\omega(E, E)$ of H (use the argument of the proof of 6.2). However, the following question is open:

Q10. Does the conclusion of Theorem 7.1 hold true under the assumption that E is an arbitrary infinite-dimensional separable Banach space?

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