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A functional differential equation in a Banach space

by BOGDAN RZEPECKI (Poznań)

Abstract. In this paper we first give theorems on the existence of ε-approximate solutions for the problem $x' = Fx$, $x(t_0) = x_0$ in the case of F being an operator of Volterra type, acting in the space of all continuous functions from an interval $I = [t_0, t_0+a]$ to a Banach space E. The results extend those of the previous works [5] and [6].

Let $h: I \to (-\infty, \infty)$ be a continuous function such that $h(t) \leqslant t$ for $t \in I$; write $m = \min\{h(t): t \in I\}$. Let $\varphi: [m, t_0] \to E$ be a continuous function; write $K = \max\{\|\varphi(t) - \varphi(t_0)\|: t \in [m, t_0]\}$. Let $r > 0$ and let $B = \{x \in E: \|x - \varphi(t_0)\| \leqslant K + r\}$. In Section II we present some results on the existence of a solution of the equation $x' = f\big(t, x(h(t))\big)$ satisfying the condition $x(t) = \varphi(t)$ for $t \in [m, t_0]$, where $f: I \times B \to E$ is a bounded continuous function.

Let $(E, \|\cdot\|)$ be an arbitrary Banach space and let $I = [t_0, t_0+a]$, $J = [t_0, t_0+h]$, where $0 < h \leqslant a$. Denote by $C(I)$ the space of all continuous functions from an interval I to E, with the usual supremum norm $|||\cdot|||$. For $X \subset E$, let $\tilde{X} = \{x \in C(I): x[I] \subset X\}$, where $x[I]$ denotes the image of the set I by the function x.

The results of this paper extend those of the previous works [5] and [6].

I. Let $B_{x_0} = \{x \in E: \|x - x_0\| \leqslant b\}$. By (PC) we shall denote the problem of finding a solution of the equation

$$x'(t) = (Fx)(t)$$

satisfying the condition

$$x(t_0) = x_0,$$

F being an operator from $\tilde{B}_{x_0}$ to $C(I)$, and the derivative being understood in the strong sense.

We introduce the following definitions:

1. Let ε be a positive number. A continuous function $u: I \to E$ is said to be an *ε-approximate solution* of the problem (PC) on the interval J, if it satisfies the following conditions:

(i) $u(t) \in B_{x_0}$ for $t \in J$ and $u(t) = u(t_0+h)$ for $t \in [t_0+h, t_0+a]$;

(ii) u has the right-hand derivative $D^+u(t)$ for $t \in [t_0, t_0+h)$ and $u(t) = x_0 + \int_{t_0}^{t} D^+u(s)\,ds$ for $t \in J$;

(iii) $\|D^+u(t) - (Fu)(t)\| \leqslant \varepsilon$ for $t \in [t_0, t_0+h)$.

2. We call an *Euler polygonal line* for (PC) on J any function $u\colon I \to E$ of the form

$$u(t) = \begin{cases} g_i(t) & \text{for } t \in [t_{i-1}, t_i],\ i = 1, 2, \ldots, n; \\ g_n(t_0+h) & \text{for } t \in [t_0+h, t_0+a], \end{cases}$$

where $t_0 < t_1 < \ldots < t_n = t_0+h$ and

$$g_0(t) = x_0 \quad \text{for } t \in I,$$

$$g_{j+1}(t) = \begin{cases} g_k(t) & \text{for } t \in [t_{k-1}, t_k],\ k = 1, 2, \ldots, j; \\ g_j(t_j) + (t-t_j)(Fg_j)(t_j) & \text{for } t \in [t_j, t_{j+1}]; \\ g_j(t_j) + (t_{j+1}-t_j)(Fg_j)(t_j) & \text{for } t \in [t_{j+1}, t_0+a] \end{cases}$$
$$(j = 0, 1, \ldots, n-1).$$

We shall give sufficient conditions for the existence of ε-approximate solutions in the case of F being an operator of Volterra type; F is said to be *of Volterra type* if for $x_1, x_2 \in \tilde{B}_{x_0}$, the equality $x_1(t) = x_2(t)$ for $t \leqslant s_0$ implies $(Fx_1)(s_0) = (Fx_2)(s_0)$.

THEOREM 1. *Let F be an operator bounded and continuous on $\tilde{B}_{x_0}$ and let $h \leqslant \min(a, M^{-1}b)$, where $M = \sup\{|||Fx|||\colon x \in \tilde{B}_{x_0}\}$. Assume, moreover, that*

1° *F is of Volterra type,*

2° *there exists a subset H of B_{x_0} such that*

$$x_0 + (t-t_0)\cdot \operatorname{conv}\Big(\bigcup\{(Fx)[J]\colon x \in \tilde{H}\}\Big) \subset H \quad \text{for all } t \in J$$

and such that all the functions belonging to $F[\mathscr{F}]$ are equicontinuous, where

$$\mathscr{F} = \{x \in C(I)\colon x(t_0) = x_0,\ x[I] \subset H,\ \|x(t)-x(s)\| \leqslant M|t-s| \text{ for } t, s \in I\}.$$

Then for any $\varepsilon > 0$ there exists an Euler polygonal line $u\colon I \to H$ which is an ε-approximate solution of (PC) on J.

Proof. Let $\varepsilon > 0$ be fixed. Since all the functions belonging to $\{Fx\colon x \in \mathscr{F}\}$ are equicontinuous, there exists a number $\delta > 0$ such that $\|(Fy)(s_1) - (Fy)(s_2)\| < \varepsilon$ for $|s_1 - s_2| < \delta$ and $y \in \mathscr{F}$. Now we divide the interval J into n parts: $t_0 < t_1 < \ldots < t_n = t_0+h$ in such a way that $\max|t_{i+1}-t_i| < \delta$.

Let us define the mappings $g_1, g_2, \ldots, g_n$ and u as in Definition 2. From assumptions 2° and 1° it follows that $u \in \mathscr{F}$ and $(Fg_i)(t_i) = (Fu)(t_i)$ for $i = 0, 1, \ldots, n-1$.

Let $0 \leqslant i \leqslant n-1$ and $t \in [t_i, t_{i+1})$; then

$$D^+ u(t) = D^+ g_{i+1}(t) = (Fg_i)(t_i),$$

$$x_0 + \int_{t_0}^{t} D^+ u(s)\,ds = x_0 + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} D^+ u(s)\,ds + \int_{t_i}^{t} D^+ u(s)\,ds$$

$$= x_0 + \sum_{k=0}^{i-1} (t_{k+1} - t_k)(Fg_k)(t_k) + (t - t_i)(Fg_i)(t_i)$$

$$= x_i + (t - t_i)(Fg_i)(t_i) = u(t)$$

and

$$\|D^+ u(t) - (Fu)(t)\| = \|(Fg_i)(t_i) - (Fu)(t)\| = \|(Fu)(t_i) - (Fu)(t)\| < \varepsilon,$$

which concludes the proof.

THEOREM 2. *Let F be an operator continuous and bounded on $\tilde{B}_{x_0}$ and let $M = \sup\{|||Fx|||: x \in \tilde{B}_{x_0}\}$. Each of the conditions which follow implies assumption 2° of Theorem 1:*

(a) *There exists a constant $k \geqslant 0$ such that*

$$\mu\big(\bigcup\{(Fx)[I]: x \in \tilde{X}\}\big) \leqslant k \cdot \mu(X)\,(^1)$$

for every subset X of B_{x_0} and $h \leqslant \min(a, M^{-1}b)$, $h \cdot k < 1$.

(b) *The operator F maps the set*

$$\{x \in C(I): x(t_0) = x_0,\ x[I] \subset B_{x_0},\ \|x(t) - x(s)\| \leqslant M|t-s|,\ t, s \in I\}$$

into a set of equicontinuous functions and $h = \min(a, M^{-1}b)$.

Proof. Condition (b) implies 2° from Theorem 1 for $H = B_{x_0}$. Modifying the proof of Daneš (cf. [7], Theorem 2, p. 797), one can prove that condition (a) implies assumption 2°.

Remark. A function $x: I \to E$ is said to be a *solution of the problem* (*PC*) on the interval J, if it is a differentiable function on J such that $x(t_0) = x_0$ and $x(t) \in B_{x_0}$ for $t \in J$, $x(t) = x(t_0 + h)$ for $t \in [t_0 + h, t_0 + a]$ and $x'(t) = (Fx)(t)$ for $t \in J$.

Let the operator F be continuous and bounded on $\tilde{B}_{x_0}$ and let $M = \sup\{|||Fx|||: x \in \tilde{B}_{x_0}\}$. Let us denote by S the set of all solutions of the problem (*PC*) for J. Each of the conditions given below implies that S is a non-empty and compact subset of $C(I)$ ([5]):

1. Condition (a) from Theorem 2 is satisfied.

2. The operator F maps every subset of $\tilde{B}_{x_0}$ consisting of equicontinuous functions into a set of equicontinuous functions and there exists

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(^{1}) μ denotes the *measure of non-compactness* due to Kuratowski (cf. [4], p. 318, [2], [3], [5]) (for a bounded set $X \subset E$ $\mu(X)$ denotes the infimum of all $r > 0$ such that there is a finite cover of X by balls of radius r).

an integrable function $p\colon I \to [0, \infty)$ such that

$$\mu\big(\{(Fx)(t)\colon x \in \mathscr{X}\}\big) \leqslant p(t)\cdot \mu\big(\{x(t)\colon x \in \mathscr{X}\}\big), \qquad t \in I,$$

for an arbitrary subset $\mathscr{X}$ of $\tilde{B}_{x_0}$ consisting of equicontinuous functions and $h = \min(a, M^{-1}b)$.

II. In this section we present some results on the existence of a solution for the Cauchy problem with lagged argument. However, we would like to emphasize that for a general differential-functional equation of Volterra type ([1], p. 127), the procedure suggested below can be carried out without any essential modifications.

Let $h\colon I \to (-\infty, \infty)$ be a continuous function such that $h(t) \leqslant t$ for every $t \in I$, let $m = \min\{h(t)\colon t \in I\}$ and let $\varphi\colon [m, t_0] \to E$ be a continuous function. Put

$$B = \{x \in E\colon \|x - \varphi(t_0)\| \leqslant K + r\}, \qquad \text{where } r > 0$$

and

$$K = \max\{\|\varphi(t) - \varphi(t_0)\|\colon t \in [m, t_0]\}.$$

By (+) we shall denote the problem of finding the solution of the equation

$$x'(t) = f\big(t, x(h(t))\big)$$

satisfying the condition

$$x(t) = \varphi(t) \qquad \text{for } t \in [m, t_0],$$

where $f\colon I \times B \to E$ is a bounded continuous function and

$$L = \sup\{\|f(t, x)\|\colon (t, x) \in I \times B\}.$$

By an integral of equation (+) in the interval J we understand a function $x\colon [m, t_0 + h] \to E$ which satisfies this equation in J and $x(t) = \varphi(t)$ for all $t \in [m, t_0]$.

Let ε be a positive number. A continuous function $u\colon [m, t_0 + h] \to E$ is said to be an *ε-approximate solution of problem* (+) on the interval J if it satisfies the following conditions:

(j) $u(t) = \varphi(t)$ for $t \in [m, t_0]$;

(jj) u has the right-hand derivative $D^+u(t)$ for $t \in [t_0, t_0 + h)$ and

$$u(t) = \varphi(t_0) + \int_{t_0}^{t} D^+u(s)\,ds \qquad \text{for } t \in J;$$

(jjj) $\big\|D^+u(t) - f\big(t, u(h(t))\big)\big\| \leqslant \varepsilon$ for $t \in [t_0, t_0 + h)$.

We introduce the following conditions:

(C.1) There exists a subset H of B such that

1° $f|_{J \times H}$ is a uniformly continuous function;

2° there exists an $1/n$-approximate solution u of problem $(+)$ on interval J such that $u[J] \subset H$.

(C.2) There exists an integrable function $p\colon I \to [0, \infty)$ such that $\mu(\{f(t, x)\colon x \in X\}) \leqslant p(t)\cdot\mu(X)$ for every $t \in I$ and every subset X of B.

(C.3) There exists a constant $k \geqslant 0$ such that $\mu(f[I \times X]) \leqslant k\cdot\mu(X)$ (²) for every subset X of B.

By S_n $(n = 1, 2, \ldots)$ we denote the set of all $1/n$-approximate solutions which are as in condition (C.1).

We obtain the following theorems:

1. *Let condition* (C.1) *be satisfied for each* $n \geqslant 1$ *and let* $\lim\limits_{n\to\infty} \mu(S_n) = 0$. *Then there exists an integral of* $(+)$ *in* J.

2. *Each of the conditions given below implies* $\lim\limits_{n\to\infty} \mu(S_n) = 0$:

(a) *Condition* 2° *from* (C.1) *is satisfied for every* $n \geqslant 1$ *and condition* (C.3) *is satisfied and* $h\cdot k < 1$.

(b) *Condition* (C.1) *is satisfied for* $H = B$ *and* $n = 1, 2, \ldots$ *and condition* (C.2) *is satisfied.*

3. *Let* $h \leqslant \min(a, L^{-1}(K+r))$ *and let condition* 1° *from* (C.1) *be satisfied and let*

$$\varphi(t) \in H \quad \text{for } t \in [m, t_0],$$

$$\varphi(t_0) + (t - t_0)\cdot\operatorname{conv}(f[J \times H]) \subset H \quad \text{for } t \in J.$$

Then for any $\varepsilon > 0$ *there exists an* ε*-approximate solution* u_ε *of* $(+)$ *on* J *such that* $u_\varepsilon(t) \in H$ *for* $t \in [m, t_0 + h]$ *and*

$$u_\varepsilon(t) = \varphi(t_0) + (t - t_j)\cdot f\big(t_j, u_\varepsilon(h(t_j))\big) + \\ + \sum_{k=0}^{j-1} (t_{j-k} - t_{j-k-1})\cdot f\big(t_{j-k-1}, u_\varepsilon(h(t_{j-k-1}))\big)$$

for $t \in [t_j, t_{j+1}]$, $j = 0, 1, \ldots, n-1$, *where* $t_0 < t_1 < \ldots < t_n = t_0 + h$ *is some partition of the interval* J.

4. *If condition* 1° *from* (C.1) *is satisfied for* $H = B$ *and* $h = \min(a, L^{-1}(K+r))$, *then the set* B *satisfies the assumptions of Theorem* 3.

If condition (C.3) *is satisfied and if* $h \leqslant \min(a, L^{-1}(K+r))$, $h\cdot k < 1$, *then there exists a compact set* H *satisfying the assumptions of Theorem* 3.

References

[1] A. Bielecki, *Równania różniczkowe zwyczajne i pewne ich uogólnienia* (skrypt), Warszawa 1961.

[2] K. Goebel, *Grubość zbiorów w przestrzeniach metrycznych i jej zastosowania w teorii punktów stałych*, Rozprawa habilitacyjna, Lublin 1970.

(²) μ — see p. 97.

[3] — and W. Rzymowski, *An existence theorem for the equations $x' = f(t, x)$ in Banach space*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 18 (1970), p. 367–370.
[4] K. Kuratowski, *Topologie*, vol. I, Warsaw 1952.
[5] B. Rzepecki, *A functional differential equation in Banach spaces* (I), Functiones et Approximatio. Commentarii Mathematici 5 (1977), p. 13–23.
[6] — *On the equation $y' = Fy$ in a Banach space*, Bull. Acad. Polon. Sci., Sér. Math. Astronom. Phys. 23 (1974), p. 11–14.
[7] S. Szufla, *Some remarks on ordinary differential equations in Banach spaces*, ibidem 16 (1968), p. 795–800.

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