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Congruences for the partition function modulo powers of 5

by

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1. Let $p(n)$ denote the number of unrestricted partitions of n and define $P(n)$ by

$$P(24n-1) = p(n)$$
$$P(n) = 0 \quad \text{if} \quad n \not\equiv -1 \pmod{24}.$$

Let

$$\psi = \psi(y) = y\prod_{n=1}^{\infty}(1-y^{24n}).$$

Then it is well known that

$$\psi^{-1} = \sum_{n=-1}^{\infty} P(n)y^n.$$

We define

$$\Big(\sum_n a(n)y^n\Big)_- = \sum_{\left(\frac{n}{5}\right)=-1} a(n)y^n$$

where $\left(\dfrac{n}{5}\right)$ is the Legendre symbol. Kolberg [3] proved that

(1) $$(\psi^{-1})_- = \frac{1}{\psi(y^5)^6}\{\psi(y)^3\psi(y^{25})^2+5\psi(y)^2\psi(y^{25})^3+10\psi(y)\psi(y^{25})^4+10\psi(y^{25})^5\}.$$

From this he could prove [4] that

(2) $$(\psi^{-1})_- \equiv (\psi^{23})_- \pmod 5.$$

Combining this with the three term recurrence relation for the coefficients of ψ^{23} found by Newman [6] we get the following congruence which was given by Atkin [1]. For q a prime > 5 and for n such that $(n/5) = -1$ we have

(3) $$q^3P(q^2n)-\left(q\left(\frac{-3n}{q}\right)+k\right)P(n)+P(q^{-2}n) \equiv 0 \pmod 5.$$

If we put $y = e^{\pi i\tau/12}$, ψ^{-1} and ψ^{23} are modular forms on the full modular group with the same multiplier system, ψ^{-1} has dimension $1/2$ and ψ^{23} is a cuspform with dimension $-23/2$. The main result of this paper is that for each $\beta \geq 0$, there exists a cuspform f_β on the full modular group, with the same multiplier system as ψ^{-1} and with dimension $-(24\cdot 5^\beta - 1)/2$, such that $(\psi^{-1})_- \equiv (f_\beta)_- (\mathrm{mod}\, 5^{\beta+1})$. A basis for the vector space Ω_β of such cuspforms is

$$\{\psi^{24\cdot 5^\beta - 1} j^m \mid m = 0, 1, \ldots, 5^\beta - 1\}$$

where

$$j = \psi^{-24}\Big(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) y^{24n}\Big)^3$$

s known as Klein's modular invariant, see e.g. [2]. In this paper we alsoi study $(\psi^{24\cdot 5^\beta - 1} j^m)_-$.

2. We shall use the following notations:

$$\psi_m = \psi(y^m), \quad T = \psi_5^6 \psi_{25}^{-6}, \quad U = \psi_5^6 \psi^{-6}, \quad V = \psi \psi_{25}^{-1}.$$

Further, if

$$\psi = \sum_{n=1}^{\infty} A(n) y^n,$$

then let

$$G_i = \sum_{n \equiv i\, (\mathrm{mod}\, 5)} A(n) y^n,$$

$$S_r = G_1^r G_0^{-r} + G_{-1}^r G_0^{-r}.$$

Kolberg [3] proved that

$$G_1 G_{-1} G_0^{-2} = -1,$$

$$V = -G_1 G_0^{-1} - G_{-1} G_0^{-1} - 1,$$

$$S_5 = -T - 11.$$

Combining these we get

LEMMA 1. (i) $V = -S_1 - 1$,

$$V^2 = S_2 + 2S_1 - 1,$$

$$V^3 = -S_3 - 3S_2 + 5,$$

$$V^4 = S_4 + 4S_3 + 2S_2 - 8S_1 - 5,$$

(ii) $T = V^5 + 5V^4 + 15V^3 + 25V^2 + 25V$.

We note that Lemma 1(ii) is the modular equation for 5.

Let $\pi(n)$ denote the exact power of 5 dividing n.

LEMMA 2. *For $k \geq 0$ we have*

$$V^{-k} = \sum_{l \geq 0} \sum_{m=0}^{4} a(k, l, m) T^{-l} V^m$$

where $a(k, l, m)$ are integers such that

(i) $a(k, l, m) = 0$ *for* $5l - m < k$,

(ii) $a(5l - m, l, m) = 1$,

(iii) $\pi(a(k, l, m)) \geq \left[\frac{5l - m - k + 1}{2}\right]$.

Proof by induction on k. It is clearly true for $k = 0$, with $a(0, 0, 0) = 1$. By Lemma 1(ii) we get

$$V^{-1} = T^{-1}(V^4 + 5V^3 + 15V^2 + 25V + 25) \tag{4}$$

which proves the lemma for $k = 1$. Suppose it is true for some $k \geq 1$. Then

$$V^{-k-1} = \sum_{l \geq 0} \sum_{m=1}^{4} a(k, l, m) T^{-l} V^{m-1} + \sum_{l \geq 0} a(k, l, 0) T^{-l-1} \sum_{m=0}^{4} a(1, 1, m) V^m.$$

Hence

$$a(k+1, l, 4) = a(k, l-1, 0),$$

$$a(k+1, l, m) = a(k, l, m+1) + a(k, l-1, 0) a(1, 1, m) \quad \text{for} \quad 0 \leq m \leq 3.$$

From this (i) and (ii) follows easily. Finally

$$\pi(a(k+1, l, 4)) = \pi(a(k, l-1, 0)) \geq \left[\frac{5l - 5 - 0 - k + 1}{2}\right]$$

$$= \left[\frac{5l - 4 - (k+1) + 1}{2}\right]$$

and

$$\pi(a(k+1, l, m)) \geq \min\left\{\left[\frac{5l - m - 1 - k + 1}{2}\right], \left[\frac{5l - 5 - k + 1}{2}\right] + \left[\frac{5 - m}{2}\right]\right\}$$

$$= \left[\frac{5l - m - (k+1) + 1}{2}\right].$$

LEMMA 3. *For $\beta \geq 0$ we have*

$$\psi^{24\cdot 5^\beta - 1} U^{-5^\beta + \gamma} \equiv \psi_{25}^{-1} T^\gamma V^{-6\gamma - 1} (\mathrm{mod}\ 5^{\beta+1}).$$

Proof. Kolberg ([5], Lemma 5) notes that

$$\psi_{p^\alpha}^{p^\beta} \equiv \psi^{p^{\alpha+\beta}} (\operatorname{mod} p^{\beta+1}).$$

Hence

$$\psi_5^{5^\beta} \equiv \psi^{5^{\beta+1}} (\operatorname{mod} 5^{\beta+1})$$

and so

$$\psi^{24\cdot 5^\beta-1} U^{-5^\beta+\gamma} \equiv \psi^{-1} U^\gamma = \psi_{25}^{-1} T^\gamma V^{-6\gamma-1} (\operatorname{mod} 5^{\beta+1}).$$

Lemma 4. *For $\gamma \geqslant 0$ there exist integers $b(\gamma, n)$ such that*

(i) $\pi(b(\gamma, n)) \geqslant \left[\frac{5n+1}{2}\right]$,

(ii) $j^{5^\beta-\gamma} \equiv \sum_{n=0}^{\infty} b(\gamma, n) U^{-5^\beta+\gamma+n} (\operatorname{mod} 5^{\beta+1})$.

Proof. Aas [7] gives the following identity:

$$j = U^{-1} + 6\cdot 5^3 + 63\cdot 5^5 U + 52\cdot 5^8 U^2 + 63\cdot 5^{10} U^3 + 6\cdot 5^{13} U^4 + 5^{15} U^5 = U^{-1} \sum_{n=0}^{6} B(n) U^n.$$

From this we get

$$j^{5^\beta} \equiv U^{-5^\beta} (\operatorname{mod} 5^{\beta+1}). \tag{5}$$

Further

$$j^{-1} \equiv U \sum_{n=0}^{\infty} b(1, n) U^n (\operatorname{mod} 5^{\beta+1}) \tag{6}$$

where

$$b(1, n) = 0 \quad \text{for} \quad n < 0, \quad b(1, 0) = 1,$$

$$\sum_{m=0}^{6} B(m) b(1, n-m) = 0. \tag{7}$$

We now prove by induction that

$$\pi(b(1, n)) \geqslant \left[\frac{5n+1}{2}\right]. \tag{8}$$

It is true for $n \leqslant 1$. Since $\pi(B(m)) = \left[\frac{5m+1}{2}\right]$, we get by (7),

$$\pi(b(1, n)) \geqslant \min_{1\leqslant m\leqslant 6} \left\{\left[\frac{5m+1}{2}\right] + \left[\frac{5n-5m+1}{2}\right]\right\} = \left[\frac{5n+1}{2}\right].$$

Next, we show by induction that

$$j^{-\gamma} \equiv U^\gamma \sum_{n=0}^{\infty} b(\gamma, n) U^n (\operatorname{mod} 5^{\beta+1}) \tag{9}$$

where

$$\pi(b(\gamma, n)) \geqslant \left[\frac{5n+1}{2}\right]. \tag{10}$$

We have proved it for $\gamma = 1$. Further

$$j^{-\gamma-1} \equiv U^{\gamma+1} \sum_{n=0}^{\infty} b(1, n) U^n \sum_{n=0}^{\infty} b(\gamma, n) U^n (\operatorname{mod} 5^{\beta+1}).$$

Hence

$$b(\gamma+1, n) = \sum_{m=0}^{n} b(1, m) b(\gamma, n-m)$$

and so

$$\pi(b(\gamma+1, n)) \geqslant \min_{0\leqslant m\leqslant n} \left\{\left[\frac{5m+1}{2}\right] + \left[\frac{5n-5m+1}{2}\right]\right\} \geqslant \left[\frac{5n+1}{2}\right].$$

Combining (5) and (9) we get Lemma 4 (ii) and (10) gives Lemma 4(i). The reason why we take congruences in (6) and (9) is that, by (8) and (10), all but finitely many of the terms are congruent 0 modulo $5^{\beta+1}$ so we do not have to think about convergence.

Lemma 5. *For $\gamma \geqslant 0$ there exist integers $c(\gamma, \lambda, m)$ such that*

(i) $c(\gamma, \lambda, m) = 0$ *if* $5\lambda \leqslant \gamma + m$,

(ii) $c(5\lambda - m - 1, \lambda, m) = 1$,

(iii) $\pi(c(\gamma, \lambda, m)) \geqslant \left[\frac{5\lambda-\gamma-m}{2}\right]$, *and*

(iv) $\psi^{24\cdot 5^\beta-1} j^{5^\beta-\gamma} \equiv \psi_{25}^{-1} \sum_{\lambda\geqslant 0} \sum_{m=0}^{4} c(\gamma, \lambda, m) T^{-\lambda} V^m (\operatorname{mod} 5^{\beta+1})$.

Proof. Combining Lemmata 2, 3, and 4 we get

$$\psi^{24\cdot 5^\beta-1} j^{5^\beta-\gamma} \equiv \sum_{n=0}^{\infty} b(\gamma, n) \psi^{24\cdot 5^\beta-1} U^{-5^\beta+\gamma+n} \equiv \psi_{25}^{-1} \sum_{n=0}^{\infty} b(\gamma, n) T^{\gamma+n} V^{-6\gamma-6n-1}$$

$$\equiv \psi_{25}^{-1} \sum_{n=0}^{\infty} b(\gamma, n) T^{\gamma+n} \sum_{l\geqslant 0} \sum_{m=0}^{4} a(6\gamma+6n+1, l, m) T^{-l} V^m.$$

Putting $\lambda = l - \gamma - n$ we get

$$c(\gamma, \lambda, m) = \sum_{l\geqslant \lambda+\gamma} b(\gamma, l-\lambda-\gamma) a(6l-6\lambda+1, l, m).$$

From Lemmata 2 and 4 we see that we get contribution to the sum only if

$$l \geqslant \lambda + \gamma \tag{11}$$

and

$$5l-m \geqslant 6l-6\lambda+1,$$

i.e.

(12) $$l \leqslant 6\lambda-m-1.$$

Hence if $6\lambda-m-1 \leqslant \lambda+\gamma-1$, then $c(\gamma,\lambda,m)=0$. This proves (i). Further

$$c(5\lambda-m-1,\lambda,m) = b(5\lambda-m-1,0)\,a(30\lambda-6m-5,\,6\lambda-m-1,\,m) = 1.$$

Finally

$$\pi\big(c(\gamma,\lambda,m)\big) \geqslant \min_{l\geqslant\lambda+\gamma}\left\{\left[\frac{5l-5\lambda-5\gamma+1}{2}\right]+\left[\frac{5l-m-6l+6\lambda-1+1}{2}\right]\right\} = \left[\frac{5\lambda-\gamma-m}{2}\right].$$

LEMMA 6. *Let*

$$W = \sum_{m=0}^{4} d_m V^m.$$

Then

(i) $W_- = (d_3-4d_4)(V^3)_- + (d_2-10d_4)(V^2)_-$,

(ii) $\big(f(y^5)W\big)_- = f(y^5)W_-$ *for any* f.

Proof. From the definition of S_r it is clear that

$$(S_r)_- = \begin{cases} S_r & \text{if } \left(\frac{r}{5}\right) = -1,\\ 0 & \text{otherwise.}\end{cases}$$

Hence (i) follows from Lemma 1(i). Further, (ii) is well known and we skip the easy proof.

LEMMA 7. *Let*

$$f = \psi_{25}^{-1}\sum_{l\geqslant\delta} T^{-l}\{r_0(l)(V^3)_- + r_1(l)(V^2)_-\}$$

where $1\leqslant\delta\leqslant 5^{\beta-1}$, *and* $r_i(l)$ *are integers such that*

$$\pi\big(r_i(l)\big) \geqslant \left[\frac{5l+t}{2}\right]+i$$

for some fixed t. *Then there exist integers* s_i, $r_i'(l)$, *not depending on* β, *such that*

$$f - \big(\psi^{24\cdot 5^\beta-1}\{s_0 j^{5^\beta-5\delta+4} + s_1 j^{5^\beta-5\delta+3}\}\big)_- \equiv \psi_{25}^{-1}\sum_{l\geqslant\delta+1} T^{-l}\{r_0'(l)(V^3)_- + r_1'(l)(V^2)_-\} \pmod{5^{\beta+1}},$$

and

$$\pi(s_i) \geqslant \left[\frac{5\delta+t}{2}\right]+i, \qquad \pi\big(r_i'(l)\big) \geqslant \left[\frac{5l+t-1}{2}\right]+i.$$

Proof. Since

$$(\psi^{24\cdot5^\beta-1} j^{5^\beta-5\delta+4})_- \equiv \psi_{25}^{-1}\Big\{T^{-\delta}(V^3)_- + c(5\delta-4,\delta,2)T^{-\delta}(V^2)_- + \sum_{l>\delta}\ldots\Big\}$$

and

$$(\psi^{24\cdot5^\beta-1} j^{5^\beta-5\delta+3})_- \equiv \psi_{25}^{-1}\Big\{T^{-\delta}(V^2)_- + \sum_{l>\delta}\ldots\Big\}$$

by Lemmata 5 and 6, the stated result follows with

$$\begin{aligned}
s_0 &= r_0(\delta),\\
s_1 &= r_1(\delta) - s_0\cdot c(5\delta-4,\delta,2),\\
r_0'(l) &= r_0(l) - s_0\{c(5\delta-4,l,3)-4c(5\delta-4,l,4)\} -\\
&\qquad - s_1\{c(5\delta-3,l,3)-4c(5\delta-3,l,4)\},\\
r_1'(l) &= r_1(l) - s_0\{c(5\delta-4,l,2)-10c(5\delta-4,l,4)\} -\\
&\qquad - s_1\{c(5\delta-3,l,2)-10c(5\delta-3,l,4)\}.
\end{aligned}$$

Here

$$\pi(s_0) = \pi\big(r_0(\delta)\big) \geqslant \left[\frac{5\delta+t}{2}\right],$$

$$\pi(s_1) \geqslant \min\left\{\left[\frac{5\delta+t}{2}\right]+1,\ \left[\frac{5\delta+t}{2}\right]+\left[\frac{5\delta-5\delta+4-2}{2}\right]\right\} = \left[\frac{5\delta+t}{2}\right]+1,$$

$$\begin{aligned}
\pi\big(r_0'(l)\big) \geqslant \min\Big\{&\left[\frac{5l+t}{2}\right],\ \left[\frac{5\delta+t}{2}\right]+\left[\frac{5l-5\delta+1}{2}\right],\ \left[\frac{5\delta+t}{2}\right]+\\
&+\left[\frac{5l-5\delta}{2}\right],\ \left[\frac{5\delta+t}{2}\right]+1+\left[\frac{5l-5\delta}{2}\right],\ \left[\frac{5\delta+t}{2}\right]+1+\left[\frac{5l-5\delta-1}{2}\right]\Big\}\\
\geqslant &\left[\frac{5l+t-1}{2}\right].
\end{aligned}$$

Similarly

$$\pi\big(r_1'(l)\big) \geqslant \left[\frac{5l+t-1}{2}\right]+1.$$

Now we can prove our main theorem.

THEOREM 1. *There exist integers* $s_0(\delta)$, $s_1(\delta)$, *not depending on* β, *such that*

(i) $\pi\big(s_i(\delta)\big) \geqslant 2\delta-2+i$,

(ii) $(\psi^{-1})_- \equiv \left(\psi^{24\cdot 5^\beta-1}\sum_{\delta\geqslant 1}\{s_0(\delta)j^{5^\beta-5\delta+4}+s_1(\delta)j^{5^\beta-5\delta+3}\}\right)_- \pmod{5^{\beta+1}}$ *for* $\beta\geqslant 0$.

Proof. By (4)

$$\psi^{-1} = \psi_{25}^{-1}V^{-1} = \psi_{25}^{-1}T^{-1}(V^4+5V^3+15V^2+25V+25).$$

Hence, by Lemma 6,

$$(\psi^{-1})_- = \psi_{25}^{-1}T^{-1}\big((V^3)_- +5(V^2)_-\big).$$

Applying Lemma 7 repeatedly, first with $\delta=1$ and $t=-4$, next with $\delta=2$ and $t=-5$ and so on we get Theorem 1.

Similarly, if we start with the expression for $\psi^{24\cdot 5^\beta-1}j^{5^\beta-\gamma}$ given in Lemma 5 and apply Lemma 7 repeatedly we get the next theorem.

THEOREM 2. *For* $\gamma>0$, $\gamma\equiv 3,4,5 \pmod 5$ *there exist integers* $t_0(\gamma,\delta)$ *and* $t_1(\gamma,\delta)$, *not depending on* β, *such that*

(i) $\pi\big(t_i(\gamma,\delta)\big)\geqslant 2(\delta-\delta_0)+\varepsilon_0+i$,

(ii) $(\psi^{24\cdot 5^\beta-1}j^{5^\beta-\gamma})_- \equiv \left(\psi^{24\cdot 5^\beta-1}\sum_{\delta\geqslant\delta_0}\{t_0(\gamma,\delta)j^{5^\beta-5\delta+4}+t_1(\gamma,\delta)j^{5^\beta-5\delta+3}\}\right)_- \pmod{5^{\beta+1}}$

or $\beta\geqslant 0$, *where* $\delta_0=\delta_0(\gamma) = -\left[\dfrac{-\gamma-3}{5}\right]$ *and*

$$\varepsilon_0=\varepsilon_0(\gamma)=\begin{cases}0 & \text{if}\quad \gamma\equiv 5\pmod 5,\\ 1 & \text{if}\quad \gamma\equiv 3,4\pmod 5.\end{cases}$$

COROLLARY. *The vector space*

$$\Omega_\beta^- = \{(f)_-\pmod{5^{\beta+1}} \mid f\in\Omega_\beta\}$$

is spanned by $(\psi^{24\cdot 5^\beta-1}j^{5^\beta-\gamma})_-$ *where* γ *runs through the set*

$$\{\gamma \mid 0<\gamma<5^\beta \,\&\, \gamma\equiv 1,2\pmod 5\}\cup$$
$$\cup\{\gamma\mid 0<\gamma<5^\beta \,\&\, 2\big(5^{\beta-1}+1-\delta_0(\gamma)\big)+\varepsilon_0(\gamma)\leqslant\beta\}.$$

The integers $s_i(\delta)$ may be computed from the formulae given in the proofs of lemmata and Theorem 1. The first few values are

$s_0(1)=1$,
$s_1(1)=149\cdot 5$,
$s_0(2)=-12,288,268,243,031\cdot 5^2$,
$s_1(2)=-9,705,652,584,447,470\cdot 5^3$.

The values are increasing very rapidly as one would expect because of the large power of 5 in $b(\gamma,n)$.

3. Using Hecke theory one can prove that the coefficients of $f\in\Omega_\beta$ satisfy recurrence relations with $\leqslant 2\dim\Omega_\beta+1 = 2\cdot 5^\beta-1$ terms, for details see [2]. Since the operator $(\ )_-$ commutes with the Hecke operators, we can show similarly that the coefficients of $(f)_-$ satisfy recurrence relations with $\leqslant 2\dim\Omega_\beta^-+1$ terms modulo $5^{\beta+1}$. Hence the corollary gives an upper bound on the number of terms in such congruences. Atkin [1] has found such three term congruences modulo 5^6, in fact (3) is true modulo 5^6.

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