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Zeta functions and Eichler integrals*

by

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1. Introduction. Let K be an algebraic number field of degree n, χ an abelian character of K and $L(s, \chi)$ the associated L-series. There has been much written about the values of $L(s, \chi)$ for integral values of the argument s. In particular, one has the following result:

THEOREM A (Klingen [5]). *Let $Q(\chi)$ denote the cyclotomic field generated by the values of the character χ and let $k = 1, 2, 3, \ldots$ Then $L(1-k, \chi) \in Q(\chi)$.*

Let $\mathfrak{b}$ be a ray class modulo $\mathfrak{f}$, where $\mathfrak{f}$ is the conductor of χ, and let

$$\zeta(s, \mathfrak{b}, \mathfrak{f}) = \sum_{A} \frac{1}{NA^s} \qquad (\operatorname{Re} s > 1),$$

where the summation is over all integral ideals in the ray class $\mathfrak{b}$. Then one clearly has

$$L(s, \chi) = \sum \chi(\mathfrak{b}) \zeta(s, \mathfrak{b}, \mathfrak{f}),$$

where the summation is over all ray classes $\mathfrak{b} \bmod \mathfrak{f}$ and χ is regarded as a ray class character $\bmod \mathfrak{f}$. In particular, Klingen's theorem is implied by

THEOREM B (Siegel [11]). $\zeta(1-k, \mathfrak{b}, \mathfrak{f}) \in Q$.

An algorithm for calculating $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$ has been given by Siegel [11], and in the special cases of the trivial character [13] and K a real quadratic field [12], Siegel actually gives a formula for $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$. Recently, Shintani [10] has specified a general procedure for obtaining formulas in general.

Our main focus in this paper is Siegel's formula [12] for the case $K = a$ real quadratic field. Siegel's approach to the formula rests on an *ad hoc* limiting argument which derives from the "Riemannsche Grenzübergang". In this paper, we rederive the formula in a somewhat more conceptual way, by relating the value of $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$ to an integral which is, essentially, an Eichler integral. The key to computing $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$

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is determining the period polynomial of the Eichler integral with respect to a unimodular substitution which is derived from the action of the unit group of K on a basis of $\mathfrak{b}$. And in Section 4, we compute the period polynomial by using the methods we have developed in [3] and [4].

The method developed here is susceptible to a number of generalizations, but we will reserve discussion of these for future papers. We also should remark that a general theory relating the values of zeta functions associated to cusp forms (for $\Gamma_0(N)$) at certain integral arguments and Eichler cohomology has recently been worked out by Razar [7] and his work is closely related to the present paper. Moreover, the computation of Section 4 has been carried out independently (by a somewhat different method) by Razar [8].

In Section 4, we exhibit an intriging relationship between the values of zeta functions at even and odd integers. This relationship arises quite naturally out of our method, although we do not fully expound on it here.

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2. Analytic formula for $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$. Our first task is to derive an integral representation for $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$. The required formula is to be found, essentially, in Siegel's work [12], p. 12. However, since the formula one finds in Siegel is not stated in precisely the form required, we have provided a sketch of the derivation here.

To begin with, let K be any algebraic number field of degree n, r = the number of real, infinite primes of K. Let $\mathfrak{f}$ be any integral K-ideal, $\mathfrak{b}$ an integral K-ideal such that $(\mathfrak{b}, \mathfrak{f}) = 1$. Let v be a signature character of K, that is

$$v\colon\ K^\times \to \{\pm 1\},$$

$$v(\alpha) = \prod_{j=1}^{r} \left(\frac{\alpha^{(j)}}{|\alpha^{(j)}|}\right)^{a_j}, \quad a_j = 0 \text{ or } 1,$$

where $\alpha \mapsto \alpha^{(j)}$ $(1 \leqslant j \leqslant r)$ denote the real embeddings of K. Set

$$m = a_1 + a_2 + \ldots + a_r.$$

Further, let $\mathfrak{d}$ = the different of K, $\mathfrak{a} = \mathfrak{b}/\mathfrak{d}\mathfrak{f}$, and

$$F(s, v) = \sum_{\substack{\mu \equiv 1 \,(\mathrm{mod}\, \mathfrak{f}) \\ \mu \in \mathfrak{b}^{-1}}}' v(\mu)\, |N(\mu)|^{-s} \qquad (\operatorname{Re} s > 1);$$

where the prime on the summation means that μ runs over a set of representatives non-associated with respect to the group $\Gamma_{\mathfrak{f}}$ of totally positive K-units which are $\equiv 1 (\mathrm{mod}\, \mathfrak{f})$. Then ([11], p. 19) $F(s, v)$ can be analytically continued as a meromorphic function to the entire s-plane, having

at most a simple pole at $s = 1$, with the pole occurring precisely when v is trivial. Moreover, $F(s, v)$ satisfies the following functional equation: Set

$$A(s) = (2^{(r-n)/2}\pi^{-n/2})^s \Gamma\left(\frac{s+1}{2}\right)^m \Gamma\left(\frac{s}{2}\right)^{r-m} \Gamma(s)^{(n-r)/2}.$$

Then, for $\operatorname{Re} s < 0$, we have

$$\text{(2.1)} \qquad i^m |d|^{1/2} N(\mathfrak{f}\mathfrak{b}^{-1}) A(s) F(s) = A(1-s) \sum_{\mathfrak{a}|\lambda}' v(\lambda)\, e^{2\pi i \operatorname{Tr}(\lambda)} |N(\lambda)|^{s-1},$$

where $\mathfrak{a} = \mathfrak{b}/\mathfrak{d}\mathfrak{f}$.

The function $F(s, v)$ is related to $\zeta(s, \mathfrak{b}, \mathfrak{f})$ via the formula

$$\text{(2.2)} \qquad \zeta(s, \mathfrak{b}, \mathfrak{f}) = N(\mathfrak{b})^{-s} 2^{-r} \sum_{v} F(s, v),$$

where the sum on right runs over all signature characters of K. From the functional equation (2.1), one can deduce that $F(s, v) = 0$ for $s = 1-k$ r$k = 1, 2, \ldots$) if either K is not totally real or v is not of the form $v(\mu) = \operatorname{sgn}(N(\mu)^k)$. Therefore, by (2.1) and (2.2), we deduce that for K totally real, $k = 1, 2, \ldots$, we have

$$\text{(2.3)} \qquad \zeta(1-k, \mathfrak{b}, \mathfrak{f}) = J_k N(\mathfrak{a}^k) \sum_{\mathfrak{a}|\lambda}' e^{2\pi i \operatorname{Tr}(\lambda)} N(\lambda)^{-k},$$

where

$$J_k = \left((2\pi i)^{-k}\Gamma(k)\right)^n \Delta^{k-1/2} f^{k-1},$$

$f = N\mathfrak{f}$, Δ = the discriminant of K, and where, in case $k = 1$, the sum on the right side of (2.3) is understood to be defined as the limit

$$\lim_{s\to 0^+} \sum_{\mathfrak{a}|\lambda}' e^{2\pi i \operatorname{Tr}(\lambda)} N(\lambda)^{-1} |N(\lambda)|^{-s}.$$

Henceforth assume that K is a real quadratic field. Let $x \mapsto x'$ denote the non-trivial automorphism of K. Consider the following integral formula:

$$\int_0^1 x^{u-1}(1-x)^{v-1}\,dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \qquad u > 0,\ v > 0.$$

Set $u = v = k$. Change variables in the integral by setting

$$w = \frac{\lambda'}{\lambda}\,\frac{x}{1-x},$$

where λ is any element of $K^\times$. This yields

$$\text{(2.4)} \qquad N(\lambda)^{-k}\frac{\Gamma(k)^2}{\Gamma(2k)} = \int_0^{\pm\infty} \frac{w^{k-1}}{(\lambda w + \lambda')^{2k}}\,dw,$$

where the $+$ sign prevails if $\lambda/\lambda' > 0$ and the $-$ sign if $\lambda/\lambda' < 0$. This formula is the key to what follows. Applying Cauchy's theorem yields

$$N(\lambda)^{-k}\frac{\Gamma(k)^2}{\Gamma(2k)} = \int_0^{i\infty} \frac{w^{k-1}}{(\lambda w+\lambda')^{2k}}\,dw,$$

where the integral is along the imaginary axis. Therefore, by (2.3), we have

$$\text{(2.5)}\quad \frac{\Gamma(k)^2}{\Gamma(2k)}\zeta(1-k, \mathfrak{b}, \mathfrak{f}) = J_k N(\mathfrak{a})^k \int_0^{i\infty} w^{k-1} \sum_{\mathfrak{a}|\lambda}{}' \frac{e^{2\pi i \operatorname{Tr}(\lambda)}}{N(\lambda w+\lambda')^{2k}}\,dw.$$

This formula is valid for $k = 2, 3, \ldots$ However, it is also valid for $k = 1$ if we understand the (conditionally convergent) sum on the right to mean(1)

$$\lim_{\varepsilon\to 0} \sum_{\mathfrak{a}|\lambda}{}' \frac{e^{2\pi i \operatorname{Tr}(\lambda)}}{N(\lambda w+\lambda')^2 |N(\lambda w+\lambda')|^{2\varepsilon}}.$$

By the Dirichlet unit theorem, $\Gamma_{\mathfrak{f}}$ is an infinite cyclic group. Let $\varepsilon_{\mathfrak{f}}$ be its (unique) generator which is > 1. The K-ideal $\mathfrak{a}$ as a $\boldsymbol{Z}$-module is of rank 2. Choose ω_1 and ω_2 such that

$$\mathfrak{a} = \boldsymbol{Z}\omega_1 + \boldsymbol{Z}\omega_2.$$

Furthermore, since $\omega_i \mathfrak{a} \subseteq \mathfrak{a}$ $(i = 1, 2)$, there exists $\sigma \in \mathrm{SL}(2, \boldsymbol{Z})$ that

$$\begin{pmatrix} \varepsilon_{\mathfrak{f}}\omega_1 \\ \varepsilon_{\mathfrak{f}}\omega_2 \end{pmatrix} = \sigma \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Since $\varepsilon_{\mathfrak{f}} \in \Gamma_{\mathfrak{f}}$, we immediately deduce that $\sigma \in \Gamma(f) =$ the principal congruence subgroup of $\mathrm{SL}(2, \boldsymbol{Z})$ of level f.

Let $\mathscr{D}$ be the interval on the imaginary axis defined by

$$\mathscr{D} = \left\{ w \mid w = i\left(\frac{\varepsilon_{\mathfrak{f}}}{\varepsilon_{\mathfrak{f}}'}\right)^x,\ 0 \leqslant x < 1 \right\}.$$

Then a standard computation, starting from (2.4), shows that

$$\text{(2.6)}\quad \frac{\Gamma(k)^2}{\Gamma(2k)}\zeta(1-k, \mathfrak{b}, \mathfrak{f}) = J_k N(\mathfrak{a})^k \int_{\mathscr{D}} w^{k-1} \sum_{\lambda\in\mathfrak{a}} \frac{e^{2\pi i \operatorname{Tr}(\lambda)}}{(\lambda w+\lambda')^{2k}}\,dw.$$

Introduce the new variable

$$z = \frac{w\omega_1 + \omega_1'}{w\omega_2 + \omega_2'}.$$

(1) The existence of the limit may be proved in the usual way, by applying the Poisson summation formula and then using elementary estimates concerning Bessel functions.

Index ω_1 and ω_2 so that $\operatorname{Im} z > 0$. (This is possible since $\operatorname{sgn}(\operatorname{Im} z) = \operatorname{sgn}(\omega_1\omega_2' - \omega_2\omega_1')$.) To express the integrand in terms of z, define $u, v \in \boldsymbol{Q}$ by

$$u = \operatorname{Tr}(\omega_1), \qquad v = \operatorname{Tr}(\omega_2).$$

Furthermore, let

$$E_k(z|u, v) = \sum_{m,n=-\infty}^{\infty} \frac{e^{2\pi i(mu+nv)}}{(mz+n)^{2k}} \qquad (k = 1, 2, 3, \ldots),$$

where for $k = 1$, the sum is defined as

$$E_1(z|u, v) = \lim_{\varepsilon\to 0} \sum_{m,n=-\infty}^{\infty} \frac{e^{2\pi i(mu+nv)}}{(m+nz)^2 |m+nz|^{2\varepsilon}}.$$

Finally, set

$$Q(z) = (\omega_2 z - \omega_1)(\omega_2' z - \omega_1').$$

Then $Q(z) \in \boldsymbol{Q}[z]$. Using these notations and equation (2.6), we readily deduce the desired integral formula, namely

THEOREM 2.1. *Let z_0 be any point in the upper half-plane. Then*

$$\zeta(1-k, \mathfrak{b}, \mathfrak{f}) = L_k \int_{z_0}^{\sigma(z_0)} E_k(z|u, v) Q(z)^{k-1}\,dz,$$

where

$$L_k = -(2\pi)^{-2k}\Gamma(2k) N(\mathfrak{b})^{k-1}$$

and where the integral is taken along any path in the upper half-plane connecting z_0 and $\sigma(z_0)$.

Proof. Let λ of (2.6) be set equal to $m\omega_1 + n\omega_2$. Then as m, n run independently over $\boldsymbol{Z}$, λ runs over $\mathfrak{a}$. This substitution yields the theorem for $z_0 = (\omega_1 i + \omega_1')/(\omega_2 i + \omega_2')$. For arbitrary z_0, merely observe that the integrand in the statement of the theorem is invariant under the transformation

$$\text{(2.7)}\qquad z \mapsto \sigma(z).$$

Therefore, if we connect z_0 to z by any path L in the upper half-plane, Cauchy's theorem and the invariance of the integral under (2.7), we have

$$\int_{z_0}^{\sigma(z_0)} = \int_{z}^{\sigma(z)} - \Big[\int_{\sigma(L)} - \int_{L}\Big] = \int_{z}^{\sigma(z)}.$$

3. Generalized Eichler integrals. In the preceding section, we expressed $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$ in terms of an integral. In this section, we build up a general theory of such integrals and show that they are closely related to the so-called Eichler integrals. In particular, our main result will allow us to evaluate $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$ in terms of the period polynomial of a certain Eichler integral.

Let Γ be a Fuchsian group of the first kind acting on the complex upper half-plane $\boldsymbol{H}$. Assume that Γ has a cusp at $i\infty$. Let $h(z)$ be an automorphic form of even integral weight $n+2$ $(n \geqslant 0)$ for Γ. An *Eichler integral* $H(z)$ of $h(z)$ is an $(n+1)$-fold iterated integral of $h(z)$. There are many choices for such an integral, corresponding to the possible choices for the constants of integration. For example, repeated integration by parts shows that one choice for $H(z)$ is

$$H(z) = \frac{1}{n!}\int_{z_0}^{z} h(\tau)(\tau-z)^n d\tau, \tag{3.1}$$

where z_0 is any point in $\boldsymbol{H}$. Another possible Eichler integral can be obtained as follows: Suppose that the Fourier expansion of $h(z)$ about the cusp $i\infty$ is given by

$$h(z) = \sum_{m=0}^{\infty} a_m e^{2\pi i m z/\lambda}.$$

Then an Eichler integral associated to $h(z)$ is given by [2]

$$H(z) = \frac{1}{(n+1)!} a_0 z^{n+1} + \left(\frac{\lambda}{2\pi i}\right)^{n+1} \sum_{m=1}^{\infty} \frac{a_m}{m^{n+1}} e^{2\pi i m z/\lambda}. \tag{3.2}$$

Eichler integrals were first introduced in [1] for the purpose of constructing what has since become known as the Eichler–Shimura cohomology. More recently, Razar [7] has exhibited a fundamental connection between Eichler integrals and the values of zeta functions attached to cusp forms at integral arguments inside the "critical strip". In this paper, we continue the work of Razar and exhibit the connection between Eichler integrals attached to Eisenstein series and the values of ray class zeta functions at integral arguments.

Let us begin by describing the main results of this section. For basic notation and facts about Eichler cohomology, see [9]. Let $p(z) \in \boldsymbol{C}[z]$, $\deg(p) = n$. Further, define the action of Γ on polynomials via

$$p^{\sigma}(z) = p(\sigma z) j(\sigma, z)^n, \quad \sigma \in \Gamma, \tag{3.3}$$

where

$$j(\sigma, z) = cz+d \quad \text{if} \quad \sigma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}.$$

(2) Of course, the two Eichler integrals (3.1) and (3.2) differ by an Eichler co-boundary. It will be important, in what follows, to make use of both Eichler integrals and to explicitly compute the co-boundary in the case of interest.

We will study integrals of the following form:

$$G_p(z) = \int_{z_0}^{z} h(\tau) p(\tau) d\tau. \tag{3.4}$$

The principal question we seek to answer in this section is: How does $G_p(z)$ transform under the action of Γ? It turns out that this question has a very simple answer, which can be most elegantly stated in the language of symmetric tensor representations, which concept we now review.

Let $u, v \in \boldsymbol{R}$. For $n > 0$ denote by $\begin{pmatrix} u \\ v \end{pmatrix}^n$ the column vector

$$\begin{pmatrix} u \\ v \end{pmatrix}^n = \begin{pmatrix} u^n \\ u^{n-1}v \\ \vdots \\ v^n \end{pmatrix}.$$

Let $\begin{pmatrix} u \\ v \end{pmatrix}^0 = 1$. For $\sigma \in \mathrm{SL}(2, \boldsymbol{R})$, set

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \sigma \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix}^n = M_n(\sigma) \begin{pmatrix} u \\ v \end{pmatrix}^n,$$

where $M_n(\sigma) \in \mathrm{SL}(n+1, \boldsymbol{R})$. The mapping

$$\mathrm{SL}(2, \boldsymbol{R}) \to \mathrm{SL}(n+1, \boldsymbol{R}),$$

$$\sigma \to M_n(\sigma)$$

is a representation called the *n-th symmetric tensor representation* of $\mathrm{SL}(2, \boldsymbol{R})$. Note that if $\sigma \in \mathrm{SL}(2, \boldsymbol{Z})$, then $M_n(\sigma) \in \mathrm{SL}(n+1, \boldsymbol{Z})$.

Define the matrix

$$P_n = \begin{pmatrix} 0 & & p_1 \\ & \therefore & \\ p_{n+1} & & 0 \end{pmatrix}$$

where

$$p_j = (-1)^{j-1}\binom{n}{j-1}, \quad 1 \leqslant j \leqslant n+1.$$

Then we have the identity

$$^t\begin{pmatrix} u \\ v \end{pmatrix}^n P_n \begin{pmatrix} w \\ z \end{pmatrix}^n = (uz - vw)^n. \tag{3.5}$$

Moreover, if ${}^tA$ denotes the transpose of the matrix A, then we have

$$ {}^tM_n(\sigma)P_nM_n(\sigma) = P_n \quad (\sigma \in \mathrm{SL}(2, \boldsymbol{R})), \tag{3.6}$$

$$ {}^tP_n = (-1)^n P_n. \tag{3.7}$$

The Eichler integral $H(z)$ satisfies the following transformation law

$$H(\sigma z)j(\sigma, z)^n = H(z) + S_\sigma(z) \quad (\sigma \in \Gamma),$$

where $S_\sigma(z)$ is a polynomial of degree $\leqslant n$.

Throughout, we write $S_\sigma(z)$ and $p(z)$ in matrix form as follows:

$$S_\sigma(z) = {}^t\binom{z}{1}^n S(\sigma), \quad p(z) = {}^t\binom{z}{1}^n P, \tag{3.8}$$

for $(n+1)$-rowed column matrices $S(\sigma)$, P. Then our main results concerning $G_p(z)$ are:

THEOREM 3.1. *Let $H(z)$ be the Eichler integral of $h(z)$ defined by*

$$H(z) = \frac{1}{n!}\int_{z_0}^{z} h(u)(z-u)^n du.$$

Then

$$G_p(\sigma z) = G_{p^\sigma}(z) + n!\,{}^tPM_n(\sigma)P_n^{-1}S(\sigma).$$

THEOREM 3.2. *Assume that $p^\sigma = p$ and let $H(z)$ be any Eichler integral of $h(z)$. Then*

$$G_p(\sigma z) = G_p(z) + n!\,{}^tPP_n^{-1}S(\sigma).$$

As preliminaries to the proof, we require four lemmas.

LEMMA 3.3. *Let $\sigma \in \mathrm{SL}(2, \boldsymbol{R})$. Then*

$$\binom{\sigma(z)}{1}^n = j(\sigma, z)^{-n}M_n(\sigma)\binom{z}{1}^n.$$

Proof. Immediate.

LEMMA 3.4. *Suppose that $p^\sigma(z) = \binom{z}{1}^n P^\sigma$. Then*

$$P^\sigma = {}^tM_n(\sigma)P.$$

Proof.

$$P^\sigma(z) = j(\sigma, z)^n p(\sigma z) = j(\sigma, z)^n\, {}^t\binom{\sigma(z)}{1}^n P = {}^t\binom{z}{1}^n {}^tM_n(\sigma)P \quad \text{(Lemma 3.3)}.$$

LEMMA 3.5. *If $p^\sigma = p$, then ${}^tM_m(\sigma)P = P$ and ${}^tM_n(\sigma^{-1})P = P$.*

Proof. The first assertion follows from the preceding lemma. The second follows since $p^\sigma = p$ implies that $p^{\sigma^{-1}} = p$.

LEMMA 3.6. *Assume that $p^\sigma = p$. The quantity $n!\,{}^tPP_n^{-1}S(\sigma)$ is independent of the choice of the Eichler integral $H(z)$.*

Remark. Lemma 3.6 asserts that the quantity $n!\,{}^tPP_n^{-1}S(\sigma)$ depends only in the Eichler cohomology class of $H(z)$. That is, the quantity is unchanged if $H(z)$ is altered by addition of a co-boundary (a polynomial of the form $R^\sigma(z) - R(z)$).

Proof. Let $H(z)$ and $H_1(z)$ be any two Eichler integrals for $h(z)$. Further, let

$$H(\sigma z)j(\sigma, z)^n = H(z) + S_\sigma(z), \quad S_\sigma(z) = {}^t\binom{z}{1}^n S(\sigma),$$

$$H_1(\sigma z)j(\sigma, z)^n = H_1(z) + S_\sigma^1(z), \quad S_\sigma^1(z) = {}^t\binom{z}{1}^n S_1(\sigma).$$

Since H and H_1 are both Eichler integrals for h, we have

$$S_\sigma(z) - S_\sigma^1(z) = k^\sigma(z) - k(z)$$

where $k(z) \in \boldsymbol{C}[z]$. Thus, if $k(z) = {}^t\binom{z}{1}^n K$, then

$$S(\sigma) - S_1(\sigma) = K^\sigma - K = {}^tM_n(\sigma)K - K \quad \text{(Lemma 3.4)}.$$

Therefore,

$$\begin{aligned} n!\,{}^tPP_n^{-1}S(\sigma) - n!\,{}^tPP_n^{-1}S_1(\sigma) &= n!\,{}^tPP_n^{-1}[S(\sigma) - S_1(\sigma)] \\ &= n!\,{}^tPP_n^{-1}[{}^tM_n(\sigma)K - K] \\ &= n!\,{}^tPP_n^{-1}\,{}^tM_n(\sigma)K - n!\,{}^tPP_n^{-1}K \\ &= n!\,{}^tPM_n(\sigma)P_n^{-1}K - n!\,{}^tPP_n^{-1}K \quad ((3.6)) \\ &= n!\,{}^tPP_n^{-1}K - n!\,{}^tPP_n^{-1}K \quad \text{(Lemma 3.6)} \\ &= 0. \end{aligned}$$

Proof of Theorems 3.1 and 3.2. Consider the following vector integral:

$$\mathfrak{F}(z) = \begin{bmatrix} \int_{z_0}^{z} \tau^n h(\tau)\,d\tau \\ \int_{z_0}^{z} \tau^{n-1} h(\tau)\,d\tau \\ \vdots \\ \int_{z_0}^{z} h(\tau)\,d\tau \end{bmatrix}.$$

Under the action of Γ, $\mathfrak{F}(z)$ satisfies the transformation law (see [9], p. 298)

$$\mathfrak{F}(\sigma z) = M_n(\sigma)\mathfrak{F}(z) + R(\sigma),$$

where the column vector $R(\sigma)$ does not depend on z. Furthermore, $G_p(z)$ and $\mathfrak{F}(z)$ are related by the equation

$$G_p(z) = {}^tP\mathfrak{F}(z).$$

Therefore

$$(3.9)\qquad \begin{aligned} G_p(\sigma z)-G_{p^\sigma}(z) &= {}^tP\mathfrak{F}(\sigma z) = {}^tP^\sigma\mathfrak{F}(z)\\ &= {}^tP[M_n(\sigma)\mathfrak{F}(z)+R(\sigma)] = {}^tPM_n(\sigma)\mathfrak{F}(z)\\ &= {}^tPR(\sigma).\end{aligned}$$

On the other hand, if $H(z)$ denotes the particular Eichler integral

$$H(z) = \frac{1}{n!}\int_{z_0}^{z} h(\tau)(z-\tau)^n d\tau,$$

then (3.3) implies that

$$H(z) = \frac{1}{n!}{}^t\binom{z}{1}^n P_n\mathfrak{F}(z).$$

Therefore, replacing z by $\sigma(z)$ in the last equation, multiplying by $j(\sigma, z)^n$, and subtracting the original equation, we obtain

$$\begin{aligned} S_\sigma(z) &= H(\sigma z)j(\sigma, z)^n - H(z)\\ &= \frac{1}{n!}j(\sigma, z)^n\,{}^t\binom{\sigma(z)}{1}^n P_n\mathfrak{F}(\sigma z) - \frac{1}{n!}{}^t\binom{z}{1}^n P_n\mathfrak{F}(z)\\ &= \frac{1}{n!}{}^t\binom{z}{1}^n {}^tM_n(\sigma)P_n\mathfrak{F}(\sigma z) - \frac{1}{n!}{}^t\binom{z}{1}^n P_n\mathfrak{F}(z) \qquad \text{(Lemma 3.3)}\\ &= \frac{1}{n!}{}^t\binom{z}{1}^n {}^tM_n(\sigma)P_n[M_n(\sigma)\mathfrak{F}(z)+R(\sigma)] = \frac{1}{n!}{}^t\binom{z}{1}^n P_n\mathfrak{F}(z)\\ &= \frac{1}{n!}{}^t\binom{z}{1}^n {}^tM_n(\sigma)P_nR(\sigma) \qquad \text{(eq. (3.6))}.\end{aligned}$$

Therefore,

$$R(\sigma) = n!P_n^{-1}{}^tM_n(\sigma^{-1})S(\sigma),$$

so that by equations (3.9) and (3.6),

$$G_p(\sigma z) - G_{p^\sigma}(z) = n!{}^tPP_n^{-1}{}^tM_n(\sigma^{-1})S(\sigma) = n!{}^tPM_n(\sigma)P_n^{-1}S(\sigma),$$

which completes the proof of Theorem 3.1.

Theorem 3.2 now follows immediately from Theorem 3.1, Lemma 3.6 and the fact that if $p^\sigma = p$, then

$$n!{}^tPM_n(\sigma)P_n^{-1}S(\sigma) = n!{}^tPP_n^{-1}S(\sigma)$$

by Lemma 3.5.

COROLLARY 3.7. *Let all notations be as above and assume that $p^\sigma = p$ and $H(z)$ is any Eichler integral associated to $h(z)$. Then*

$$\int_z^{\sigma(z)} h(u)p(u)\,du = n!{}^tPP_n^{-1}S(\sigma).$$

We close this section by applying Corollary 3.7 to the integral

$$\int_{z_0}^{\sigma(z_0)} E_k(z\,|\,u, v)Q(z)^{k-1}dz$$

of Theorem 2.1. It is trivial to check that for $k \geqslant 2$, the function $E_k(z\,|\,u, v)$ is an automorphic form of weight $n+2 = 2k$ for the group $\Gamma = \Gamma(f)$. The case $k = 1$ is more delicate. If u and v are not both integers, then $E_k(z\,|\,u, v)$ is an automorphic form of weight $n+2 = 2$ for the group $\Gamma = \Gamma(f)$. However, if both u and v are integers, then $E_k(z\,|\,u, v)$ is not even holomorphic, let alone an automorphic form. The proofs of the statements for $k = 1$ can be patterned after the argument given in [14], pp. 63–68. The case u, v both integers corresponds to $\mathfrak{f} = (1)$, so that $\zeta(s, \mathfrak{b}, \mathfrak{f})$ is just the zeta function of the principal ideal class of K. And the functional equation for this zeta function ([6], p. 67) implies that

$$\zeta\big(0, (1), (1)\big) = 0.$$

Therefore, let us henceforth assume that either (a) $k > 1$ *or* (b) $k = 1$ *and* $\mathfrak{f} \neq (1)$. Modulo this assumption, $E_k(z\,|\,u, v)$ is an automorphic form of weight $n+2 = 2k$.

LEMMA 3.8. *The polynomial $p(z) = Q(z)^{k-1}$ has degree $n = 2k-2$. Let $\sigma \in \mathrm{SL}(2, \mathbf{Z})$ be defined by*

$$\begin{pmatrix}\varepsilon_\mathfrak{f}\omega_1\\ \varepsilon_\mathfrak{f}\omega_2\end{pmatrix} = \sigma\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}.$$

Then $p^\sigma = p$.

Proof. Let $\sigma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$. Then

$$\begin{pmatrix}\varepsilon_\mathfrak{f}^{-1}\omega_1\\ \varepsilon_\mathfrak{f}^{-1}\omega_2\end{pmatrix} = \begin{pmatrix} d & -b\\ -c & a\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}.$$

Moreover,

$$\begin{aligned} p^\sigma(z) &= \{[\omega_2(az+b)-\omega_1(cz+d)][\omega_2'(az+b)-\omega_1'(cz+d)]\}^{k-1}\\ &= (\varepsilon_\mathfrak{f}^{-1}\varepsilon_\mathfrak{f}'^{-1})^{k-1}[(\omega_2 z-\omega_1)(\omega_2' z-\omega_1')]^{k-1}\\ &= p(z) \qquad (\text{since } N(\varepsilon_\mathfrak{f}) = +1).\end{aligned}$$

Thus, finally deduce,

THEOREM 3.9. *Assume that either $k > 1$ or $k = 1$ and $\mathfrak{f} = 1$. Let $\sigma \in \mathrm{SL}(2, \mathbf{Z})$ be such that*

$$\varepsilon_{\mathfrak{f}}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \sigma\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix},$$

and let $S_\sigma(z)$ be the σ-period of any Eichler integral associated to $E_k(z|u, v)$. Further, let $p(z) = Q(z)^{k-1}$,

$$p(z) = {}^t\begin{pmatrix}z\\ 1\end{pmatrix}^n P, \qquad S_\sigma(z) = {}^t\begin{pmatrix}z\\ 1\end{pmatrix}^n S(\sigma).$$

Then

$$\zeta(1-k, \mathfrak{b}, \mathfrak{f}) = I_k\, {}^tPP_n^{-1}S(\sigma),$$

where

$$I_k = \frac{N(\mathfrak{b})^{k-1}\Gamma(2k)^2}{2^{2k}(2k-1)}\,\pi^{-2k}.$$

4. Computation of the period polynomial. Throughout this section, $E_k(z|u, v)$ will be considered only in case either $k > 1$ or $k = 1$ and u, v are not both integers. This restriction guarantees that $E_k(z|u, v)$ is an automorphic form for a certain subgroup of finite index in $\mathrm{SL}(2, \mathbf{Z})$. By using the Poisson summation formula (plus a limiting argument if $k = 1$), we obtain the following Fourier expansion around the cusp at $i\infty$:

$$E_k(z|u, v) = \Lambda(u) + \frac{(2\pi i)^{2k}}{\Gamma(2k)} \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{2\pi inv} \sum_{\substack{m=-\infty\\ n(m+u)>0}}^{\infty} |m+u|^{2k-1} e^{2\pi in(m+u)z},$$

where

$$\Lambda(u) = \sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty} \frac{e^{2\pi imu}}{|m|^{2k}}.$$

Thus, by (3.2), we find that an Eichler integral of $E_k(z|u, v)$ is given by

$$(4.1)\qquad H_k(z) = H_k(z|u, v)$$
$$= \frac{\Lambda(u)}{\Gamma(2k)} z^{2k-1} + \frac{2\pi i}{\Gamma(2k)} \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{2\pi inv} \sum_{m=-\infty}^{\infty} \frac{e^{2\pi in(m+u)z}}{|n|^{2k-1}}.$$

The main result of this section is to calculate the period polynomial $S_\sigma(z)$ for this Eichler integral with respect to any $\sigma \in \Gamma(f)$. There are many methods available for accomplishing this. The one used here is a quite general method based on the following general principle: Any functional equation involving automorphic forms (or Eichler integrals) should be a reflection of the functional equation of an appropriate zeta function. The method used here is an adaptation of the methods used in [3] and [4] and is capable of much greater generality than exhibited here. The zeta functions appropriate to this computation are the partial zeta functions which have recently figured in the work of Coates and Sinnot [2] on p-adic L-functions of real quadratic fields.

Our plan, in outline, is as follows: We express the non-constant part of the Eisenstein series $(H_k(z))$ in terms of a sum of functions $(G_{p,q}(\omega))$ whose Mellin transforms are reasonably elementary zeta functions. We use the Mellin inversion theorem to express $H_k(z)$ in terms of these zeta functions. By shifting the line of integration in the resulting formula and at the same time applying the functional equation of the zeta functions, we arrive at the transformation formula for $H_k(z)$. The period polynomial is computed from the residues of the poles passed over in shifting the line of integration. Unfortunately the details of the calculation are somewhat tedious to carry out.

Let $\sigma = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$. If $c = 0$, then $a = d = \pm 1$ and we immediately see that

$$S_\sigma(z) = \frac{\Lambda(u)}{\Gamma(2k)}\{(az+b)^{2k-1} - z^{2k-1}\}.$$

Thus, it suffices to assume that $c \neq 0$, which we do throughout the remainder of this section. In fact, by replacing σ by $-\sigma$, we may assume that $c > 0$.

Set

$$w = \frac{cz+d}{i},$$

so that

$$z = -\frac{d}{c} + \frac{iw}{c}, \qquad \sigma(z) = \frac{a}{c} + \frac{i}{cw}.$$

Further, let

$$u = \frac{h}{f}, \qquad v = \frac{g}{f}.$$

Then, in terms of these notations, we have

$$H_k(z)-\frac{A(u)}{\Gamma(2k)}z^{2k-1} = \frac{2\pi i}{\Gamma(2k)}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\sum_{\substack{m=-\infty\\ n(m+u)>0}}^{\infty} e^{2\pi i n v}\frac{1}{|n|^{2k-1}}e^{2\pi i n(m+u)\left(-\frac{d}{c}+\frac{iw}{c}\right)}$$

$$= \frac{2\pi i}{\Gamma(2k)}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\sum_{\substack{n=-\infty\\ n(m+u)>0}}^{\infty} e^{2\pi i n v}\frac{1}{|n|^{2k-1}}e^{\frac{2\pi i}{f}n(mf+h)\left(-\frac{d}{c}+\frac{iw}{c}\right)}$$

$$= \frac{2\pi i}{(2k)}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{2\pi i n g/f}\sum_{\substack{m=-\infty\\ m\equiv h\,(\mathrm{mod}\, f)\\ mn>0}}^{\infty}\frac{1}{|n|^{2k-1}}e^{\frac{2\pi i}{f}nm\left(-\frac{d}{c}+\frac{iw}{c}\right)},$$

$$= \frac{2\pi i}{\Gamma(2k)}\sum_{p\,(\mathrm{mod}\, cf)} e^{2\pi i g p/f}\sum_{\substack{q\,(\mathrm{mod}\, cf)\\ q\equiv h\,(\mathrm{mod}\, f)}} e^{-2\pi i q p d/cf}G_{p,q}(w),$$

where

$$G_{p,q}(w) = \sum_{\substack{n=-\infty\\ n\equiv p\,(\mathrm{mod}\, cf)}}^{\infty}{}' \sum_{\substack{m=-\infty\\ m\equiv q\,(\mathrm{mod}\, cf)\\ nm>0}}^{\infty}\frac{1}{|n|^{2k-1}}e^{-2\pi m n w/cf},$$

where the prime on the summation means that the summand 0 is omitted. Let θ_ε ($\varepsilon = 0, 1$) denote the two signature characters of the real field, that is

$$\theta_\varepsilon\colon \boldsymbol{R}^\times \to \{\pm 1\},$$

$$\theta_\varepsilon(x) = \operatorname{sgn}(x)^\varepsilon.$$

Then

$$(4.2)\quad H_k(z)-\frac{A(u)}{\Gamma(2k)}z^{2k-1} = \frac{\pi i}{\Gamma(2k)}\sum_{\varepsilon=0,1}\sum_{p\,(\mathrm{mod}\, cf)}\sum_{\substack{q\,(\mathrm{mod}\, cf)\\ q\equiv h\,(\mathrm{mod}\, f)}} e^{2\pi i\frac{gpc-qpd}{cf}}G_{p,q,\varepsilon}(w)$$

where

$$G_{p,q,\varepsilon}(w) = \sum_{\substack{n=-\infty\\ n\equiv p\,(\mathrm{mod}\, cf)}}^{\infty}{}' \sum_{\substack{m=-\infty\\ m\equiv q\,(\mathrm{mod}\, cf)}}^{\infty}{}' \frac{\theta_\varepsilon(m)\theta_\varepsilon(n)}{|n|^{2k-1}}e^{-2\pi|mn|w/cf}.$$

Let $\Phi_{p,q,\varepsilon}(s)$ denote the Mellin transform of $G_{p,q,\varepsilon}(w)$:

$$\Phi_{p,q,\varepsilon}(s) = \int_0^\infty G_{p,q,\varepsilon}(w)w^{s-1}dw \qquad (\mathrm{Re}\, s>0).$$

Then a simple calculation shows that

$$(4.3)\qquad \Phi_{p,q,\varepsilon}(s) = \left(\frac{2\pi}{cf}\right)^{-s}\Gamma(s)\zeta_\varepsilon(s+2k-1, p, cf)\zeta_\varepsilon(s, q, cf),$$

where

$$\zeta_\varepsilon(s, p, r) = \sum_{\substack{m=-\infty\\ m\equiv p\,(\mathrm{mod}\, r)}}^{\infty}{}' \frac{\theta_\varepsilon(m)}{|m|^s} \qquad (\mathrm{Re}\, s>1).$$

The function $\zeta_\varepsilon(s, p, r)$ was studied in [3]. Here is a summary of its properties: $\zeta_\varepsilon(s, p, r)$ can be analytically continued to a meromorphic function in the entire s-plane with a single possible pole at $s=1$. The pole occurs if and only if $\varepsilon=0$ and has residue $2/r$. Furthermore, we have the functional equation

$$(4.4)\quad \left(\frac{\pi}{r}\right)^{-s/2}\Gamma\left(\frac{s+\varepsilon}{2}\right)\zeta_\varepsilon(s, p, r)$$
$$= (-i)^\varepsilon r^{-1/2}\left(\frac{\pi}{r}\right)^{-(1-s)/2}\Gamma\left(\frac{1-s+\varepsilon}{2}\right)\sum_{\lambda\,(\mathrm{mod}\, r)} e^{2\pi i\lambda p/r}\zeta_\varepsilon(1-s, \lambda, r).$$

Furthermore, the function on the left side of the functional equation is analytic except for poles at $s=1$ ($\varepsilon=0$, all p) and $s=0$ ($\varepsilon=0$, $p\equiv 0\,(\mathrm{mod}\, r)$).

LEMMA 4.1. *The function $\Phi_{p,q,\varepsilon}(s)$ is meromorphic in the entire s-plane and satisfies the functional equation*

$$\Phi_{p,q,\varepsilon}(s) = \frac{(-1)^{k-1}}{cf}\sum_{\alpha,\beta\,(\mathrm{mod}\, cf)} e^{2\pi i\frac{\alpha p+\beta q}{cf}}\Phi_{\beta,\alpha,\varepsilon}(-s-2k+2).$$

Proof. Replace r by cf in (4.4). Then multiply the resulting equations corresponding to the arguments (s, p, cf) and $(s+2k-1, q, cf)$, to get

$$(4.5)\quad \left(\frac{\pi}{cf}\right)^{-s-k+1/2}\Gamma\left(\frac{s+2k-1+\varepsilon}{2}\right)\Gamma\left(\frac{s+\varepsilon}{2}\right)\zeta_\varepsilon(s+2k-1, p, cf)\zeta_\varepsilon(s, q, cf)$$
$$= \frac{(-1)^\varepsilon}{cf}\left(\frac{\pi}{cf}\right)^{-(1-s-k+1/2)}\Gamma\left(\frac{2-s-2k+\varepsilon}{2}\right)\Gamma\left(\frac{1-s+\varepsilon}{2}\right)\times$$
$$\times\sum_{\alpha,\beta\,(\mathrm{mod}\, cf)} e^{2\pi i\frac{\alpha p+\beta q}{cf}}\zeta_\varepsilon(2-s-2k, \alpha, cf)\zeta_\varepsilon(1-s, \beta, cf).$$

Now since

$$\Gamma\left(\frac{s+2k-1+\varepsilon}{2}\right) = \frac{s+2k-3+\varepsilon}{2}\,\frac{s+2k-5+\varepsilon}{2}\cdots\frac{s-1+\varepsilon}{2}\Gamma\left(\frac{s+1+\varepsilon}{2}\right),$$

we have

$$(4.6)\qquad \Gamma\left(\frac{s+2k-1+\varepsilon}{2}\right)\Gamma\left(\frac{s+\varepsilon}{2}\right)$$
$$= 2^{-k+1}(s+2k-3+\varepsilon)(s+2k-5+\varepsilon)\ldots(s+1+\varepsilon)\Gamma\left(\frac{s+1+\varepsilon}{2}\right)\Gamma\left(\frac{s+\varepsilon}{2}\right)$$
$$= \sqrt{\pi}2^{2-k-s-\varepsilon}(s+2k-3+\varepsilon)(s+2k-5+\varepsilon)\ldots(s+1+\varepsilon)\Gamma(s+\varepsilon)$$
$$= \sqrt{\pi}2^{2-k-s-\varepsilon}(s+2k-3+\varepsilon)(s+2k-5+\varepsilon)\ldots(s+1+\varepsilon)s^{\varepsilon}\Gamma(s).$$

By replacing s with $-s-2k+2$ in the last formula, we obtain

$$(4.7)\qquad \Gamma\left(\frac{2-s-2k+\varepsilon}{2}\right)\Gamma\left(\frac{1-s+\varepsilon}{2}\right)$$
$$= \sqrt{\pi}(-1)^{k-1+\varepsilon k+s-\varepsilon}(s+1+\varepsilon)(s+3+\varepsilon)\ldots$$
$$\ldots(s+2k-3+\varepsilon)s^{\varepsilon}\Gamma(-s-2k+2).$$

Inserting (4.6) and (4.7) into (4.5) and dividing both sides by

$$(s+1+\varepsilon)(s+3+\varepsilon)\ldots(s+2k-3+\varepsilon)s^{\varepsilon},$$

we get the desired functional equation.

Note that the function on the left side of (4.5) is regular except for possible poles at $\varepsilon = 1, 0$ and $s = -2k+2$ and these poles occur precisely in case $\varepsilon = 0$ and $s = 0$ under the further condition that $p \equiv 0 \pmod{cf}$. Since, to get $\Phi_{p,q,\varepsilon}(s)$, we divided by

$$(s+1+\varepsilon)(s+3+\varepsilon)\ldots(s+2k-3+\varepsilon)s^{\varepsilon},$$

we see that

LEMMA 4.2. *$\Phi_{p,q,\varepsilon}(s)$ is analytic except for possible poles at*

$$s = \begin{cases} 0,1,-1,-3,\ldots,-2k+3,-2k+1,-2k+2 & (\varepsilon=0),\\ 0,-2,-4,\ldots,-2k+2 & (\varepsilon=1).\end{cases}$$

Let us now make use of the functional equation of Lemma 4.1. By the Mellin inversion theorem,

$$G_{p,q,\varepsilon}(w) = \frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\Phi_{p,q,\varepsilon}(s)w^{-s}\,ds.$$

Apply Cauchy's theorem to shift the line of integration to the line $\operatorname{Re} s = -2k+\frac{1}{2}$. To do so, we must take into account the residues of the poles of the integrand in the strip $-2k+\frac{1}{2} < \operatorname{Re} s < \frac{3}{2}$. By Lemma 4.2, these poles belong to the set

$$S(\varepsilon) = \begin{cases}\{-1,-3,\ldots,-2k+1\}\cup\{0,1,-2k+2\} & \text{if } \varepsilon=0,\\ \{0,-2,-4,\ldots,-2k+2\} & \text{if } \varepsilon=1.\end{cases}$$

Therefore, we obtain

$$G_{p,q,\varepsilon}(w) = \frac{1}{2\pi i}\int_{-2k+1/2-i\infty}^{-2k+1/2+i\infty}\Phi_{p,q,\varepsilon}(s)w^{-s}\,ds + \sum_{s\in S(\varepsilon)}\operatorname{Res}\Phi_{p,q,\varepsilon}(s)w^{-s}$$
$$= \frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\Phi_{p,q,\varepsilon}(-s-2k+2)w^{s+2k-2}\,ds + \sum_{s\in S(\varepsilon)}\operatorname{Res}\Phi_{p,q,\varepsilon}(s)w^{-s}.$$

Applying Lemma 4.1, we obtain

$$(4.8)\qquad G_{p,q,\varepsilon}(w)$$
$$= \frac{(-1)^{k-1}w^{2k-2}}{cf}\sum_{\alpha,\beta\,(\mathrm{mod}\,cf)} e^{2\pi i\frac{p\alpha+q\beta}{cf}}\frac{1}{2\pi i}\int_{3/2-i\infty}^{3/2+i\infty}\Phi_{\beta,\alpha,\varepsilon}(s)w^{s}\,ds +$$
$$+\sum_{s\in S(\varepsilon)}\operatorname{Res}\Phi_{p,q,\varepsilon}(s)w^{-s}$$
$$= \frac{(-1)^{k-1}w^{2k-2}}{cf}\sum_{\alpha,\beta\,(\mathrm{mod}\,cf)} e^{2\pi i\frac{p\alpha+q\beta}{cf}}G_{\beta,\alpha,\varepsilon}(w^{-1}) + \sum_{s\in S(\varepsilon)}\operatorname{Res}\Phi_{p,q,\varepsilon}(s)w^{-s}.$$

Thus, combining (4.2) and (4.8), we find that

$$(4.9)\qquad H_k(z) - \frac{\Lambda(u)}{\Gamma(2k)}z^{2k-1}$$
$$= \frac{(-1)^{k-1}w^{2k-2}\pi i}{cf\Gamma(2k)}\sum_{\varepsilon=0,1}\sum_{p,q\,(\mathrm{mod}\,cf)}\sum_{\alpha,\beta\,(\mathrm{mod}\,cf)} e^{2\pi i\frac{gpc-qpd+p\alpha+q\beta}{cf}}G_{\beta,\alpha,\varepsilon}(w^{-1}) +$$
$$+\frac{\pi i}{\Gamma(2k)}\sum_{\varepsilon=0,1}\sum_{s\in S(\varepsilon)}\operatorname{Res}\Bigg(\sum_{\substack{p,q\,(\mathrm{mod}\,cf)\\ q\equiv h\,(\mathrm{mod}\,f)}} e^{2\pi i\frac{gpc-qpd}{cf}}\Phi_{p,q,\varepsilon}(s)w^{-s}\Bigg).$$

We now study the two terms on the right hand side of (4.9). As to the first one,

$$\sum_{\substack{p,q\,(\mathrm{mod}\,cf)\\ q\equiv h\,(\mathrm{mod}\,f)}}\sum_{\alpha,\beta\,(\mathrm{mod}\,cf)} e^{2\pi i\frac{gpc-qpd+p\alpha+q\beta}{cf}}G_{\beta,\alpha,\varepsilon}(w^{-1})$$
$$= \sum_{\alpha,\beta\,(\mathrm{mod}\,cf} G_{\beta,\alpha,\varepsilon}(w^{-1})\sum_{p\,(\mathrm{mod}\,cf)} e^{2\pi i(gc+\alpha)p/cf}\sum_{\substack{q\,(\mathrm{mod}\,cf)\\ q\equiv h\,(\mathrm{mod}\,cf)}} e^{2\pi i(\beta-pd)q/cf}.$$

It is easy to see that the innermost sum is 0 if $c \nmid \beta - pd$ and equals

$$ce^{2\pi i h(\beta - pd)/cf}$$

otherwise. However, the condition $c \nmid \beta - pd$ is equivalent to

$$p \equiv \beta a \pmod{c}$$

since $ad - bc = 1$. Therefore,

$$\sum_{\substack{p,q \pmod{cf} \\ q \equiv h \pmod{f}}} \sum_{\alpha,\beta \pmod{cf}} e^{2\pi i \frac{gpc - qpd + p\alpha + q\beta}{cf}} G_{\beta,\alpha,\varepsilon}(w^{-1})$$
$$= c \sum_{\alpha,\beta \pmod{cf}} G_{\beta,\alpha,\varepsilon}(w^{-1}) \sum_{\substack{p \pmod{cf} \\ p \equiv \beta a \pmod{c}}} e^{2\pi i[(gc+\alpha)p + h(\beta - pd)]/cf}$$
$$= c \sum_{\alpha,\beta \pmod{cf}} G_{\beta,\alpha,\varepsilon}(w^{-1}) e^{2\pi i h\beta/cf} \sum_{\substack{p \pmod{cf} \\ p \equiv \beta a \pmod{c}}} e^{2\pi i(gc+\alpha-dh)p/cf}$$
$$= cf \sum_{\substack{\alpha,\beta \pmod{cf} \\ gc+\alpha-dh \equiv 0 \pmod{f}}} G_{\beta,\alpha,\varepsilon}(w^{-1}) e^{2\pi i h\beta/cf} e^{2\pi i(gc+\alpha-dh)\beta a/cf}$$
$$= cf \sum_{\substack{\alpha,\beta \pmod{cf} \\ \alpha \equiv h \pmod{cf}}} G_{\beta,\alpha,\varepsilon}(w^{-1}) e^{2\pi i(h\beta + g\beta ca + \alpha\beta a - dh\beta a)/cf}$$
$$= cf \sum_{\substack{\alpha,\beta \pmod{cf} \\ \alpha \equiv h \pmod{cf}}} G_{\beta,\alpha,\varepsilon}(w^{-1}) e^{2\pi i(g\beta c + a\alpha\beta)/cf}$$
$$(\text{since } ad \equiv 1 \pmod{cf},\ a \equiv 1 \pmod{f}).$$

Therefore, by (4.2) and the formulae relating w with z and $\sigma(z)$, we see that (4.9) implies that

$$(4.10)\qquad H_k(z) - \frac{\Lambda(u)}{\Gamma(2k)} z^{2k-1}$$
$$= j(\sigma, z)^{2k-1}\left\{H_k(\sigma z) - \frac{\Lambda(u)}{\Gamma(2k)}\sigma(z)^{2k-1}\right\} + \sum_{\varepsilon=0,1}\sum_{j \in S(\varepsilon)} R_{j,\varepsilon},$$

where

$$R_{j,\varepsilon} = \frac{\pi i}{\Gamma(2k)} \operatorname{Res}_{s=j}\left(\sum_{\substack{p,q \pmod{cf} \\ q \equiv h \pmod{f}}} e^{2\pi i \frac{gpc - qpd}{cf}} \Phi_{p,q,\varepsilon}(s) w^{-s}\right).$$

To complete our derivation of the transformation law for $H_k(z)$, let us now compute the residues $R_{j,\varepsilon}$. We do this in the following three lemmas.

LEMMA 4.3.

$$R_{1,0} = j(\sigma, z)^{-1} \frac{\Lambda(u)}{\Gamma(2k)} c^{-2k+1}.$$

Proof. The pole at $s = 1$ arises because of the simple pole of $\zeta_0(s, q, cf)$ there. Therefore,

$$R_{1,0} = \frac{\pi i}{\Gamma(2k)} \sum_{p,q \pmod{cf}} e^{2\pi i \frac{gpc - qpd}{cf}} \zeta_0(2k, p, cf) \frac{2}{cf}\left(\frac{2\pi}{cf}\right)^{-1} w^{-1}$$
$$= \frac{iw^{-1}}{\Gamma(2k)} \sum_{p \pmod{cf}} e^{2\pi i gp/f} \zeta_0(2k, p, cf) \sum_{\substack{q \pmod{cf} \\ q \equiv h \pmod{f}}} e^{-2\pi i qpd/cf}$$
$$= \frac{iw^{-1}c}{\Gamma(2k)} \sum_{\substack{p \pmod{cf} \\ p \equiv 0 \pmod{c}}} e^{2\pi i(gpc + hpd)/cf} \zeta_0(2k, p, cf)$$
$$= \frac{iw^{-1}c^{-2k+1}}{\Gamma(2k)} \sum_{r \pmod{f}} e^{2\pi i(gc + hd)r/f} \zeta_0(2k, cr, cf)$$
$$= \frac{iw^{-1}c^{-2k+1}}{\Gamma(2k)} \sum_{r \pmod{f}} e^{2\pi i hr/f} \zeta_0(2k, r, f)$$
$$(\text{since } c \equiv 0 \pmod{f},\ d \equiv 1 \pmod{f})$$
$$= \frac{iw^{-1}c^{-2k+1}}{\Gamma(2k)} \sideset{}{'}\sum_{m=-\infty}^{\infty} \frac{e^{2\pi i um}}{|m|^{2k}} = j(\sigma, z)^{-1} \frac{\Lambda(u)}{\Gamma(2k)} c^{-2k+1}.$$

Let us introduce the quantity

$$C_k(u, v) = \begin{cases} 0 & \text{if} \quad u \notin \mathbf{Z} \text{ or } k = 1, \\ \dfrac{\pi i}{\Gamma(2k)} \displaystyle\sideset{}{'}\sum_{m=-\infty}^{\infty} \frac{e^{2\pi i m v}}{|m|^{2k-1}} & \text{if} \quad u \in \mathbf{Z} \text{ and } k > 1. \end{cases}$$

Then, we have

LEMMA 4.4. $R_{0,0} = C_k(u, v)$ *if* $k > 1$, $= 0$ *if* $k = 1$.

Proof. If $k > 1$, then $\zeta_0(s + 2k - 1, p, cf)$ and $\zeta_0(s, q, cf)$ are both regular for $s = 0$. The residue is then contributed by the simple pole of $\Gamma(s)$ at $s = 0$. Therefore,

$$R_{0,0} = \frac{\pi i}{\Gamma(2k)} \sum_{\substack{q \pmod{cf} \\ q \equiv h \pmod{f}}} \zeta_0(0, q, cf) \sum_{p \pmod{cf}} e^{2\pi i(gc - qd)p/cf} \zeta_0(2k-1, p, cf).$$

However, $\zeta_0(0, q, cf) = 0$ if $cf \nmid q$. From this it follows that

$$R_{0,0} = \begin{cases} 0 & \text{if} \quad f \nmid h, \\ \dfrac{\pi i}{\Gamma(2k)} \displaystyle\sum_{p (\bmod cf)} e^{2\pi i gp/cf} \zeta_0(2k-1, p, cf) & \text{if} \quad f | h \end{cases}$$

$$= C_k(u, v).$$

Assume now that $k = 1$. Then

$$R_{0,0} = \frac{\pi i}{\Gamma(2k)} \sum_{\substack{q (\bmod cf) \\ q \equiv h (\bmod f)}} \operatorname{Res}_{s=0} \left(\left(\frac{2\pi}{cf} \right)^{-s} \Gamma(s) \zeta_0(s, q, cf) \times \right.$$

$$\left. \times \sum_{p (\bmod cf)} e^{2\pi i (gc - qd)p/cf} \zeta_0(s+1, p, cf) \right).$$

Note that the function $\zeta_0(s+1, p, cf)$ has a simple pole at $s = 0$ with residue $2/cf$. Therefore, if $qd \not\equiv gc (\bmod cf)$, the inner sum is regular at $s = 0$. But this last condition is equivalent to $q \not\equiv gc (\bmod cf)$. On the other hand, if $q \equiv gc (\bmod cf)$, then the inner sum equals $\zeta(s+1)$, which has a simple pole at $s = 0$ with residue 1. Ostensibly, in this latter circumstance, there is a double pole since $\Gamma(s)$ also has a pole at $s = 0$. However, note that the double pole never occurs since if $q \equiv gc (\bmod cf)$ and $q \equiv h (\bmod f)$, then $h \equiv gc \equiv 0 (\bmod f)$. Furthermore $\zeta_0(0, q, cf) = 0$ if $cf \nmid q$. Thus, a double pole could exist only if $q \equiv 0 (\bmod cf)$. But this would imply that $gc \equiv 0 (\bmod cf)$ and $g \equiv 0 (\bmod f)$. Therefore, both u, v are in $\mathbf{Z}$, contrary to our assumption. Therefore, we may compute $R_{0,0}$ as follows:

$$R_{0,0} = \frac{\pi i}{\Gamma(2k)} \sum_{\substack{q (\bmod cf) \\ q \equiv h (\bmod f) \\ q \not\equiv gc (\bmod cf)}} \zeta_0(0, q, cf) \cdot \lim_{s \to 0} \sum_{p (\bmod cf)} e^{2\pi i (gc - qd)p/cf} \zeta_0(s+1, p, cf) +$$

$$+ \frac{\pi i}{\Gamma(2k)} \sum_{\substack{q (\bmod cf) \\ q \equiv h (\bmod f) \\ q \equiv gc (\bmod cf)}} \lim_{s \to 0} \left(\frac{2\pi}{cf} \right)^{-s} \Gamma(s) \zeta_0(s, q, cf).$$

Let $\delta(h) = 1$ if $h \equiv 0 (\bmod f)$, $= 0$ otherwise. Then

$$R_{0,0} = \frac{\pi i}{\Gamma(2k)} \delta(h) \zeta_0(0, 0, cf) \sum_{m=-\infty}^{\infty}{}' \frac{e^{2\pi i m g/f}}{|m|} +$$

$$+ \frac{\pi i}{\Gamma(2k)} \delta(h) \lim_{s \to 0} \left(\frac{2\pi}{cf} \right)^{-s} \Gamma(s) \zeta_0(s, gc, cf).$$

By the functional equation for $\zeta_0(s, gc, cf)$, we readily deduce that

$$\lim_{s \to 0} \left(\frac{2\pi}{cf} \right)^{-s} \Gamma(s) \zeta_0(s, gc, cf) = \frac{1}{2} \sum_{m=-\infty}^{\infty}{}' \frac{e^{2\pi i m g/f}}{|m|}.$$

Thus, since $\zeta_0(0, 0, cf) = \zeta(0) = -\frac{1}{2}$, we have $R_{0,0} = 0$.

LEMMA 4.5. $R_{-2k+2,0} = -C_k(u, v) j(\sigma, z)^{2k-2}$ $(k > 1)$.

Proof. By Lemma 4.1,

$$R_{-2k+2,0} = \frac{\pi i (-1)^{k-1}}{cf \Gamma(2k)} \operatorname{Res}_{s=-2k+2} \left(\sum_{\substack{p, q, \alpha, \beta (\bmod cf) \\ q \equiv h (\bmod f)}} e^{2\pi i (gpc - qpd + \alpha p + \beta q)/cf} \times \right.$$

$$\left. \times \Phi_{\beta, \alpha, 0}(-s-2k+2) w^{-s} \right)$$

$$= \frac{\pi i (-1)^k}{cf \Gamma(2k)} w^{2k-2} \lim_{s \to -2k+2} \left(\sum_{\substack{\alpha, \beta, q (\bmod cf) \\ q \equiv h (\bmod f)}} e^{2\pi i \beta q/cf} \zeta_0(-s+1, \beta, cf) \times \right.$$

$$\left. \times \zeta_0(-s-2k+2, \alpha, cf) \sum_{p (\bmod cf)} e^{2\pi i (gc - qd + \alpha)p/cf} \right).$$

But the innermost sum is 0 unless

$$gc - qd + \alpha \equiv 0 (\bmod cf) \Leftrightarrow \alpha \equiv qd - gc (\bmod cf).$$

Therefore,

$$R_{-2k+2,0} = \frac{\pi i (-1)^k}{\Gamma(2k)} w^{2k-2} \lim_{s \to -2k+2} \left(\sum_{\substack{\beta, q (\bmod cf) \\ q \equiv h (\bmod f)}} e^{2\pi i \beta q/cf} \zeta_0(s+1, \beta, cf) \times \right.$$

$$\left. \times \zeta_0(-s-2k+2, qd - gc, cf) \right).$$

However,

$$\zeta_0(0, r, cf) = \begin{cases} 0 & \text{if} \quad cf \nmid r, \\ 1 & \text{if} \quad cf | r \end{cases}$$

(use the functional equation). However,

$$qd - gc \equiv 0 (\bmod cf) \Leftrightarrow q \equiv gca (\bmod cf) \Leftrightarrow q \equiv gc (\bmod cf).$$

Therefore, since $f | c$,

$$R_{-2k+2,0} = \begin{cases} 0 & \text{if} \quad f \nmid h, \\ \dfrac{\pi i (-1)^k}{\Gamma(2k)} w^{2k-2} \displaystyle\lim_{s \to -2k+2} \sum_{\beta (\bmod cf)} e^{2\pi i \beta g/f} \zeta_0(-s+1, \beta, cf) & \text{if} \quad f | h \end{cases}$$

$$= -C_k(u, v) j(\sigma, z)^{2k-2}.$$

The poles of $\Phi_{p,q,\varepsilon}(s)w^{-s}$ other than those considered in Lemmas 4.3–4.5 above, are all of the form

$$s=-m,\quad 0\leqslant m\leqslant 2k-1,\quad m+\varepsilon \text{ odd}.$$

LEMMA 4.6. *Assume that* $0\leqslant m\leqslant 2k-1$, $m+\varepsilon$ *odd. Then*

$$R_{-m,\varepsilon}=j(\sigma,z)^m\times$$
$$\times\frac{(-1)^k2^{2k-1}\pi^{2k}}{k\Gamma(2k)^2}\binom{2k}{m+1}\sum_{r(\mathrm{mod}\,c)}P_{m+1}\left(\frac{u+r}{c}\right)P_{2k-m-1}\left(v-\frac{u+r}{c}d\right)$$

where $P_r(x)=B_r(x-[x])$ *is the* r*-th Bernoulli polynomial*[3] mod 1.

Proof. Since, for $m\geqslant 0$, we have

$$\mathrm{Res}_{s=m}\Gamma(s)=\frac{(-1)^m}{m!},$$

we deduce that

$$R_{-m,\varepsilon}=\frac{\pi i}{\Gamma(2k)}\frac{(-1)^m}{m!}\left(\frac{2\pi}{cf}\right)^m w^m\sum_{\substack{q(\mathrm{mod}\,cf)\\ q\equiv h(\mathrm{mod}\,f)}}\zeta_\varepsilon(-m,q,cf)\times$$
$$\times\sum_{p(\mathrm{mod}\,cf)}e^{2\pi i(qc-qd)p/cf}\zeta_\varepsilon(2k-m-1,p,cf)$$
$$=\frac{\pi i}{\Gamma(2k)}\frac{(-1)^m}{m!}\left(\frac{2\pi}{cf}\right)^m w^m\sum_{\substack{q(\mathrm{mod}\,cf)\\ q\equiv h(\mathrm{mod}\,f)}}\zeta_\varepsilon(-m,q,cf)\sum_{t=-\infty}^{\infty}{}'\frac{e^{2\pi i(qc-qd)t/cf}}{t^{2k-m-1}}.$$

It is well known ([11], p. 17) that for positive integral r, we have

$$P_r(x)=-\frac{r!}{(2\pi i)^r}\sum_{t=-\infty}^{\infty}{}'\frac{e^{2\pi itx}}{t^r}\qquad(r\geqslant 1).$$

Therefore,

$$R_{-m,\varepsilon}=w^m\frac{\pi i}{\Gamma(2k)}\frac{(-1)^{m+1}}{m!}\left(\frac{2\pi}{cf}\right)^m\frac{(2\pi i)^{2k-m-1}}{(2k-m-1)!}\times$$
$$\times\sum_{\substack{q(\mathrm{mod}\,cf)\\ q\equiv h(\mathrm{mod}\,f)}}\zeta_\varepsilon(-m,q,cf)P_{2k-m-1}\left(v-\frac{qd}{cf}\right).$$

[3] Note that we normalize B_1, the first Bernoulli number to be $-\frac{1}{2}$, so that $B_1(x)=x-\frac{1}{2}$.

However, by the functional equation for $\zeta_\varepsilon(s,q,cf)$, we have

$$\zeta_\varepsilon(-m,q,cf)$$
$$=(-i)^\varepsilon\pi^{-1/2-m}(cf)^m\frac{\Gamma\left(\frac{1+m+\varepsilon}{2}\right)}{\Gamma\left(\frac{-m+\varepsilon}{2}\right)}\sum_{\lambda(\mathrm{mod}\,cf)}e^{2\pi i\lambda q/cf}\zeta_\varepsilon(1+m,\lambda,cf)$$
$$=(-i)^\varepsilon\pi^{-1/2-m}(cf)^m\frac{\Gamma\left(\frac{1+m+\varepsilon}{2}\right)}{\Gamma\left(\frac{-m+\varepsilon}{2}\right)}\sum_{t=-\infty}^{\infty}{}'\frac{e^{2\pi itq/cf}}{t^{1+m}}$$
$$=-\frac{2}{m+1}(cf)^mP_{m+1}\left(\frac{q}{cf}\right).$$

Therefore, we see that

$$R_{-m,\varepsilon}=w^m\frac{i^{2k+m}2^{2k-1}\pi^{2k}}{k\Gamma(2k)^2}\binom{2k}{m+1}\sum_{\substack{q(\mathrm{mod}\,cf)\\ q\equiv h(\mathrm{mod}\,f)}}P_{m+1}\left(\frac{q}{cf}\right)P_{2k-m-1}\left(v-\frac{qd}{cf}\right).$$

Now, if we set $q=h+rf$, where r runs modulo c, and replace w by $i^{-1}j(\sigma,z)$, the desired formula for $R(-m,\varepsilon)$ follows.

Let us now adopt the following notations:

$$S_m(\sigma\,|\,u,v)=\sum_{r(\mathrm{mod}\,c)}P_m\left(v-\frac{u+r}{c}d\right)P_{2k-m}\left(\frac{u+r}{c}\right),$$

$$M_k=\frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{k\Gamma(2k)^2},$$

$$R(z)=\frac{A(u)}{\Gamma(2k)}z^{2k-1}-\frac{A(u)}{\Gamma(2k)}\left(z-\frac{a}{c}\right)^{2k-1}+C_k(u,v).$$

A simple calculation shows that

$$R^\sigma(z)=\frac{A(u)}{\Gamma(2k)}j(\sigma,z)^{2k-2}\sigma(z)^{2k-1}+\frac{A(u)}{\Gamma(2k)}j(\sigma,z)^{-1}c^{1-2k}+C_k(u,v)j(\sigma,z)^{2k-2}.$$

Therefore, (4.10) and Lemmas 4.3–4.5 imply that

$$(4.11)\qquad H_k(\sigma z)j(\sigma, z)^{2k-2} = H_k(z) + M_k \sum_{m=0}^{2k-1} j(\sigma, z)^m \binom{2k}{m+1} S_{2k-m-1}(\sigma|u, v) + [R^\sigma(z) - R(z)] - \frac{\Lambda(u)}{\Gamma(2k)}\left(z - \frac{a}{c}\right)^{2k-1}.$$

To bring this formula into final form, we note that for $m = 2k-1$,

$$(4.12)\qquad M_k \binom{2k}{m+1} S_{2k-m-1}(\sigma|u, v) = M_k S_0(\sigma|u, v) = M_k \sum_{r \,(\mathrm{mod}\, c)} P_{2k}\left(\frac{u+r}{c}\right)$$
$$= \frac{(2k)!}{(2\pi i)^{2k}} M_k \sum_{m=-\infty}^{\infty}{}' \frac{1}{m^k} \sum_{r\,(\mathrm{mod}\, c)} e^{2\pi i m(u+r)/c}$$
$$= -\frac{(2k)!}{(2\pi i)^{2k}} M_k c^{1-2k} \Lambda(u) = \frac{\Lambda(u)}{\Gamma(2k)} c^{1-2k}.$$

Thus, combining equations (4.11) and (4.12), we have

THEOREM 4.6. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(f)$, $c > 0$. *Let* $H_k(z) = E_k(z|u, v) - \Lambda(u)$. *Then*

$$H_k(\sigma z) j(\sigma, z)^{2k-2} = H_k(z) + M_k \sum_{m=0}^{2k-1} c^m \binom{2k}{m+1} S_{2k-m-1}(\sigma|u, v) \left(z + \frac{d}{c}\right)^m - M_k S_0(\sigma|u, v) c^{2k-1} \left(z - \frac{a}{c}\right)^{2k-1} + [R^\sigma(z) - R(z)],$$

where

$$S_m(\sigma|u, v) = \sum_{r\,(\mathrm{mod}\, c)} P_m\left(v - \frac{u+r}{c} d\right) P_{2k-m}\left(\frac{u+r}{c}\right),$$

$P_m(x)$ = *the m-th Bernoulli polynomial modulo* 1, *and*

$$M_k = \frac{(-1)^{k-1} 2^{2k-1} \pi^{2k}}{k\Gamma(k)^2},$$

$$R(z) = \frac{\Lambda(u)}{\Gamma(2k)} z^{2k-1} - \frac{\Lambda(u)}{\Gamma(2k)} \left(z - \frac{a}{c}\right)^{2k-1} + \begin{cases} 0 & \text{if} \quad u \notin \mathbf{Z} \text{ or } k = 1 \\ \dfrac{\pi i}{\Gamma(2k)} \displaystyle\sum_{m=-\infty}^{\infty} \frac{e^{2\pi i m v}}{|v|^{2k-1}} & \text{if} \quad u \in \mathbf{Z} \text{ and } k > 1. \end{cases}$$

Remark 4.7. By Theorem 4.6, the period polynomial $S_\sigma(z)$ is given by

$$S_\sigma(z) = M_k \sum_{m=0}^{2k-1} c^{-m} \binom{2k}{m+1} S_{2k-m-1}(\sigma|u, v) \left(z + \frac{d}{c}\right)^m - M_k S_0(\sigma|u, v) c^{2k-1} \left(z - \frac{a}{c}\right)^{2k-1} + [R^\sigma(z) - R(z)]$$
$$= M_k \sum_{m=0}^{2k-2} c^{-m} \binom{2k}{m+1} S_{2k-m-1}(\sigma|u, v) \left(z + \frac{d}{c}\right)^m + M_k S_0(\sigma|u, v) c^{2k-1} \left[\left(z + \frac{d}{c}\right)^{2k-1} - \left(z - \frac{a}{c}\right)^{2k-1}\right] + [R^\sigma(z) - R(z)].$$

It is clear from the second representation that $\deg S_\sigma(z) \leqslant 2k-2$.

Remark 4.8. The term $R^\sigma(z) - R(z)$ is an Eichler co-boundary. And therefore by Lemma 3.6, this term may be ignored in computing the value of $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$. However, this term is potentially very interesting since it contains information about the values of zeta functions at the "wrong parity". For example, if $u = v = 0$, $\mathfrak{f} = 1$, $k > 1$, then

$$R(z) = \frac{2\zeta(2k)}{\Gamma(2k)} \left[z^{2k-1} - \left(z - \frac{a}{c}\right)^{2k-1}\right] + \frac{2\pi i}{\Gamma(2k)} \zeta(2k-1).$$

We shall explore the connection between the co-boundary $R^\sigma(z) - R(z)$ and the values of zeta functions at odd integers in a future paper.

5. Final formula for $\zeta(1-k, \mathfrak{b}, \mathfrak{f})$. Let $H_k^*(z) = H_k(z) - R(z)$. Then H_k^* is an Eichler integral of $E_k(z|u, v)$ with period polynomial

$$S_\sigma^*(z) = S_\sigma(z) - [R^\sigma(z) - R(z)].$$

Set

$$S_\sigma^*(z) = {}^t\binom{z}{1}^n S^*(\sigma) \qquad (n = 2k-2),$$
$$p(z) = Q(z)^{k-1} = {}^t\binom{z}{1}^n P.$$

Further, suppose that

$$S^*(\sigma) = \begin{pmatrix} \alpha_n \\ \alpha_{n-1} \\ \vdots \\ \alpha_0 \end{pmatrix}, \qquad P = \begin{pmatrix} \beta_n \\ \beta_{n-1} \\ \vdots \\ \beta_0 \end{pmatrix}.$$

Then by Lemma 3.6 and Theorem 3.9, we have

$$(5.1)\qquad \zeta(1-k, \mathfrak{b}, \mathfrak{f}) = I_k {}^t P P_n^{-1} S^*(\sigma) = I_k \sum_{r=0}^{n} (-1)^{r+1} \alpha_r \beta_{n-r} \binom{n}{r}^{-1}.$$

Next, we compute α_r, the coefficients of the polynomial $S_\sigma^*(z)$. By Theorem 4.6,

$$S_\sigma^*(z) = M_k \sum_{m=0}^{2k-1} c^m \binom{2k}{m+1} S_{2k-m-1}(\sigma|u,v)\left(z+\frac{d}{c}\right)^m - M_k S_0(\sigma|u,v)c^{2k-1}\left(z-\frac{a}{c}\right)^{2k-1}$$

$$= \sum_{r=0}^{2k-2} z^r \cdot M_k \sum_{m=0}^{2k-1} c^m \left(\frac{d}{c}\right)^{m-r}\binom{m}{r}\binom{2k}{m+1} S_{2k-m-1}(\sigma|u,v) + M_k S_0(\sigma|u,v)c^{2k-1}\sum_{r=0}^{2k-2} z^r\cdot(-1)^r\binom{2k-1}{r}\left(\frac{a}{c}\right)^{2k-1-r}.$$

Therefore, for $0 \leqslant r \leqslant 2k-2$,

$$\alpha_r = M_k \sum_{m=0}^{2k-1} c^m\left(\frac{d}{c}\right)^{m-r}\binom{m}{r}\binom{2k}{m+1} S_{2k-m-1}(\sigma|u,v) + M_k S_0(\sigma|u,v)c^{2k-1}(-1)^r\binom{2k-1}{r}\left(\frac{a}{c}\right)^{2k-1-r}.$$

Therefore, by equation (5.1), we have

$$M_k^{-1}I_k^{-1}\zeta(1-k,\mathfrak{b},\mathfrak{f}) = \sum_{r=0}^{2k-2}\sum_{m=0}^{2k-1}(-1)^{r+1}c^m\left(\frac{d}{c}\right)^{m-r}\binom{m}{r}\binom{2k}{m+1}\binom{2k-2}{r}^{-1}\times \beta_{2k-2-r}S_{2k-m-1}(\sigma|u,v) - \sum_{r=0}^{2k-2}c^{2k-1}\left(\frac{a}{c}\right)^{2k-1-r}\binom{2k-1}{r}\binom{2k-2}{r}^{-1}\beta_{2k-2-r}S_0(\sigma|u,v).$$

Replace r by $2k-2-r$, m by $2k-1-m$ and reverse the order of summation in the first term on the right to obtain

(5.2) $\quad M_k^{-1}I_k^{-1}\zeta(1-k,\mathfrak{b},\mathfrak{f})$

$$= \sum_{m=0}^{2k-1}(-1)^m c^{2k-1-m}S_m(\sigma|u,v)\binom{2k}{m}\cdot\sum_{r=0}^{2k-2}\binom{2k-1-m}{r+1-m}\binom{2k-2}{r}^{-1}\left(\frac{d}{c}\right)^{r-m+1}\beta_r - S_0(\sigma|u,v)c^{2k-1}\sum_{r=0}^{2k-2}\left(\frac{a}{c}\right)^{r+1}\beta_r\binom{2k-1}{r+1}\binom{2k-2}{r}^{-1}$$

$$= 2k(2k-1)\sum_{m=0}^{2k-1}(-1)^m\frac{c^{2k-1-m}}{m!(2k-m)}\times S_m(\sigma|u,v)\sum_{r=m}^{2k-2}\frac{r!}{(r-m+1)!}\left(-\frac{d}{c}\right)^{r-m+1}\beta_r - (2k-1)S_0(\sigma|u,v)c^{2k-1}\sum_{r=0}^{2k-2}\left(\frac{a}{c}\right)^{r+1}\left(\frac{\beta_r}{r+1}\right).$$

Thus, let us set

$$R_k(z) = \int_{a/c}^{z} Q(z)^{k-1}dz = \sum_{r=0}^{2k-2}\frac{\beta_r}{r+1}z^{r+1} - \sum_{r=0}^{2k-2}\frac{\beta_r}{r+1}\left(\frac{a}{c}\right)^{r+1}.$$

Then, from equation (5.2), we readily deduce

THEOREM 5.1. *Let $\mathfrak{b}$ be an integral ideal prime to $\mathfrak{f}$,*

$$\mathfrak{a} = \mathfrak{b}/\mathfrak{d}\mathfrak{f} = \omega_1\boldsymbol{Z}+\omega_2\boldsymbol{Z},\quad Q(z) = (\omega_2 z-\omega_1)(\omega_2' z-\omega_1'),\quad \Gamma = \langle\varepsilon_{\mathfrak{f}}\rangle,$$

$$\begin{pmatrix}\varepsilon_{\mathfrak{f}}\omega_1\\ \varepsilon_{\mathfrak{f}}\omega_2\end{pmatrix} = \sigma\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix},\qquad \sigma = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\boldsymbol{Z}).$$

Furthermore, let $u = \mathrm{Tr}(\omega_1)$, $v = \mathrm{Tr}(\omega_2)$,

$$S_m(\sigma|u,v) = \sum_{r(\mathrm{mod}\,c)} P_m\left(v-\frac{u+r}{c}d\right)P_{2k-m}\left(\frac{u+r}{c}\right),$$

$$R_k(z) = \int_{a/c}^{z}Q(z)^{k-1}dz,$$

where $P_m(x)$ is the m-th Bernoulli polynomial modulo 1. *Then we have*

$$\zeta(1-k,\mathfrak{b},\mathfrak{f}) = (-1)^k N(\mathfrak{b})^{k-1}\sum_{m=0}^{2k-1}(-1)^m\frac{c^{2k-1-m}}{m!(2k-m)}\times R_k^{(m)}\left(-\frac{d}{c}\right)S_m(\sigma|u,v).$$

COLLORARY 5.2. $\zeta(1-k,\mathfrak{b},\mathfrak{f})\in\boldsymbol{Q}$.

Theorem 5.1 was first enunciated by Siegel [12], and has proved to be of use in constructing the p-adic L-functions of real quadratic fields. (See Coates and Sinnot [2].) Siegel derived his formula in a rather *ad hoc* way, using a limiting argument resembling Riemann's "Grenzübergang". In case $k = 1$, it was necessary for Siegel to resort to using Kronecker's limit formulas. Our argument is, of course, quite different and points toward a connection between values of zeta functions and automorphic integrals which is not obvious from Siegel's work. Moreover, the present method admits of a number of extensions, which will be discussed in subsequent papers.

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Über die Entwicklung multiplikativer Funkiionen nach Ramanujan-Summen

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1. Einleitung. Die Ramanujan-Summen $C_q(n)$ sind definiert durch

$$C_q(n) = \sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}} \exp\left(2\pi i\,\frac{a}{q}\,n\right) = \sum_{d|(q,n)} d\mu\left(\frac{q}{d}\right) \qquad (q,n\in \boldsymbol{N}).$$

Sie wurden erstmalig von Ramanujan [9] in die Zahlentheorie eingeführt, der sie zur Entwicklung zahlentheoretischer Funktionen $f\colon \boldsymbol{N}\to\boldsymbol{C}$ in der Gestalt

$$f(n) = \sum_{q=1}^{\infty} a_q C_q(n) \tag{1}$$

verwendete. Ähnliche Entwicklungen finden sich bei Hardy [7]. Die ersten allgemeinen Aussagen über die Ramanujan-Entwicklung (1) zahlentheoretischer Funktionen stammen von Wintner [14] und Delsarte [5].

Es sei $f' = f*\mu$ die Faltung(1) von f mit der Möbiusfunktion μ, und es gelte ($\tau(n) = \sum_{d|n} 1$)

$$\sum_{n=1}^{\infty} \tau(n)\frac{|f'(n)|}{n} < \infty. \tag{2}$$

Dann existieren die Entwicklungskoeffizienten(2)

$$a_q = \frac{1}{\varphi(q)} \lim_{x\to\infty} \frac{1}{x} \sum_{n\leqslant x} f(n) C_q(n), \tag{3}$$

und die zugehörige Reihe (1) konvergiert absolut gegen $f(n)$ für jedes $n\in\boldsymbol{N}$.

(1) Für je zwei zahlentheoretische Funktionen f_1, f_2 wird die Faltung f_1*f_2 durch $(f_1*f_2)(n) = \sum_{d|n} f_1(d) f_2\left(\frac{n}{d}\right)$ $(n\in\boldsymbol{N})$ erklärt.

(2) Hinreichend für die Existenz der Entwicklungskoeffizienten (3) ist die Konvergenz der Reihe $\sum_{n=1}^{\infty}\frac{|f'(n)|}{n}$ (vgl. [5], [14]).