Separable extensions of first countable spaces

by

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Abstract. B. Fitzpatrick, J. W. Ott and G. M. Reed raised the following questions:

QUESTION 1. Can each Moore space with weight at most c be embedded in a separable Moore space?

QUESTION 2. Can each first countable space with weight at most c be embedded in a separable first countable space?

We show that these questions are *independent* of the ZFC axioms for set theory.

We also prove the following theorem, which is well known for completely regular spaces.

THEOREM. *Let $\varkappa$ be an infinite cardinal. Any space with weight $\leqslant 2^{\varkappa}$ which is Hausdorff or regular can be embedded in a space with density $\varkappa$ of the same type.*

1. Introduction. It is well known that a completely regular space can be embedded in a separable completely regular space if and only if its weight is at most c. In 1969 J. W. Ott proved that every metrizable space of weight at most c can be embedded in a separable Moore space, [16] (see [18], Thm. 5 for an easy proof) and raised the following

QUESTION 1. Can each Moore space with weight at most c be embedded in a separable Moore space?

When investigating this question, G. M. Reed, [20] asked

QUESTION 2. Can each first countable Hausdorff space with weight at most c be embedded in a separable first countable Hausdorff space?

The purpose of this paper is to show that these questions cannot be answered in ZFC. It is convenient to introduce the following statements, where $\varkappa$ is an infinite cardinal.

$D(\varkappa)$: Let $\mathscr{B}$ be a family of subsets of any set X, with $|\mathscr{B}|\leqslant\varkappa$. Then there is a function $h\colon \mathscr{B}\to\mathscr{P}(N)$ ($\mathscr{P}(N)$ is the power set of N) such that
1) $h(B)$ is infinite for nonempty $B\in\mathscr{B}$,

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2) if $A, B \in \mathscr{B}$ are disjoint, $h(A) \cap h(B)$ is finite, and
3) if $A \in \mathscr{B}$ and if $\mathscr{F} \subseteq \mathscr{B}$ is finite, and if $A \subseteq \bigcup \mathscr{F}$, then $h(A) - \bigcup \{h(B)\colon B \in \mathscr{F}\}$ is finite.

$T(\varkappa)$: If S is a set of cardinality $\leqslant \varkappa$, and if $\mathscr{A}$ is a family of $\leqslant \varkappa$ subsets of S, then there is a separable metrizable topology μ on S such that each member of $\mathscr{A}$ is an F_σ with respect to μ.

One can show the negation of $T(c)$ to be consistent with ZFC, see Section 2, hence the following example shows that it is consistent with ZFC that the answer to the above questions is in the negative.

1.1. Example [$\neg T(\varkappa)$]. There exists a zero-dimensional Moore space with cardinality and weight $\varkappa$ which cannot be embedded in a separable first countable Hausdorff space.

The construction of this example depends on the following lemma which is of independent interest, see Appendix.

1.2. Lemma. *Let X be a first countable separable (or, more generally, sequentially separable) Hausdorff space. Then there is a separable metrizable topology μ on X such that any two disjoint open subsets of X are contained in disjoint sets which are F_σ with respect to μ.*

On the other hand, it is known that $D(\omega_1)$ is true in ZFC, see Section 2, so CH, the Continuum Hypothesis, implies $D(c)$. Consequently the following theorem shows that it is consistent with ZFC that the answer to the above questions is in the positive.

1.3. Theorem [$D(\varkappa)$]. *Any space with weight $\leqslant \varkappa$ which is first countable, or quasi-developable, or developable and which also is Hausdorff, or regular, or completely regular, or zerodimensional, or locally compact, or compact, can be embedded in a separable space of the same type.*

Note that the theorem gives an absolute result for spaces with weight $\leqslant \omega_1$. Special cases of the theorem were known already: G. M. Reed has shown that any space with weight $\leqslant \omega_1$ which is first countable or developable, and locally compact or compact, can be embedded in a separable space of the same type ([20], Thm. 1.7, 1.8 and 2.1); he did not observe however that his method also works for completely regular spaces, see also Remark 2.6.

We also include a proof of the following theorem which seems to be new.

1.4. Theorem. *Let $\varkappa$ be an infinite cardinal. Any space with weight $\leqslant 2^{\varkappa}$ which is Hausdorff or regular can be embedded in a space with density $\varkappa$ of the same type.*

Earlier G. M. Reed had shown that MA+$\neg$CH implies that every Hausdorff space of cardinality ω_1 can be embedded in a separable Hausdorff space ([20], Thm. 1.10). Since a space of cardinality ω_1 has weight $\leqslant 2^{\omega_1}$, and $2^{\omega_1} = c$ under MA+$\neg$CH, this is an immediate consequence of Theorem 1.4.

Our conventions and definitions are fairly standard; for undefined notions we refer to [10]. We review only the most important facts. Regular spaces, zero-dimensional spaces and locally compact spaces are taken to be Hausdorff. A space is *quasi-developable* (*developable*) if there is a countable family $\underline{\mathscr{G}}$ consisting of open families (open covers) such that for each $x \in X$ the family

$$\{\mathrm{St}(x, \mathscr{G})\colon \mathscr{G} \in \underline{\mathscr{G}}\} - \{\emptyset\}$$

is a neighborhood base for x in X (recall that $\mathrm{St}(x, \mathscr{G}) = \bigcup \{G \in \mathscr{G}\colon x \in G\}$). A *Moore space* is a regular developable space. N is the set (or space) of positive integers. A cardinal is an initial ordinal, an ordinal is the set of smaller ordinals, ω is ω_0 and $c = 2^{\omega}$.

2. $D(\varkappa)$ and $T(\varkappa)$. Bukovský [5] proved that the statement

(*) $2^{\omega} = 2^{\omega_1}$ and there is no uncountable separable metrizable Q-set

is consistent with ZFC. (A *Q-set* is a space every subset of which is an F_σ.) The following fact has been brought to the authors' attention by G. M. Reed.

2.1. Theorem. (*) *implies* $\neg T(c)$.

Proof. $\mathscr{P}(\omega_1)$, the power set of ω_1, has cardinality 2^{ω} if $2^{\omega} = 2^{\omega_1}$, so $T(c)$ would imply the existence of an uncountable separable metrizable Q-set. ∎

The statement $T(\varkappa)$ will be investigated more thoroughly in [19], in particular it will be shown that a result of Rothberger, [22], implies the following

2.2. Theorem. $\omega_2 \leqslant 2^{\omega} < 2^{\omega_1} = \omega_{\omega_2}$ *implies* $\neg T(c)$.

So $T(c)$ is not true in various models of set theory.

We now turn our attention to $D(\varkappa)$. We may assume that the family $\mathscr{B}$ in the statement of $D(\varkappa)$ is closed under finite intersections, under complementation and that $\emptyset \in B$. Then $D(\omega_1)$ is an immediate consequence of [6], 4.12. Hence we have the following

2.3. Theorem. $D(\omega_1)$ *is true in* ZFC.

2.4. Remark. Example 1.1 and Theorem 1.3 show that $D(\varkappa)$ implies $T(\varkappa)$ for every $\varkappa$. Consequently $T(\omega_1)$ is true in ZFC. We do not know if $D(\varkappa)$ and $T(\varkappa)$ are equivalent for all $\varkappa$. See also Remark 2.8.

We now give topological equivalents of $D(\varkappa)$.

2.5. Theorem. *The following conditions are equivalent for every infinite cardinal $\varkappa$:*

(a) $D(\varkappa)$,

(b) *for every compact Hausdorff space Y of weight $\leqslant \varkappa$ there is a Hausdorff compactification bN of N such that $bN - N$ and Y are homeomorphic,*

(c) *every compact Hausdorff space Y of weight $\leqslant \varkappa$ is a continuous image of $\beta N - N$, and*

(d) *every zero-dimensional compact space Y of weight $\leqslant \varkappa$ is a continuous image of $\beta N - N$.*

Proof. (a)→(b). We prove this implication in detail in Section 4, here we only outline the construction of bN. We may assume that $Y \cap N = \emptyset$. Let $\mathscr{B}$ be a base for Y with $|\mathscr{B}| \leqslant \varkappa$ which is closed under finite unions. Let h be as in the statement of $D(\varkappa)$. We may assume that $h(Y) = N$. The family

$$\{B \cup (h(B)-F)\colon B \in \mathscr{B}, F \subseteq N \text{ finite}\} \cup \{\{n\}\colon n \in N\}$$

is a base for a topology on $bN = N \cup Y$. This topology is Hausdorff since $h(A) \cap h(B)$ is finite for disjoint $A, B \in \mathscr{B}$, and bN is compact since Y is compact, and $h(A) - \bigcup \{h(B)\colon B \in \mathscr{F}\}$ is finite whenever $A \in B$, $\mathscr{F} \subseteq \mathscr{B}$ is finite and $A \subseteq \bigcup \mathscr{F}$.

(b)→(c) if bN is as under (b), there is a continuous $f\colon \beta N \to bN$ such that $f(n) = n$ for $n \in N$. Then f maps $\beta N - N$ onto $bN - N$.

(c)→(d) Trivial.

(d)→(a) Let $\mathscr{B}$ be a family of subsets of a set X, with $|\mathscr{B}| \leqslant \varkappa$. We may assume that $\mathscr{B}$ is closed under finite intersections and complementation. We also may assume that $\mathscr{B}$ is point-separating, i.e. for any two distinct $x, y \in X$ there is a $B \in \mathscr{B}$ which contains only one of x and y. (Argument: if not, define an equivalence relation $\sim$ on X by

$$x \sim y \quad \text{iff} \quad x \in B \Leftrightarrow y \in B \quad \text{for every } B \in \mathscr{B}$$

and do the obvious thing.) Hence $\mathscr{B}$ is a base for a zero-dimensional Hausdorff topology on X. Since the members of $\mathscr{B}$ are clopen in X (when equipped with this topology) there is, as is well known, a Hausdorff compactification Y of X with weight $\leqslant \varkappa$ such that

(α) $$\mathrm{Cl}_Y B \cap \mathrm{Cl}_Y(X-B) = \emptyset, \quad \text{for all } B \in \mathscr{B}.$$

Let $f\colon \beta N - N \to Y$ be any continuous surjection. Recall that each clopen subset of $\beta N - N$ has the form $A' = (\mathrm{Cl}_{\beta N} A) - N$ for some $A \subseteq N$, that

(β) $$A' \cap B' = (A \cap B)',$$

(γ) $$A' \cup B' = (A \cup B)',$$

(δ) if $A' \subseteq B'$ then $A - B$ is finite.

Define a map $h\colon \mathscr{B} \to \mathscr{P}(N)$ by choosing for each $B \in \mathscr{B}$ a $h(B) \subseteq N$ such that

$$h(B)' = f^{-1}[\mathrm{Cl}_Y B].$$

One can easily check that h has all properties required. ■

2.6. Remarks. I. I. Parovičenko proved that condition (c) holds in ZFC for $\varkappa = \omega_1$ [17]. His proof contains a small gap, but the result is known to be true. This theorem also follows from our Theorems 2.3 and 2.5.

K. D. Magill observed that if a Hausdorff space S is a continuous image of $\beta X - X$ for some locally compact space X, then S is homeomorphic to $bX - X$ for some Hausdorff compactification bX of X, [15]. (This explains conditions (c) and (d).) S. P. Franklin and M. Rajagopalan combined these results of Parovičenko and Magill to obtain condition (b) of Theorem 2.5 for $\varkappa = \omega_1$ [11].

G. M. Reed used condition (b) of Theorem 2.5 for $\varkappa = \omega_1$ to obtain the results mentioned in the Introduction immediately following Theorem 1.3. That theorem derives conclusions from $D(\varkappa)$. So our method is in some sense equivalent to Reed's.

For our next result we need the following two combinatorial statements, where $\varkappa$ is an infinite cardinal.

$S(\varkappa)$: If $\mathscr{S}$ and $\mathscr{T}$ are families of subsets of some countable set K with $|\mathscr{S}| + |\mathscr{T}| < \varkappa$, and if $S \cap \bigcap \mathscr{F}$ is infinite for every $S \in \mathscr{S}$ and finite $\mathscr{F} \subseteq \mathscr{T}$, then there is an infinite $J \subseteq K$ such that $S \cap J$ is infinite for each $S \in \mathscr{S}$, and $T - J$ is finite for each $T \in \mathscr{T}$.

$P(\varkappa)$: As $S(\varkappa)$ with $\mathscr{S} = \{K\}$.

It is known that MA implies both $P(c)$ and $S(c)$, see e.g. [23]; $S(\varkappa)$ clearly implies $P(\varkappa)$, but the converse also is true, [7].

2.7. Theorem. *$P(\varkappa)$ implies $D(\lambda)$ for $\lambda < \varkappa$. In particular, Martin's Axiom implies $D(\lambda)$, for $\lambda < c$.* ■

We check condition (b) of Theorem 2.5. Let Y be a compact Hausdorff space with weight $\lambda < \varkappa$. Let I be the closed unit interval. We may assume that Y is a subspace of I^λ, we may even assume that Y is nowhere dense (simply make sure that $y(0) = 0$ for all $y \in Y$).

Since $P(\varkappa)$ implies $\varkappa \leqslant c$, I^λ is separable. Since Y is nowhere dense, we therefore can find a countable dense subset K of I^λ which misses Y. Let $\mathscr{B}$ be a base for I^λ with $|\mathscr{B}| = \lambda$ which is closed under finite unions. It is easy to check that the families

$$\mathscr{S} = \{B \cap K\colon B \in \mathscr{B},\ B \cap Y \neq \emptyset\}$$

and

$$\mathscr{T} = \{B \cap K\colon B \in \mathscr{B},\ Y \subseteq B\}$$

satisfy the hypothesis of $S(\varkappa)$. Since $P(\varkappa)$ implies $S(\varkappa)$, there is a $J \subseteq K$ as in the conclusion of $S(\varkappa)$. Let $Z = Y \cup J$.

It remains to show that Z is compact and that J is a dense set of isolated points of Z, for then Z is homeomorphic to a Hausdorff compactification of N.

Each set of the form $B \cap K$, with $B \in \mathscr{B}$ and $B \cap Y \neq \emptyset$, has infinite intersection with J, hence J is dense in Z. Any member of $\mathscr{B}$ that includes Y contains all but finitely many points of J. Since $\mathscr{B}$ is closed under finite unions and Y is compact, a moments reflection shows that this implies that Z is compact and also that the points of J are isolated in Z. ■

2.8. Remark. We do not know if $P(\varkappa)$ implies $D(\varkappa)$, we do not even know if MA implies $D(c)$. On the other hand, it will be shown in [19] that MA implies $T(c)$, cf. Remark 2.4.

3. No if $\neg T(c)$. Recall that a subset S of a space X is *sequentially dense* iff each point of X is the limit of some convergent sequence of points from S; also recall that a space X is *sequentially separable* if it has a countable sequentially dense subset. βN is an easy example of a separable space that is not sequentially separable. The following lemma is the key to the construction of Example 1.1. Other applications will be given in Appendix.

5.1. LEMMA. *Let X be a sequentially separable Hausdorff space. Then there is a separable metrizable topology μ on X such that any two disjoint open sets of X are contained in disjoint sets which are F_σ with respect to μ.*

Proof. Let S be a countable sequentially dense subset of X, let $D(S)$ be S with the discrete topology, and let P be the product $D(S)^\omega$. P is a separable metrizable space (which is homeomorphic to the irrationals). Points of P are sequences in S, hence we can choose for each $x \in X$ a $c(x) \in P$ which converges to x. Let M be $c[X]$. Then $c\colon X \to M$ is a bijection. Hence $\mu = \{c^{-1}[U]\colon U \text{ open in } M\}$ is a separable metrizable topology on X. We show that it has the property required.

For U open in X define $m(U) \subseteq M$ by

$$m(U) = \{p \in M\colon \exists n \forall k > n \ (p_k \in U)\}.$$

Clearly $c[U] \subseteq m(U)$, and if V also is open in X, then $m(U) \cap m(V) = m(U \cap V)$, hence $m(U) \cap m(V) = \emptyset$ if $U \cap V = \emptyset$.

We complete the proof by showing that each $m(U)$ is an F_σ in M. To this end it suffices to prove that if $A \subseteq X$ and $n \in \omega$ are arbitrary, then

$$F = \{x \in M\colon \forall k > n \ (x_k \in A)\}$$

is closed in M. Indeed, let $x \in M - F$. Then $x_k \notin A$ for some $k > n$. But then $\{y \in M\colon y_k = x_k\}$ is open in M and disjoint from F. ■

We need one more, easy, lemma.

3.2. LEMMA. *Let $\{\tau_k\colon k \in \omega\}$ be a family of separable metrizable topologies on a set X. Then there is a separable metrizable topology τ on X with $\bigcup_{k\in\omega} \tau_k \subseteq \tau$.*

Sketch of proof. Consider the diagonal $\{\langle x, x, \ldots\rangle\colon x \in X\}$ in $\prod_{k\in\omega} \langle X, \tau_k\rangle$. ■

3.3. EXAMPLE. $[\neg T(\varkappa)]$. There is a zero-dimensional Moore space with cardinality and weight $\varkappa$ that cannot be embedded in a first countable separable Hausdorff space (in fact, not even in a sequentially separable Hausdorff space).

Construction. If $\varkappa > \mathfrak{c}$ there is nothing to prove: the discrete space with $\varkappa$ points will do, so we assume $\varkappa \leqslant \mathfrak{c}$.

Let T be a set with cardinality $\varkappa$ and let $\{T_\alpha\colon \alpha \in \varkappa\}$ be a family of subsets of T such that for no separable metrizable topology τ on T it is the case that every T_α is an F_σ with respect to τ.

Let $\mathscr{A} = \{A_\alpha\colon \alpha \in \varkappa\}$ be an almost disjoint family of infinite subsets of ω (almost disjoint means that $A_\alpha \cap A_\beta$ is finite whenever $\alpha \neq \beta$). It is well known that such families exist.

Topologize $Y = \varkappa \times 2 \cup T \times \omega$ as follows (we assume $\varkappa \times 2$ and $T \times \omega$ are disjoint). Points of $T \times \omega$ are isolated, and a basic neighborhood of a point $\langle \alpha, i\rangle \in \varkappa \times 2$ has the form

$$U(\alpha, i, n) = \begin{cases} \{\langle \alpha, i\rangle\} \cup T_\alpha \times \{k \in A_\alpha\colon k > n\} & \text{if } i = 0, \\ \{\langle \alpha, i\rangle\} \cup (T - T_\alpha) \times \{k \in A_\alpha\colon k > n\} & \text{if } i = 1. \end{cases}$$

Since $\mathscr{A}$ is almost disjoint, Y is easily seen to be zero-dimensional. It is obvious that Y is developable.

Suppose it were possible to embed Y in a sequentially separable space X. Let μ be as in Lemma 3.1, and for $n \in \omega$ let μ_n be the subspace topology on $T \times \{n\}$. For each $n \in \omega$ the family

$$\tau_n = \{U \subseteq T\colon U \times \{n\} \in \mu_n\}$$

is a separable metrizable topology on T. We will prove

(*) for each $\alpha \in \varkappa$ there is a k such that T_α is an F_σ with respect to τ_k

which is impossible because of Lemma 3.2.

To prove (*) let $\alpha \in \varkappa$ be arbitrary. Since X is Hausdorff the points $\langle \alpha, 0\rangle$ and $\langle \alpha, 1\rangle$ have disjoint neighborhoods, V_0 and V_1, in X. Find $n \in \omega$ such that $U(\alpha, i, n) \subseteq V_i$ for $i = 0, 1$, and pick $k \in A_\alpha$ with $k > n$. Then $T_\alpha \times \{k\}$ and $(T - T_\alpha) \times \{k\}$ are contained in V_0 and V_1, respectively. But V_0 is contained in a set, which is F_σ with respect to μ, that misses V_1. It follows that $T_\alpha \times \{k\}$ is an F_σ with respect to μ_k. In view of the way the τ_k's are defined this proves (*). ■

3.4. Comments. In [9] the same technique is used to construct small spaces all compactifications of which contain a homeomorph of βN. The idea of splitting the index set $\varkappa$ has been inspired by examples of G. M. Reed, [21], and is quite essential: If we do not split $\varkappa$ we get (a space looking like) the subspace $Y_0 = \varkappa \times \{0\} \cup T \times \omega$ of Y. If $\mathscr{A}$ is constructed carefully, Y_0 is submetrizable, and it is known that every submetrizable first countable (developable) Hausdorff space can be embedded in a separable first countable (developable) Hausdorff space, [20], Thm. 1.4. On the other hand, it is not difficult to show, using the same methods as above, that Y_0 cannot be embedded in a separable sequentially separable *regular* space. This answers Reed's question of whether every Moore space that can be embedded in a separable developable Hausdorff space, can be embedded in a separable Moore space, [20], 4.3, in the negative.

4. Yes if $D(\varkappa)$. Let X be a space with weight $\leqslant \varkappa$. Let $\mathscr{B}$ be a base for X with $|\mathscr{B}| \leqslant \varkappa$. We first construct a separable extension Y of X and then show how various properties of X and $\mathscr{B}$ determine which properties Y has.

4.1. Construction of Y. We may assume that $X \cap N = \emptyset$. Let $h\colon \mathscr{B} \to \mathscr{P}(N)$ be as in the statement of $D(\varkappa)$. If $B \in \mathscr{B}$, we write B^* for $B \cup h(B)$, if $\mathscr{A} \subset \mathscr{B}$ we write $\mathscr{A}^*$ for

$$\{B^* - F\colon B \in \mathscr{A}, F \subseteq N \text{ finite}\} \cup \{\{x\}\colon x \in N\}.$$

Note

(a) $B^* \cap N$ is infinite for nonempty $B \in \mathscr{B}$,

(b) if $A \in \mathscr{B}$, $\mathscr{F} \subseteq \mathscr{B}$ finite and $A \subseteq \bigcup \mathscr{F}$, then $A^* - \bigcup \{B^*: B \in \mathscr{F}\}$ is a finite subset of N,

(c) if $A, B \in \mathscr{B}$ are disjoint, then $A^* \cap B^*$ is a finite subset of N.

Clearly $\mathscr{B}^*$ is a base for some topology on $Y = X \cup N$. The subspace topology on X coincides with the original topology, and N is dense in Y (because of (a)), so Y is a separable extension of X. It is useful to observe the following facts

(d) if $B \in \mathscr{B}^*$ then $\mathrm{Cl}_Y B^* = B^* \cup \mathrm{Cl}_X B$, (because of (c)),

(e) if $\mathscr{A} \subseteq \mathscr{B}$ is a base for X, then $\mathscr{A}^*$ is a base for Y, and

(f) all points of N are isolated.

4.2. *Y is Hausdorff if X is.* This follows from (c) and (f).

4.3. *Y is regular if X is.* This follows from (d) and (f).

4.4. *Y is zero-dimensional if X is* (for suitable $\mathscr{B}$). We have to assume that $\{B \in \mathscr{B}: B \text{ closed}\}$ is a base for X, the result then follows from (d), (e) and (f).

4.5. *Y is completely regular if X is* (for suitable $\mathscr{B}$). Let $Q = \{q(n): n \in \omega\}$ be the rationals between 0 and 1. We have to assume that if $x \in B \in \mathscr{B}$ for some $x \in X$, then there is a $\{B_s: s \in Q\} \subseteq \mathscr{B}$ such that

(1) $$x \in B_s \subset B \quad \text{for all } s \in Q,$$

(2) $$\text{if} \quad s < t \quad \text{then} \quad \mathrm{Cl}_X B_s \subseteq B_t.$$

Let $x \in U \in \mathscr{B}^*$ for some $x \in X$. Find $B \in \mathscr{B}$ and finite $F \subseteq N$ with $U = B^* - F$. Choose a $\{B_s: s \in Q\}$ satisfying (1) and (2). With an easy induction on n we can find for each $n \in \omega$ a $U_{q(n)} \subseteq Y$ of the form $(B_{q(n)} - G) \cup H$, with $G, H \subseteq$ finite, such

(3) $$U_s \subseteq U_t \subseteq U \quad \text{for all } s, t \in Q \text{ with } s < t.$$

It follows from (d) that

(4) $$\text{if} \quad s < t \quad \text{then} \quad \mathrm{Cl}_Y U_s \subset U_t.$$

The proof of Urysohn's Lemma now shows that there is a continuous $f: Y \to R$ with $f(x) = 0$, $f[Y - U] = \{1\}$ (cf. [10], Theorem 1.5.10).

4.6. *Y is locally compact if X is.* This easily follows from (b), (d) and (f).

If $X \in \mathscr{B}$ then we may assume that $h(X) = N$, for otherwise define h' and N' by $N' = h(X)$, $h'(B) = N' \cap h(B)$. Therefore the following makes sense.

4.7. *Y is compact if X is* (provided $X \in \mathscr{B}$ and $h(X) = N$). Similar to 4.6.

4.8. *Y is first countable if X is.* This easily follows from (b).

4.9. *Y is quasi-developable if X is* (for suitable $\mathscr{B}$). A space S is quasi-developable if and only if the following is true

(*) There are open families $\mathscr{V}_n$, $n \in \omega$, such that for each $x \in S$ and each neighborhood U of x there is an $n \in \omega$ such that there is exactly one $V \in \mathscr{V}_n$ with $x \in V$, and $V \subseteq U$ for this V.

This statement is implicit in the proof of Bennett and Lutzer's ([3], Prop. 7). Find such $\mathscr{V}_n$'s for X. We have to assume that $\bigcup_n \mathscr{V}_n \subseteq B$. For $n \in \omega$ and $k \in N$ define

$$\mathscr{W}_{nk} = \{V^* - \{1, \ldots, k\}: V \in \mathscr{V}_n\} \cup \{\{k\}\}.$$

For each $x \in Y$ and each neighborhood U of x in Y there are $n \in \omega$ and $k \in N$ such that $x \in W$ for exactly one $W \in \mathscr{W}_{nk}$, and $W \subseteq U$, this follows from (b) and (f). Hence Y is quasi-developable.

4.10. *Y is developable if X is* (for suitable $\mathscr{B}$). This follows from 4.9 since a space is developable iff it is quasi-developable and perfect (= open sets are F_σ's), [2], and clearly Y is perfect if X is, because X is closed in Y and N is countable. (Alternatively, since a space is developable iff it is quasi-developable and perfect, it follows from the result mentioned in 4.7 that a space S is developable iff

(**) There are open *covers* $\mathscr{V}_n$, $n \in \omega$, such that for each $x \in S$ and each neighborhood U of x there is an $n \in \omega$ such that there is exactly one $V \in \mathscr{V}_n$ with $x \in V$, and $V \subseteq U$ for this V.

Now proceed as under 4.7, but define $\mathscr{W}_{nk}$ by

$$\mathscr{W}_{nk} = \{V^* - \{1, \ldots, k\}: W \in \mathscr{V}_n\} \cup \\ \cup \{\{x\}: x \in N - \bigcup \{V^* - \{1, \ldots, k\}: V \in \mathscr{V}_n\}\}.$$

4.11. Remark. In 4.4, 4.5, 4.8 and 4.9 we required $\mathscr{B}$ to *include* certain families (this might be avoidable, but that is of no interest), so we can make sure that any combination of properties considered is transferred from X to Y.

4.12. Remark. A more efficient but less natural way of achieving 4.5 is to compactify X first and then apply 4.7. We leave the details to the reader.

4.13. Remark. Note that given any space X we can construct Y as in 4.1 provided there is for some base $\mathscr{B}$ of X a map $h: \mathscr{B} \to \mathscr{P}(N)$ as in the statement of $D(\varkappa)$.

Positive results concerning separable extensions of first countable spaces which do not require additional set theoretical axioms can be found in [18].

5. Proof of Theorem 1.4. Let X be a space with weight $\leqslant 2^\varkappa$, let $\mathscr{B}$ be a base for X with $|\mathscr{B}| \leqslant 2^k$. Let D be a set, disjoint from X, with $|D| = \varkappa$. It is well known that there is a family $\mathscr{I}$ of subsets of D with $|\mathscr{I}| = 2^\varkappa$ such that

(α) for any two disjoint finite subsets $\mathscr{F}$ and $\mathscr{G}$ of $\mathscr{I}$ the set $\bigcap \mathscr{F} - \bigcup \mathscr{G}$ is infinite (actually: has cardinality $\varkappa$), ([10], Exercise 3.6.F). For $B \in \mathscr{B}$ choose $D_B \in \mathscr{I}$ in such a way that $D_A \neq D_B$ if $A \neq B$. For $A, B \in \mathscr{B}$ define $D(B, A)$ by

$$C(B, A) = \begin{cases} D_A & \text{if} \quad B \subseteq A\,, \\ D - D_A & \text{if} \quad B \subseteq X - A\,, \\ D & \text{otherwise.} \end{cases}$$

For $B \in \mathscr{B}$ let

$$\mathscr{I}_B = \{\bigcap \{D(B, A)\colon A \in \mathscr{F}\}\colon \mathscr{F} \text{ a finite subcollection of } \mathscr{B}\}\,.$$

It follows from (α) that

$$(\beta) \qquad \forall B \in \mathscr{B}\, \forall I \in \mathscr{I}_B \quad (I \text{ is infinite})$$

and it is easy to see that

$$(\gamma) \qquad \forall A, B, B' \in \mathscr{B} \quad (B \subseteq B' \rightarrow D(B, A) \subseteq D(B', A))\,.$$

Clearly (γ) implies that

$$\mathscr{B}^* = \{B \cup (I-F)\colon I \in \mathscr{I}_B, B \in \mathscr{B}, F \subseteq D \text{ finite}\} \cup \{\{x\}\colon x \in D\}$$

is a base for some topology on $Y = X \cup D$. The subspace topology on X coincides with the original topology of X, and D is dense in Y because of (β). The definition of the $D(B, A)$'s shows

$$(\delta) \qquad \forall A, B \in \mathscr{B} \quad (A \cap B = \emptyset \rightarrow D(B, A) \cap D(A, A) = \emptyset)\,.$$

Since the points of D are isolated, it follows that Y is Hausdorff if X is. It follows from (δ) that

$$(\varepsilon) \quad \forall B \in \mathscr{B}\, \forall I \in \mathscr{I}_B\, \forall \text{ finite } F \subseteq D \big(I \subseteq D(B, B) \rightarrow \mathrm{Cl}_Y(B \cup (I-F)) = (\mathrm{Cl}_X B) \cup (I-F)\big)\,.$$

One easily deduces that Y is regular if X is. ■

5.1. Remark. Theorem 1.4 is well known for completely regular spaces, for every completely regular space X with weight $\leqslant 2^{\varkappa}$ can be embedded in $I^{2^{\varkappa}}$, where I is the closed unit interval, and $I^{2^{\varkappa}}$ has a dense subspace with cardinality $\varkappa$.

6. Open questions. The following questions are open in ZFC.

6.1. Can every first countable compact Hausdorff space be embedded in a separable first countable space which is Hausdorff? is regular? is compact Hausdorff? Recall that first countable compact Hausdorff spaces have cardinality $\leqslant \mathfrak{c}$, hence have weight $\leqslant \mathfrak{c}$, [1], so they can be embedded in a separable compact Hausdorff space.

6.2. Can a Moore space with weight $\leqslant \mathfrak{c}$ be embedded in a separable first countable (developable?) Hausdorff (regular?) space provided it is locally compact, or has a point-countable base, or is metacompact?

6.3. Can a first countable Hausdorff or regular space X with weight $\leqslant \mathfrak{c}$ be embedded in a separable first countable Hausdorff (regular? — only if X is of course) space provided X is locally compact, or has a point-countable base, or has a σ-point-finite base, or is metacompact?

7. Appendix. In this section we show how Lemma 3.1 can be used to prove part of the following theorems.

Recall the following definitions. A *Q-set* (*λ-set*) is a space every (countable) subset of which is a G_δ. A space is *pseudonormal* if any two disjoint closed sets, one of which is countable, have disjoint neighborhoods. If $\varkappa$ is a cardinal, a space is *$\varkappa$-compact* if it does not contain a closed discrete subset with $\varkappa$ points.

7.1. Theorem. *The following conditions on a cardinal $\varkappa$ are equivalent*:

(a) *there is a separable normal Moore space which is not $\varkappa$-compact*,

(b) *there is a sequentially separable normal space which is not $\varkappa$-compact, and*

(c) *there is a separable metrizable Q-set of cardinality $\varkappa$.*

Proof. (a)→(b) is trivial.

(b)→(c). Let X be a sequentially separable normal space which contains a closed discrete subset M with cardinality $\varkappa$. Let μ be as in Lemma 3.1. Let $S \subseteq M$ be arbitrary. Since X is normal, there are disjoint open sets U_0 and U_1 in X with $S \subseteq U_0$ and $M - S \subseteq U_1$. There is a subset F of X with $U_0 \subseteq F \subseteq X - U_1$ which is an F_σ with respect to μ. Clearly $M \cap F = S$. Consequently μ induces a separable metrizable topology on M which makes M a Q-set.

(c)→(a). This is due to Bing ([4], Ex. E). ■

7.2. Theorem. *The following conditions on a cardinal $\varkappa$ are equivalent*:

(a) *there is a separable pseudonormal Moore space which is not $\varkappa$-compact*,

(b) *there is a sequentially separable pseudonormal space which is not $\varkappa$-compact, and*

(c) *there is a separable metrizable λ-set of cardinality $\varkappa$.*

Proof. (a)→(b) is trivial.

(b)→(c) the same argument as above.

(c)→(a) F. D. Tall observed that the proof of (c)→(a) of Theorem 7.1 also proves this implication, [25]. ■

7.3. Remarks. The implication (a)→(c) in Theorem 7.1 is due to R. W. Heath, [13]. Our Lemma 3.1 (= 1.2) and its use in 7.1 (and 7.2) generalize and simplify his method.

We do not know if the implication (a)→(c) of Theorem 7.2 has been explicitly stated before. It is a classical result that uncountable separable λ-sets exist, [14]. However, we do not know if a separable metrizable λ-set of cardinality $\mathfrak{c}$ exists in ZFC, although such spaces can easily be constructed, using Theorem 7.2, under set theoretic assumptions much weaker than MA, [8].

Added in proof. Professor K. Kunen showed that the existence of a separable metrizable λ-set of cardinality c is independent of ZFC (cf. [19]).

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