Riemannian manifolds with many Killing vector fields

by

Ann K. Stehney * and **Richard S. Millman** (Carbondale, Ill.)

Abstract. In this paper we consider Riemannian manifolds with many Killing vector fields (infinitesimal isometries), specifically those for which the Killing fields span each tangent space and those parallelized by Killing fields, which we call almost-Killing spaces and Killing spaces respectively. We show in Section 2 that for a complete, connected Riemannian manifold, homogeneity follows from the transitivity of the Killing fields on a single tangent space. This is the homogeneous analogue of a theorem of Cartan for Riemannian symmetric spaces. Restricting attention to Killing spaces in Section 3, we compare the following additional assumptions about Killing fields $X_1, \ldots, X_n$ which are a parallelization for a Riemannian n-manifold: (1) each X_i has constant norm, (2) each inner product $\langle X_i, X_j\rangle$ is constant, (3) the vector fields $X_1, \ldots, X_n$ generate an n-dimensional Lie algebra, and (4) $X_1, \ldots, X_n$ commute. We generalize some results of J. D'Atri and H. Nickerson on parallelizations satisfying (2), which they called Killing frames. Finally we discuss in Section 4 the implications of Section 2 for other geometric structures whose automorphism groups are Lie groups.

In this paper we consider Riemannian manifolds with many Killing vector fields (infinitesimal isometries), specifically those for which the Killing fields span each tangent space and those parallelized by Killing fields, which we call almost-Killing spaces and Killing spaces respectively.

Ambrose and Singer [1], Nomizu [7] and [8], Singer [9], and others have given various criteria for the homogeneity of a Riemannian manifold. Approaching this question in terms of the existence of global Killing fields we show in Section 2 that for a complete, connected Riemannian manifold, homogeneity follows from the transitivity of the Killing fields on a single tangent space. This is the homogeneous analogue of a theorem of Cartan (see [6]) for Riemannian symmetric spaces.

Restricting attention to Killing spaces in Section 3, we compare the following additional assumptions about Killing fields $X_1, \ldots, X_n$ which are a parallelization for a Riemannian n-manifold: (1) each X_i has constant norm, (2) each inner product $\langle X_i, X_j\rangle$ is constant, (3) the vector fields $X_1, \ldots, X_n$ generate an n-dimensional Lie

* First author supported by a Faculty Research Award from Wellesley College. She wishes to thank the Enrico Fermi Institute and Department of Mathematics of the University of Chicago for their hospitality during 1974–75.

algebra, and (4) $X_1, \ldots, X_n$ commute. We generalize some results of J. D'Atri and H. Nickerson [2] on parallelizations satisfying (2), which they called Killing frames.

Finally we discuss in Section 4 the implications of Section 2 for other geometric structures whose automorphism groups are Lie groups.

1. Preliminaries. We let M denote a smooth n-manifold with a fixed Riemannian metric tensor g and associated Levi–Civita connection ∇. The tangent space to M at a point m shall be denoted by M_m, the value at m of a vector field X by $X(m)$, and the Lie derivative with respect to X by L_X.

A vector field X on M is called a *Killing vector field* if the local 1-parameter group of diffeomorphisms generated by X consists of isometries of M. This is equivalent to the condition that its derivative $A_X = L_X - \nabla_X$ be a skew endomorphism on each tangent space, i.e., for all vector fields Y and Z, $g(\nabla_Y X, Z) = -g(\nabla_Z X, Y)$. The Killing vector fields on M form a Lie algebra which will be denoted $\hat{I}(M)$.

If M is connected and complete, there is a correspondence between the Killing vector fields on M and the global isometries of M. In particular, every Killing field on M generates a global 1-parameter group of isometries of M and the isometries of M form a (locally compact) Lie group $I(M)$ in the compact-open topology, whose Lie algebra is naturally isomorphic to $\hat{I}(M)$.

A connected Riemannian manifold M is called a *Riemannian homogeneous space* if $I(M)$ acts transitively on M. It follows that M is isometric to $I(M)/H$ with an $I(M)$-invariant metric, where H is the isotropy subgroup of $I(M)$ at any point.

Remarks. 1. A proper subgroup G of $I(M)$ may act transitively on M, so that M may have several representations as a homogeneous space. If G is not closed in $I(M)$, then the isotropy subgroup of G need not be compact. In any case, there is a G-invariant metric on $G/H \cap G$ for which M is isometric to this quotient.

2. If M is not complete in the Riemannian metric, then $I(M)$ cannot act transitively on M, for any homogeneous space G/H is complete in any G-invariant metric.

3. In questions of the transitivity of $I(M)$, only those isometries which arise from Killing vector fields are relevant. For if a group of diffeomorphisms acts transitively on a connected manifold, then so does its identity component.

2. Transitivity of the Killing vector fields. We consider a manifold with sufficiently many Killing fields to span each tangent space. We first recall a general fact about vector fields which does not involve the Riemannian structure.

PROPOSITION 2.1. *Let $m \in M$ be given. If $X_1, \ldots, X_n$ are smooth vector fields on M such that $\{X_i(m)\}$ span M_m, then each point in some neighborhood of m lies on the integral curve through m of some linear combination $\sum c_i X_i$, $c_i \in \mathbf{R}$.*

In particular, Proposition 2.1 implies that if the Killing fields span the tangent space at m, then each point in some neighborhood of m lies on the integral curve through m of some Killing field. This is the key to the following theorem which provides a local criterion for homogeneity.

THEOREM 2.2. *If M is a complete, connected Riemannian manifold, the following are equivalent:*

(1) *M is Riemannian homogeneous,*

(2) *for each $m \in M$, $\{X(m)\colon X \in \hat{I}(M)\}$ spans M_m, and*

(3) *for some $m \in M$, $\{X(m)\colon X \in \hat{I}(M)\}$ spans M_m.*

Proof. That (1) implies (2) is well known. Since (2) implies (3), it suffices to show that (3) implies (1).

By 2.1, the orbit 0 of m under $I(M)$ contains a neighborhood of m, hence is open. But 0 is also closed since in general the orbit of a point of M under (any closed subgroup of) $I(M)$ is a closed submanifold. ■

In considering Riemannian symmetric spaces, B. Kostant [6] has called condition (3) the transitivity of $\hat{I}(M)$ at m.

From the proof of Theorem 2.2, we obtain

COROLLARY 2.3. *Let M be a complete, connected Riemannian manifold and $X_1, \ldots, X_n$ Killing vector fields on M which span some tangent space M_m. Then the smallest closed subgroup of $I(M)$ whose Lie algebra contains $X_1, \ldots, X_n$ is transitive on M.*

The size alone of the isometry group of M gives little information about whether M is homogeneous. There exist Riemannian homogeneous n-manifolds with $\dim I(M)$ as small as n (the standard torus). Inspection of the classification, in [5] for example, shows that manifolds for which $\dim I(M) > \frac{1}{2}n(n-1)$, i.e., $\dim I(M) = \frac{1}{2}n(n-1)+1$ or $\frac{1}{2}n(n+1)$, are all Riemannian homogeneous. The following construction yields manifolds with $\dim I(M) = \frac{1}{2}n(n-1)$ which are not. Let M_1 be $\mathbf{R}^{n-1}$, S^{n-1}, $\mathbf{RP}^{n-1}$, or H^{n-1} with its usual metric as a space of constant curvature, and M_2 be $\mathbf{R}^1$ or S^1 with a metric with discrete isometry group. Then $M = M_1 \times M_2$, with the product metric, is not Riemannian homogeneous since there are no Killing vector fields in the direction of the second factor. Here $\dim I(M) = \dim I(M_1) = \frac{1}{2}n(n-1)$. Note that M is homogeneous but not Riemannian homogeneous. Finally, the Kervaire sphere of dimension $n = 8k+1$ also has a large isometry group, of dimension $\frac{1}{8}(n^2+7)$, [3], but it is not a homogeneous space.

3. Killing spaces. We define a *Killing space* to be a Riemannian manifold M which admits a parallelization $\mathbf{X} = \{X_1, \ldots, X_n\}$ by Killing vector fields. We shall first consider the following conditions on such a Killing parallelization $\mathbf{X}$ and the implications between them.

(1) *Each $X_i \in \mathbf{X}$ has constant norm on M*, i.e., its integral curves are geodesics.

(2) *For $X_i, X_j \in \mathbf{X}$, the inner product $g(X_i, X_j)$ is constant on M.*

In this case, the Gram–Schmidt process, applied to $\mathbf{X}$ at one point to obtain orthonormal vectors, yields an orthonormal Killing parallelization globally. We shall assume this has been done. J. Wolf has considered this condition in a different context in [11] and [12].

(3) *The Lie algebra generated by* $\{X_1, \ldots, X_n\}$ *has dimension* n.

(4) *The vector fields* $X_1, \ldots, X_n$ *commute.*

Clearly (4) implies (3), and (2) implies (1). We shall see that (3) implies neither (2) nor (1). A local result shows that (4) implies (2):

LEMMA 3.1. *If commuting Killing fields* $X_1, \ldots, X_n$ *are independent on an open set* U, *then* $g(X_i, X_j)$ *is constant on* U.

Proof. More is actually true. If vector fields Y and Z both commute with a Killing field X, then $Xg(Y, Z) = 0$. ■

The following lemma shows that orthonormal Killing spaces (condition 2) are characterized by $\nabla_{X_i} X_j = \frac{1}{2}[X_i, X_j]$, or equivalently $\nabla_{X_i} X_j = -\nabla_{X_j} X_i$. Its proof is a direct computation.

LEMMA 3.2. *If* X *and* Y *are Killing vector fields on* M, *then for any vector field* Z,

$$g(\nabla_X Y, Z) = \tfrac{1}{2}\{g([X, Y], Z) - Zg(X, Y)\}.$$

Therefore if $g(X, Y)$ *is constant,* $\nabla_X Y = \frac{1}{2}[X, Y]$, *so* $\nabla_X Y$ *is a Killing field and* $\nabla_X Y = A_X Y$.

These lemmas may be used to obtain the converse to a well known result about flat manifolds:

PROPOSITION 3.3. *If commuting Killing vector fields* $X_1, \ldots, X_n$ *are independent on an open set* U, *then the metric is flat on* U.

For orthonormal Killing spaces, condition (3) can be expressed in terms of the derivatives A_{X_i} of the Killing fields, as follows.

LEMMA 3.4. *Let* $\{X_1, \ldots, X_n\}$ *be an orthonormal Killing parallelization of* M. *Then* $X_1, \ldots, X_n$ *generate an* n*-dimensional Lie algebra if and only if* $\nabla_{X_k} \nabla_{X_j} X_i = A_{X_k} \circ A_{X_j}(X_i)$ *for all* $1 \leqslant i, j, k \leqslant n$.

Proof. Applying 3.2, $X_m g([X_i, X_j], X_k) = 2g(\nabla_{X_k} \nabla_{X_j} X_i - A_{X_k} \circ A_{X_j}(X_i), X_m)$. ■

Manifolds with local orthonormal Killing fields have been studied by J. D'Atri and H. Nickerson [2]. They show for example that all sectional curvatures are non-negative and that the manifold is locally symmetric.

Counterexamples to other implications between conditions (1)–(4) are provided by Killing spaces which admit no Killing parallelizations of certain types:

1. A Lie group G with left-invariant metric is parallelized by Killing (i.e., right-invariant) vector fields which generate a Lie algebra of dimension n, but which do not commute unless G is abelian and which do not have constant norm unless the metric is also right-invariant. Hence (3) implies neither (1), (2), nor (4).

2. S^7 with its usual metric has global orthonormal Killing fields which arise from its multiplicative structure as the unit Cayley numbers [2] and hence shares many properties of a compact Lie group with a bi-invariant metric. However no Killing parallelization of S^7 generates a 7-dimensional Lie algebra (see Corollary 3.6), hence (2) does not imply (3) in general. As remarked in [2], (2) does imply (3) for $n \leqslant 6$.

It is false locally that (1) implies (2): on S^7 with the orthonormal Killing parallelization of [2], replace X_3 by $[X_1, X_2]$ and consider the set on which the resulting fields are independent. $[X_1, X_2]$ has constant norm but does not meet $X_4, \ldots, X_7$ at constant angles. We do not know whether every complete manifold which admits a geodesic Killing parallelization (1) also admits an orthonormal Killing parallelization (2).

For Killing spaces, we may eliminate the hypothesis that G be closed in Corollary 2.3, as follows.

THEOREM 3.5. *Let* M *be a complete, connected Killing space. If* G *is the connected Lie subgroup of* $I(M)$ *whose Lie algebra is generated by the Killing parallelization, then* G *is transitive on* M.

Proof. Since the Killing fields span each tangent space M_m, the proof of Theorem 2.2 shows that each orbit $G(m)$ is open in M. Orbits either coincide or are disjoint, so each orbit is also closed, hence all of M. ■

If $\dim G = \dim M$, the isotropy subgroup of G at any point is discrete. Hence

COROLLARY 3.6. *If* M *is complete and a Killing parallelization of* M *generates a Lie algebra of dimension* n, *then* M *is naturally isometric to* G/D *with a* G*-invariant metric, where* D *is a discrete subgroup of* G.

4. Applications to other geometric structures. The properties of isometry groups needed to show Riemannian homogeneity are shared by the automorphism groups of certain other geometric structures. Let M be a smooth, connected n-manifold with a geometric structure which we shall denote by $*$, for instance a G-structure in the sense of Chern (see [4]). Let $I(*)$ denote the set of automorphisms of the structure. Then M is called $*$*-homogeneous* if $I(*)$ acts transitively on M, and a vector field on M is called $*$*-Killing* if the local diffeomorphisms it generates lie in $I(*)$. Let $\hat{I}(*)$ denote the set of complete $*$-Killing vector fields on M. In lieu of the completeness assumption in the Riemannian case, we shall require that $I(*)$ be a (finite-dimensional) Lie group whose Lie algebra is isomorphic to $\hat{I}(*)$. Among such structures are Riemannian, pseudo-Riemannian, conformal, almost complex, and almost Hermitian structures.

The proof of 2.2 yields

THEOREM 4.1. *If the orbit of each point of* M *under* $I(*)$ *is a closed submanifold of* M, *then the following are equivalent:*

(1) M *is* $*$*-homogeneous,*

(2) *for each* $m \in M$, $\{X(m)\colon X \in \hat{I}(*)\}$ *spans* M_m, *and*

(3) *for some* $m \in M$, $\{X(m)\colon X \in \hat{I}(*)\}$ *spans* M_m.

COROLLARY 4.2. *Suppose that the orbit of each point of* M *under a closed subgroup of* $I(*)$ *is a closed submanifold. If* $X_1, \ldots, X_n \in \hat{I}(*)$ *span the tangent space at one point, then the smallest closed subgroup of* $I(*)$ *whose Lie algebra contains* $X_1, \ldots, X_n$ *is transitive on* M.

To show that the first hypothesis of 4.2 is necessary, even if the orbit of a point under $I(*)$ is closed, let * be the pseudo-Riemannian metric g on $\boldsymbol{R}^2$ defined by $g(X, X) = -g(Y, Y) = 1$ and $g(X, Y) = 0$ for $X = \partial/\partial x$ and $Y = \partial/\partial y$. The vector fields $X_1 = X+Y$ and $X_2 = xY+yX$ generate a 2-dimensional Lie algebra and are linearly independent on $\boldsymbol{R}^2$ except along the line $x = y$. Now the diffeomorphisms generated by X_1 and X_2 are respectively the translations φ_t: $(x, y) \to (x+t, y+t)$ and the "boosts" about the origin

$$\psi_t: (x, y) \to (x\cosh t + y\sinh t, y\cosh t + x\sinh t),$$

which are isometries. The full isometry group $I(*)$, of dimension 3, is transitive on $\boldsymbol{R}^2$, but the subgroup generated by φ_t and ψ_t is not transitive: its orbits are the line $x = y$ and the two open half-planes on either side of it. This subgroup is closed, since its complement in $I(*)$ consists of those isometries which take some points across the line $x = y$.

The generalizations of 3.5 and 3.6 require no assumption about the topology of the orbits.

THEOREM 4.3. *If $X_1, \ldots, X_n \in \hat{I}(*)$ parallelize M, then the connected subgroup of $I(*)$ generated by $X_1, \ldots, X_n$ is transitive on M.*

COROLLARY 4.4. *If $X_1, \ldots, X_n \in \hat{I}(*)$ parallelize M and generate an n-dimensional subalgebra, then M is the quotient of a Lie group by a discrete subgroup.*

The properties of parallelizations by structure-preserving vector fields depend on the structure. A Killing parallelization of a compact Riemannian n-manifold will in general generate a Lie algebra of dimension $>n$. For complex structures, this cannot happen: it is not difficult to show than any parallelization of a compact complex manifold by holomorphic vector fields generates a Lie algebra of the same (real) dimension as the manifold. From this and Corollary 4.4, we recover the theorem of H. C. Wang [10] that a compact complex manifold is the quotient of a Lie group by a discrete subgroup if and only if it is parallelized by holomorphic vector fields.

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WELLESLEY COLLEGE
SOUTHERN ILLINOIS UNIVERSITY AT CARBONDALE

Accepté par la Rédaction le 4. 4. 1977