A characterization of unicoherence in terms of separating open sets

by

J. H. V. Hunt * (México)

Abstract. We show that in a connected locally connected space some separation properties that are well-known to characterize unicoherence when phrased in terms of closed sets also characterize unicoherence when phrased in terms of open sets.

1. Introduction. In [4] this question was raised.

QUESTION. *Are the following properties equivalent in a connected locally connected space X:*

(i) *X is unicoherent,*

(ii) *if L is a set which separates points p, q, then some component of L separates p, q,*

(iii) *if L is a set which separates X, then some component of L separates X?*

It is well known that the answer to this question is affirmative when L is a closed set. This is partially proved in Theorem 1 of [7], and the part that is missing can easily be supplied using standard arguments on connected locally connected spaces. In this paper we answer the question affirmatively when L is an open set([1]). This result was announced in [3]. It is similar to Stone's theorem on "open-unicoherence" in [7] in that properties that are well-known to characterize unicoherence in terms of closed sets are also shown to characterize unicoherence in terms of open sets.

In § 2 we prove the theorem, placing parts of the argument in two lemmas. In Lemma 1 a construction is given which is valid in any connected locally connected space. The main part of the argument appears in the *principal lemma*, which is of independent interest, as it is used in [4] as well. The proof of the theorem itself is then reduced to standard arguments on connected locally connected spaces. We conclude the paper in § 3 with some remarks on the question mentioned above.

---

* The author is supported by the Multinational Program of Mathematics of the O. A. S.

([1]) If the space is in addition normal, then this result follows from the previous special case for closed sets. However, the hypotheses of the question do not allow any separation axioms on the space.

The author would like to thank Dr. A. García-Máynez and the referee for suggesting several improvements to the proof, and J. Alarcón for pointing out that the original proof of Lemma 1 was incorrect.

2. The proof. Capital letters and small letters refer to subsets and points, respectively, of a topological space X. We take $\cup$, $\cap$ to have precedence over $-$; e.g., $L-M\cup N$ means $L-(M\cup N)$.

The sets P, Q are separated if $P\cap\bar{Q}=\emptyset=\bar{P}\cap Q$. In this case we shall sometimes say that P is *separated from* Q. A set L *separates points* p, q if there are separated sets P, Q containing p, q respectively, such that $X-L=P\cup Q$. A set L *separates* X if its complement is not connected.

The case of the following construction in which U is an open set and $A=\{p,q\}$ is a two-point set will be used in the proof of the principal lemma.

LEMMA 1. *Let U be a proper subset of a connected locally connected space X, and let $\{U_\lambda\}_\lambda$ be the collection of all the components of U. Let A be a non-empty subset of $X-U$ and, for each λ, let V_λ be the union of U_λ and all the components of $X-U_\lambda$ which do not meet A. Then for any pair of sets V_λ, V_μ at least one of the following hree relations holds*:

$$V_\lambda\supset V_\mu,\quad V_\mu\supset V_\lambda,\quad V_\lambda\cap V_\mu=\emptyset.$$

Proof. Consider two distinct components U_λ, U_μ of U. Let S be the union of all the components of $X-U_\lambda\cup U_\mu$ which are separated from U_μ, and let T be the union of all the components of $X-U_\lambda\cup U_\mu$ which are separated from U_λ. Notice that it is a consequence of the connectedness and local connectedness of X that no component of $X-U_\lambda\cup U_\mu$ is separated from both U_λ and U_μ, and this implies that S, T are disjoint and that $X-U_\lambda\cup S$, $X-U_\mu\cup T$ are connected. Now we consider three possibilities.

Firstly suppose that $X-V_\lambda\subset S$. In this case $X-V_\lambda\subset X-U_\mu\cup T$, as $S\cap T=\emptyset$ implies that $S\subset X-U_\mu\cup T$. Since $X-U_\mu\cup T$ is a connected subset of $X-U_\mu$ which contains A (because $A\subset X-V_\lambda$), it is contained in a component of $X-V_\mu$. Consequently $X-V_\lambda\subset X-V_\mu$, and so $V_\lambda\supset V_\mu$.

Secondly, the supposition that $X-V_\mu\subset T$ leads by similar reasoning to the conclusion that $V_\mu\supset V_\lambda$.

Thirdly suppose that $X-V_\lambda\not\subset S$, $X-V_\mu\not\subset T$. Since $X-V_\lambda\not\subset S$, $X-U_\lambda\cup S$ is a connected subset of $X-U_\lambda$ which meets $X-V_\lambda$, and so it is contained in some component of $X-U_\lambda$ lying in $X-V_\lambda$; i.e., $X-U_\lambda\cup S\subset X-V_\lambda$. Consequently $U_\lambda\cup S\supset V_\lambda$. Similarly, $X-V_\mu\not\subset T$ implies that $U_\mu\cup T\supset V_\mu$. Since $U_\lambda\cup S$, $U_\mu\cup T$ are disjoint, $V_\lambda\cap V_\mu=\emptyset$. This proves the lemma.

A connected space X is *unicoherent* if for each pair of connected closed sets M, N such that $X=M\cup N$, $M\cap N$ is connected.

A set L in a connected space X is *simple* if it is connected and does not separate X. This terminology is taken from [7]. Observe that in a connected space every complementary component of a connected set is simple (e.g., see p. 140 of [6]).

The *frontier* of a set L is denoted by $\mathrm{Fr}L$, i.e., $\mathrm{Fr}L=\bar{L}\cap\overline{X-L}$. The main part of the argument is in the following lemma.

PRINCIPAL LEMMA. *Let U be an open set which separates two points p, q in a connected locally connected unicoherent space X, and let P, Q be disjoint closed sets containing p, q, respectively, such that $X-U=P\cup Q$. Let $\{U_\lambda\}_\lambda$ be the collection of all the components of U and, for each λ, let P_λ, Q_λ be the components of $X-U_\lambda$ that contain p, q, respectively. Then, for some λ, either*

$$\mathrm{Fr}P_\lambda\subset P,\quad \mathrm{Fr}\,Q_\lambda\subset Q$$

or

$$\mathrm{Fr}P_\lambda\subset Q,\quad \mathrm{Fr}\,Q_\lambda\subset P.$$

Proof. We adopt the above notation, and suppose that the lemma is false. To see what this means, notice that P_λ, Q_λ are simple sets (see the observation above). Thus, by the unicoherence of X and Theorem 1 (ii) of [7], $\mathrm{Fr}P_\lambda$, $\mathrm{Fr}\,Q_\lambda$ are non-empty connected subsets of $\mathrm{Fr}\,U_\lambda$. However, since the components of U are open, $\mathrm{Fr}\,U_\lambda$ is a subset of $P\cup Q$. It follows that each one of $\mathrm{Fr}P_\lambda$, $\mathrm{Fr}\,Q_\lambda$ is a subset of either P or Q. Thus the negation of the lemma means that, for each λ, either $\mathrm{Fr}P_\lambda\cup\mathrm{Fr}\,Q_\lambda\subset P$ or $\mathrm{Fr}P_\lambda\cup\mathrm{Fr}Q_\lambda\subset Q$.

Let $V_\lambda=X-P_\lambda\cup Q_\lambda$, for each λ. Then V_λ is a non-empty open set whose complement contains p, q, and so $\mathrm{Fr}\,V_\lambda$ is a non-empty subset of $\mathrm{Fr}P_\lambda\cup\mathrm{Fr}\,Q_\lambda$. Thus the negation of the lemma means that, for each λ, either $\mathrm{Fr}\,V_\lambda\subset P$ or $\mathrm{Fr}\,V_\lambda\subset Q$.

Now let $W_\lambda=\bigcup\{V_\mu|\ V_\mu\supset V_\lambda\}$, for each λ. Then W_λ is a non-empty open set whose complement contains p, q, and so $\mathrm{Fr}\,W_\lambda\neq\emptyset$. We claim that, for each λ, either $\mathrm{Fr}\,W_\lambda\subset P$ or $\mathrm{Fr}\,W_\lambda\subset Q$.

For suppose that $\mathrm{Fr}\,W_\lambda\not\subset P$. Select a point $x\in\mathrm{Fr}\,W_\lambda-P$, and let N be a connected open set containing x which does not meet P. Then $N\cap V_\nu\neq\emptyset$, for some $V_\nu\supset V_\lambda$. Thus, whenever $V_\mu\supset V_\nu$, $N\cap V_\mu\neq\emptyset$ and $x\in N-V_\mu$; consequently $N\cap\mathrm{Fr}\,V_\mu\neq\emptyset$. This implies by the second last paragraph that $\mathrm{Fr}\,V_\mu\subset Q$, for each $V_\mu\supset V_\nu$. However $W_\lambda=\bigcup\{V_\mu|\ V_\mu\supset V_\nu\}$, because $\{V_\mu|\ V_\mu\supset V_\lambda\}$ is totally ordered by inclusion, by Lemma 1. Thus $\mathrm{Fr}\,W_\lambda\subset\overline{\bigcup\{\mathrm{Fr}\,V_\mu|\ V_\mu\supset V_\nu\}}$, by Theorem 1, p. 236 of [6]. It follows that $\mathrm{Fr}\,W_\lambda\subset Q$, which proves the claim([2]).

We terminate the proof by showing that X is not connected. For this purpose let $P'=P-\bigcup_\lambda W_\lambda$, $Q'=Q-\bigcup_\lambda W_\lambda$. Then P', Q' are disjoint closed sets containing p, q, respectively. We have remarked that each family $\{V_\mu|\ V_\mu\supset V_\lambda\}$ is totally ordered by inclusion, from which it follows that $\{W_\lambda\}_\lambda$ is a collection of disjoint sets. Thus each $\mathrm{Fr}\,W_\lambda$, besides being a subset of either P or Q, is also a subset of either P' or Q'. Now let P'' be the union of P' and all the sets W_λ whose frontiers lie in P', and let Q''

([2]) In Example 10, p. 835 of [1] the *topological limit* of a partially ordered collection of subsets of a topological space is defined. In these terms $\mathrm{Fr}\,W_\lambda$ is the topological limit of the chain $\{\mathrm{Fr}\,V_\mu|\ V_\mu\supset V_\lambda\}$, the order being induced by the inclusion relation on $\{V_\mu|\ V_\mu\supset V_\lambda\}$. We have not made use of this definition for reasons of economy.

be the union of Q' and all the sets W_λ whose frontiers lie in Q'. Then P'', Q'' are disjoint non-empty sets whose union is X. Furthermore, since P', Q' are closed, it follows from Theorem 1, p. 236 of [6] that $\mathrm{Fr}P''\subset P'$, $\mathrm{Fr}\,Q''\subset Q'$. Thus P'', Q'' are closed. This contradiction to the connectedness of X completes the proof.

Now we prove the result claimed in § 1.

THEOREM. *In a connected locally connected space X the following properties are equivalent*:

(i) *X is unicoherent,*

(ii) *if U is an open set which separates points p, q, then some component of U separates p, q,*

(iii) *if U is an open set which separates X, then some component of U separates X.*

Proof. In order to prove that (i) implies (ii), let X be a connected locally connected unicoherent space, and let U be an open set which separates two given points p, q. Let $X-U = P\cup Q$, where P, Q are disjoint closed sets containing p, q, respectively. By the principal lemma, there is a component U_λ of U such that the components P_λ, Q_λ of $X-U_\lambda$ containing p, q, respectively, satisfy either $\mathrm{Fr}P_\lambda\subset P$, $\mathrm{Fr}\,Q_\lambda\subset Q$ or $\mathrm{Fr}P_\lambda\subset Q$, $\mathrm{Fr}\,Q_\lambda\subset P$.

We show that U_λ separates p, q. Notice that the frontier of each component of $X-U_\lambda$ is a non-empty subset of either P or Q (the reasoning is the same as in the first paragraph of the proof of the principal lemma, where the special cases of $\mathrm{Fr}P_\lambda$, $\mathrm{Fr}\,Q_\lambda$ were considered). Thus let P' be the union of all the components of $X-U_\lambda$ whose frontiers are contained in P, and let Q' be the union of all the components of $X-U_\lambda$ whose frontiers are contained in Q. Then $X-U_\lambda = P'\cup Q'$, and P', Q' are disjoint sets one of which contains P_λ and the other of which contains Q_λ; i.e., one of which contains p and the other of which contains q. Also, by Theorem 1, p. 236 of [6], $\mathrm{Fr}P'\subset P$, $\mathrm{Fr}\,Q'\subset Q$, and so $\mathrm{Fr}P'$, $\mathrm{Fr}\,Q'$ are disjoint. Since P', Q' are disjoint, it now follows that their closures are disjoint. This implies that P', Q' are closed, because the complement of their union is open. Consequently U_λ separates p, q.

Since (ii) implies (iii) trivially, we turn to proving that (iii) implies (i). Thus suppose that (iii) holds, but that X is a connected locally connected space which is not unicoherent. Then there are connected closed sets M, N such that $X = M\cup N$ and $M\cap N = A\cup B$, where A, B are disjoint non-empty closed sets. We assert that there is a component C of $X-N$ such that $A\cap \mathrm{Fr}\,C\neq\emptyset\neq B\cap\mathrm{Fr}\,C$. For suppose that this is not the case. Then it follows from the connectedness and local connectedness of X that the frontier of each component of $X-N$ is a non-empty subset of either A or B. Thus let A' be the union of A and all the components of $X-N$ whose frontiers are contained in A, and let B' be the union of B and all the components of $X-N$ whose frontiers are contained in B. Then A', B' are disjoint non-empty sets whose union is M. Further, since A, B are closed, it follows from Theorem 1, p. 236 of [6] that $\mathrm{Fr}\,A'\subset A$, $\mathrm{Fr}\,B'\subset B$. Thus A', B' are closed. However this contradicts the connectedness of M and so proves our assertion. So now select a component C of $X-N$ whose frontier meets both A and B, and put $U = C\cup(X-\overline{C})$. Then it follows from the connectedness and local connectedness of X that no component of U separates X. Yet $X-U = \mathrm{Fr}\,C$, which is not connected. This contradicts (iii) and completes the proof.

We remark that, just as it is an easy matter to prove that (iii) implies (i) in the above theorem, so it is also easy to prove that (i) implies (iii). That is, the direct proof that (i), (iii) are equivalent is straightforward.

3. Remarks on the question. We return to the question mentioned in § 1. As we remarked there, it is well-known that the answer to this question is affirmative when L is a closed set. From this one deduces an affirmative answer to the question if the space X is in addition completely normal. Thus it is really being asked whether complete normality can be dropped from this last result.

The question was originally raised in [4] because it follows from the Phragmen–Brouwer theorem of that paper (which was announced as Theorem 1 in [3]) that a connected locally connected unicoherent space X has properties (ii), (iii) for any set L with a finite number of components. Dr. A. García-Máynez points out that such a space also has properties (ii), (iii) for any set L whose complement has a finite number of components (property (ii) is (b) of Theorem (3.3) of [2] and property (iii) follows from Lemma (3.2) of [2] and the Phragmen–Brouwer theorem of [4]).

An affirmative answer to the question would not only subsume the Phragmen–Brouwer theorem of [4]; it would also enable us to prove the main theorem of [5] without any separation axioms on the space at all.

References

[1] E. Čech, *Topological Spaces*, revised ed. by Z. Frolík and M. Katětov, Publishing House of the Czechoslovak Academy of Sciences, Prague, and Interscience Publishers, John Wiley & Sons, London–New York, 1966.

[2] A. García-Máynez, *Some results on unicoherent spaces*, Anales del Instituto de Matemáticas de la Universidad Autónoma Nacional de México 16 (1976), pp. 117–133.

[3] J. H. V. Hunt, *The Phragmen-Brouwer theorem for separated sets*, Abstracts of Communications, International Congress of Mathematicians, Vancouver, 1974, p. 193.

[4] — *The Phragmen-Brouwer theorem for separated sets*, Bol. Soc. Mat. Mex. 19 (1974), pp. 26–35.

[5] — and E. D. Tymchatyn, *A theorem on involutions on unicoherent spaces*, to appear.

[6] K. Kuratowski, *Topology*, Vol. II, Polish Scientific Publishers and Academic Press, New York–London–Warszawa 1968.

[7] A. H. Stone, *Incidence relations in unicoherent spaces*, Trans. Amer. Math. Soc. 65 (1949), pp. 427–447.

CENTRO DE INVESTIGACIÓN Y DE ESTUDIOS AVANZADOS DEL IPN
México

Accepté par la Rédaction le 10. 10. 1977