Superextensions of metrizable continua are Hilbert cubes

by

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Abstract. We prove that the superextension λX of X is a Hilbert cube if and only if X is a nondegenerate metrizable continuum, thus proving a conjecture of de Groot.

0. Introduction. The aim of this paper is to prove the generalized de Groot conjecture (stated in Verbeek [27]) that the superextension λX of X is a Hilbert cube if and only if X is a nondegenerate metrizable continuum. In our previous paper [14] we proved that the superextension of the closed unit interval λI is a Hilbert cube. With a similar technique we show here that if X is a finite topological sum of closed unit segments that the maximal connected superextension μX of X is a Hilbert cube. Then, by an approximation technique, this result is used to prove that superextensions of finite connected graphs are Hilbert cubes. By a similar approximation technique, using deep results of Curtis and Schori [9], [10], it then follows that λX is a Hilbert cube for every nondegenerate connected polyhedron. By a result of van Mill and Van de Vel [20] this fact suffices to prove the generalized de Groot conjecture.

Throughout this paper joint results of Marcel Van de Vel and the author concerning subbase convexity theory are used extensively. We would like to thank Marcel Van de Vel for his stimulating enthusiasm and for many helpful comments.

1. The spaces μX. Let $\mathcal{S}$ be a subbase for the closed subsets of a topological space. A subsystem $\mathcal{M} \subset \mathcal{S}$ is called a *linked system* provided that every two of its members meet. A *maximal linked system*, or, mls, is a linked system $\mathcal{M} \subset \mathcal{S}$ not properly contained in another linked system $\mathcal{N} \subset \mathcal{S}$. Let $\lambda(X, \mathcal{S})$ denote the collection of mls's in $\mathcal{S}$. For each $A \subset X$ define $A^+ \subset \lambda(X, \mathcal{S})$ by

$$A^+ := \{\mathcal{M} \in \lambda(X, \mathcal{S}) \mid \exists M \in \mathcal{M}: \mathcal{M} \subset A\}.$$

We topologize $\lambda(X, \mathcal{S})$ by taking the collection $\mathcal{S}^+ = \{S^+ \mid S \in \mathcal{S}\}$ is a closed subbase. With this subbase $\lambda(X, \mathcal{S})$ is called the *superextension of X with respect to $\mathcal{S}$*. The subbase $\mathcal{S}^+$ has the property that every linked subcollection of it has

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nonempty intersection. Such a subbase is called *binary*. $\lambda(X, \mathcal{S})$ is Hausdorff whenever $\mathcal{S}$ is *normal*, i.e. disjoint subbase sets are separated by disjoint complements of subbase sets. It is easily seen that $\mathcal{S}$ is normal iff $\mathcal{S}^+$ is normal. A subbase $\mathcal{S}$ for X is called a *T_1-subbase* if for each $x \in X$ and $S \in \mathcal{S}$ with $x \notin S$ there is an $S_0 \in \mathcal{S}$ such that $x \in S_0$ and $S_0 \cap S = \emptyset$. If $\mathcal{S}$ is a T_1-subbase then the mapping $i\colon X \to \lambda(X, \mathcal{S})$ defined by $i(x) := \{S \in \mathcal{S} \mid x \in S\}$ is an embedding. We will always identify X and $i[X]$. If $\mathcal{S}$ consists of all the closed subsets of X then $\lambda(X, \mathcal{S})$ is denoted by λX and is called the *superextension of* X. In this paper we will prove that λX is homeomorphic to the Hilbert cube if and only if X is a nondegenerate metrizable continuum, thus proving a conjecture of De Groot. For more information concerning superextensions, see Verbeek [27] and van Mill [16].

For the remainder of this section let X be a finite topological sum of nondegenerate continua. Let 2^X be the hyperspace of X. (Recall that for any topological space X the hyperspace 2^X is the space with underlying set the set of all nonvoid closed subsets of X topologized by taking the collection

$$\{\langle F\rangle \mid F \in 2^X\} \cup \{\langle F, X\rangle \mid F \in 2^X\}$$

as a closed subbase, where, for all $A_1, \ldots, A_n \subset X$ the set $\langle A_1, \ldots, A_n\rangle \subset 2^X$ is defined by

$$\langle A_1, \ldots, A_n\rangle = \{F \in 2^X \mid F \subset \bigcup_{i \leqslant n} A_i \text{ and } F \cap A_i \neq \emptyset \text{ for all } i \leqslant n\}.$$

It is well known that 2^X is compact iff X is compact (cf. Michael [13]). This fact will be used without explicit reference in the remaining part of this paper. For many strong results concerning hyperspaces, see [7], [8], [9], [10], [22], [23], [24] and [28]). Define

$$\mathcal{T}(X) := \{A \in 2^X \mid A \text{ is not open}\} \cup \{\emptyset, X\}.$$

1.1. Lemma. *$\mathcal{T}(X)$ is a normal closed T_1-subbase for X.*

Proof. That $\mathcal{T}(X)$ is a closed subbase is trivial. Also, since X has no isolated points, $\mathcal{T}(X)$ is a T_1-subbase. Hence it suffices to prove that $\mathcal{T}(X)$ is normal. To this end, take $T_0, T_1 \in \mathcal{T}(X)$ such that $T_0 \cap T_1 = \emptyset$. It is easy to construct a surjective Urysohn mapping $f\colon X \to I$ such that $f[T_i] = i$ $(i \in \{0, 1\})$. Then, since f is surjective, $\{f^{-1}[0, \frac{1}{2}], f^{-1}[\frac{1}{2}, 1]\} \subset \mathcal{T}(X)$; consequently $\{f^{-1}[0, \frac{1}{2}], f^{-1}[\frac{1}{2}, 1]\}$ is the desired covering of X by elements of $\mathcal{T}(X)$. ■

For simplicity of notation write $\mu X := \lambda(X, \mathcal{T}(X))$. Notice that $\mu X = \lambda X$ in case X is connected.

We need the following result (cf. van Mill [16], Theorem 2.5.1).

1.2. Theorem. *Let X be a topological space and let $\mathcal{S}$ be a normal closed T_1-subbase for X. Then the following properties are equivalent:*

(i) *$\lambda(X, \mathcal{S})$ is connected;*

(ii) *$\lambda(X, \mathcal{S})$ is connected and locally connected;*

(iii) *for all nonvoid $S_0, S_1 \in \mathcal{S}$: $(S_0 \cap S_1 = \emptyset \Rightarrow S_0 \cup S_1 \neq X)$.* ■

1.3. Corollary. *Let X be a finite sum of nondegenerate metrizable continua. Then μX is an* AR.

Proof. $\mathcal{T}(X)$ clearly satisfies condition (iii) of Theorem 1.2. Consequently $\lambda(X, \mathcal{T}(X))$ is connected. In addition $\lambda(X, \mathcal{T}(X))$ is metrizable by a result of Verbeek [27]. Hence $\lambda(X, \mathcal{T}(X))$ is a connected metrizable space with a binary normal subbase (cf. Lemma 1.1). However such a space is an AR by a result of van Mill [14]. ■

We think of μX as the *maximal* connected superextension of X. It can be shown that for every connected superextension $\lambda(X, \mathcal{S})$ of X with respect to a normal T_1-closed subbase $\mathcal{S}$ there is a continuous surjection $f\colon \mu X \to \lambda(X, \mathcal{S})$ which extends the identity on X and which in addition respects the canonical convexity structures of μX and $\lambda(X, \mathcal{S})$ (cf. van Mill and Van de Vel [19]).

In Section 6 we will show that μX is a Hilbert cube if X is a finite sum of nondegenerate metrizable continua.

2. Convexity preserving mappings. Let X be a space which possesses a binary normal closed subbase $\mathcal{S}$. A nonempty closed subset $C \subset X$ is called *$\mathcal{S}$-closed* provided there is a subfamily $\mathcal{C} \subset \mathcal{S}$ such that $C = \bigcap \mathcal{C}$. Let $H(X, \mathcal{S})$ be the set of all $\mathcal{S}$-closed subsets of X. We topologize $H(X, \mathcal{S})$ by regarding it to be a subspace of the hyperspace 2^X of X. For each $A \subset X$ define $I_{\mathcal{S}}(A) \in H(X, \mathcal{S})$ by

$$I_{\mathcal{S}}(A) := \bigcap \{S \in \mathcal{S} \mid A \subset S\}.$$

The set $I_{\mathcal{S}}(A)$ is called the *$\mathcal{S}$-closure* of A and $I_{\mathcal{S}}$ is called the *convex closure operator*. If A is a two point set, say $A = \{x, y\}$, then we usually write $I_{\mathcal{S}}(x, y)$ instead of $I_{\mathcal{S}}(\{x, y\})$. Let $L(X, \mathcal{S})$ denote the subspace of the hyperspace $2^{H(X, \mathcal{S})}$, consisting of all nonempty, closed (in $H(X, \mathcal{S})$), and linked systems $\mathcal{L} \subset H(X, \mathcal{S})$. We need the following results (cf. van Mill and Van de Vel [17]).

2.1. Theorem. *Let X be a topological space which possesses a binary normal closed subbase $\mathcal{S}$. Then*

(i) *The convex closure operator $I_{\mathcal{S}}\colon 2^X \to H(X, \mathcal{S})$ is continuous;*

(ii) *the mapping $p\colon X \times H(X, \mathcal{S}) \to X$ defined by*

$$\{p(x, A)\} = \bigcap_{a \in A} I_{\mathcal{S}}(x, a) \cap A$$

is continuous (p is called the nearest-point mapping of X);

(iii) *a closed set $A \subset X$ is $\mathcal{S}$-closed iff $\forall x, y \in A$: $I_{\mathcal{S}}(x, y) \subset A$;*

(iv) *the intersection operator $\bigcap\colon L(X, \mathcal{S}) \to H(X, \mathcal{S})$ which sends each $\mathcal{L} \in L(X, \mathcal{S})$ onto $\bigcap \mathcal{L}$, is continuous.* ■

Notice that for each $A \in H(X, \mathcal{S})$ the restriction of p to $X \times \{A\}$ is a retraction of X onto A. This is a very useful result and already had a variety of applications (cf. Van de Vel [26], van Mill and Van de Vel [18], van Mill [15], Szymański [25]).

The convex closure operator of μX (resp. λX) with respect to its canonical binary normal subbase $\mathcal{T}(X)^+ = \{T^+ | \ T \in \mathcal{T}(X)\}$ (resp. $(2^X)^+ = \{F^+ | \ F \in 2^X\}$) will simply be denoted by I.

Let X and Y be spaces, and let $\mathcal{S}$ and $\mathcal{T}$ be binary normal subbases for, respectively, X and Y. A function $f\colon X \to Y$ is called a *convexity preserving map* (briefly: a cp *map*) relative to $\mathcal{S}$ and $\mathcal{T}$ if for each $T \in H(Y, \mathcal{T})$ it is true that $f^{-1}(T) \in H(X, \mathcal{S}) \cup \{\emptyset\}$ (cf. van Mill and Van de Vel [19]). In this case we shall write

$$f\colon (X, \mathcal{S}) \to (Y, \mathcal{T}).$$

Notice that a cp mapping is automatically continuous.

If X is a space, then it is easily seen that each $\mathcal{M} \in \lambda X$ regarded as subspace of 2^X is closed in 2^X. This suggests a mapping from λX into 2^{2^X} which sends each $\mathcal{M} \in \lambda X$ onto $\mathcal{M} \in 2^{2^X}$. This mapping is obviously one to one and it is quite surprising that for compact Hausdorff X it is also continuous (cf. van Mill and Van de Vel [20]). Hence for compact Hausdorff X we may regard λX to be a subspace of 2^{2^X}. Often it is useful to do so (cf. Section 6 and the following theorem).

2.2. THEOREM. *Let X and Y be compact Hausdorff spaces and let $F\colon X \to 2^Y$ be continuous. Fix $y \in Y$. Then the function $f\colon \lambda X \to \lambda Y$ defined by*

$$f(\mathcal{M}) := p\Big(y, \bigcap_{M \in \mathcal{M}} \big(\bigcup F(M)\big)^+\Big)$$

is a cp *mapping.*

Proof. Let $k(\lambda Y)$ be the space $H(\lambda Y, (2^Y)^+)$. Define a mapping $F^*\colon 2^X \to 2^Y$ by $F^*(A) := \bigcup F[A]$. It is easily seen that this mapping is continuous. This mapping extends to a continuous mapping $H\colon 2^{2^X} \to 2^{2^Y}$ defined by $H(\mathcal{A}) := F^*[\mathcal{A}]$. Take $\mathcal{M} \in \lambda X \subset 2^{2^X}$. We claim that $H(\mathcal{M})$ is a linked system. Indeed, since $H(\mathcal{M}) = \{\bigcup F[M] | \ M \in \mathcal{M}\}$, this is a simple consequence of the linkedness of $\mathcal{M}$. Define a mapping $\varphi\colon 2^Y \to k(\lambda Y)$ by $\varphi(A) := A^+$. It is easily seen that this mapping is an embedding. This mapping extends to a mapping $\varphi^*\colon 2^{2^Y} \to 2^{k(\lambda Y)}$ defined in the obvious way. We now have the following mappings

$$\lambda X \xrightarrow{i} 2^{2^X} \xrightarrow{H} 2^{2^Y} \xrightarrow{\varphi^*} 2^{k(\lambda Y)}$$

where i is the inclusion. Let $\psi := \varphi^* \circ H \circ i$. Then $\psi[\lambda X] \subset L(\lambda Y, (2^Y)^+)$ as was pointed out above (notice that $A \cap B \neq \emptyset$ implies that $\varphi(A) \cap \varphi(B) \neq \emptyset$). Regard ψ to be a mapping from λX to $L(\lambda Y, (2^Y)^+)$. Then consider the composition

$$\lambda X \xrightarrow{\psi} L(\lambda Y, (2^Y)^+) \xrightarrow{\cap} k(\lambda Y) \xrightarrow{p(y, -)} \lambda Y.$$

Since the intersection operator is continuous, cf. Theorem 2.1(iv), we conclude that f being the composition of ψ, $\cap$ and $p(y, -)$, is continuous (cf. Theorem 2.1(ii)).

Let us prove now that f is cp. Indeed, take $T \in 2^Y$ and assume that there are $\mathcal{M}, \mathcal{N} \in f^{-1}(T^+)$ and $\mathcal{P} \in I(\mathcal{M}, \mathcal{N})$ such that $\mathcal{P} \in I(\mathcal{M}, \mathcal{N}) - f^{-1}(T^+)$. We will derive a contradiction; by the binarity of the subbase $(2^Y)^+$ there are two cases:

Case 1. There is $P \in \mathcal{P}$ such that $\big(\bigcup F(P)\big)^+ \cap T^+ = \emptyset$.

Suppose that $P \notin \mathcal{M}$ and that $P \notin \mathcal{N}$. Take $M \in \mathcal{M}$ and $N \in \mathcal{N}$ such that $P \cap M = \emptyset = N \cap P$. Then $P \cap (M \cup N) = \emptyset$ and since $\mathcal{P} \in I(M \cup N) \subset (M \cup N)^+$ this is a contradiction. Hence we may assume that, without loss of generality, $P \in \mathcal{M}$. Then $f(\mathcal{M}) \in \big(\bigcup F(P)\big)^+ \cap T^+$, which is impossible.

Case 2. There is an $S \in 2^Y$ such that

(i) $y \in S$ and $S \cap \bigcup F(P) \neq \emptyset$ for all $P \in \mathcal{P}$;

(ii) $S \cap T = \emptyset$.

Since $f(\mathcal{M}) \in T^+$ and since $S^+ \cap T^+ = \emptyset$ we can consider two cases:

(a) $\exists M \in \mathcal{M}\colon \bigcup F(M) \cap S = \emptyset$;

(b) $\exists S_0 \in 2^Y$ such that $S_0 \cap \bigcup F(M) \neq \emptyset$ for all $M \in \mathcal{M}$ and $y \in S_0$ and moreover $S_0 \cap S = \emptyset$.

However case (b) cannot occur since $y \in S_0 \cap S_1$. Hence we may assume that there is an $M \in \mathcal{M}$ such that $\bigcup F(M) \cap S = \emptyset$. In the same way there is an $N \in \mathcal{N}$ such that $\bigcup F(N) \cap S = \emptyset$. Then $F(M) \cup F(N) \subset \langle \bigcup F(M) \cup \bigcup F(N) \rangle$ and consequently $M \cup N \subset F^{-1}\langle \bigcup F(M) \cup \bigcup F(N) \rangle$. Since $I(\mathcal{M}, \mathcal{N}) \subset (M \cup N)^+$ we conclude that $M \cup N \in \mathcal{P}$. Hence

$$f(\mathcal{P}) \in \big(\bigcup \langle \bigcup F(M) \cup \bigcup F(N) \rangle\big)^+ = \big(\bigcup F(M) \cup \bigcup F(N)\big)^+$$

and since $S \in f(\mathcal{P})$ this contradicts the linkedness of $f(\mathcal{P})$.

By the continuity of f the set $f^{-1}(T^+)$ is closed in λX. By the characterization of Theorem 2.1(iii) we conclude that $I(f^{-1}(T^+)) = f^{-1}(T^+)$ for all $T \in 2^Y$. This obviously implies that f is a cp mapping. ■

The proof of the above theorem is unexpectedly difficult. There are other theorems concerning cp mappings which have a more straightforward proof. For later use let us mention one (cf. van Mill and Van de Vel [19]).

2.3. THEOREM. *Let $\mathcal{S}$ and $\mathcal{T}$ be normal T_1-subbases for the spaces X and Y, respectively, and let $f\colon X \to Y$ be a mapping such that $f^{-1}(T) \in \mathcal{S}$ for each $T \in \mathcal{T}$. Then there is a canonical* cp *mapping $\lambda(f)\colon \lambda(X, \mathcal{S}) \to \lambda(Y, \mathcal{T})$ extending f. Moreover $\lambda(f)$ is the unique* cp *mapping which extends f.* ■

That in the above circumstances there is a *continuous mapping* $\bar{f}\colon \lambda(X, \mathcal{S}) \to \lambda(Y, \mathcal{T})$ was first proved by G. A. Jensen (see Verbeek [27]). The mapping $\bar{f}$ can be described as follows

$$\{\bar{f}(\mathcal{M})\} = \bigcap \{T^+ | \ T \in \mathcal{T} \text{ and } f^{-1}(T) \in \mathcal{M}\}.$$

The mapping $\lambda(f)$ of Theorem 2.3 is simply Jensen's $\bar{f}$. Hence the only new fact is that $\bar{f}$ is a cp mapping and that it is unique as cp mapping. This is of interest since simple examples show that there may exist more than one *continuous* extension. The mappings $\lambda(f)$ will be called *Jensen mappings.*

In [14] a normal T_1-subbase $\mathcal{S}$ for a topological space X was called *supernormal* provided that for all $S \in \mathcal{S}$ and $A \in 2^X$ with $S \cap A = \emptyset$ there is an $S_0 \in \mathcal{S}$ such that

$A \subset S_0$ and $S \cap S_0 = \emptyset$. If $\mathcal{T}$ is a normal T_1-subbase for X and if $\mathcal{S}$ is a supernormal subbase for X such that $\mathcal{S} \subset \mathcal{T}$ then the Jensen extension

$$\lambda(\mathrm{id})\colon \lambda(X, \mathcal{T}) \to \lambda(X, \mathcal{S}),$$

which extends the identity on X, can be described very easily. In [14] it was shown that for each $\mathcal{M} \in \lambda(X, \mathcal{T})$ the set $\mathcal{M} \cap \mathcal{S}$ is an mls in $\mathcal{S}$ and hence determines a point of $\lambda(X, \mathcal{S})$. It is straightforward to prove that $\lambda(\mathrm{id})(\mathcal{M}) = \mathcal{M} \cap \mathcal{S}$ for all $\mathcal{M} \in \lambda(X, \mathcal{T})$. This observation will be used several times in the sequel. In addition, notice that $\lambda(\mathrm{id})$ is onto.

Also notice that if X is a sum of finitely many nondegenerate continua the subbase $\mathcal{T}(X)$ for X defined in Section 1 is supernormal.

Finally we mention another useful fact concerning cp mappings (cf. van Mill and Van de Vel [19]).

2.4. THEOREM. *Let $f\colon (X, \mathcal{S}) \to (Y, \mathcal{T})$ be a* cp *surjection. Then for all $A \in H(X, \mathcal{S})$ we have that $f[A] \in H(Y, \mathcal{T})$.* ■

The proof of this theorem heavily relies on the fact that $\mathcal{S}$ and $\mathcal{T}$ both are binary *normal* subbases.

2.5. COROLLARY. *Let $f\colon (X, \mathcal{S}) \to (Y, \mathcal{T})$ be a* cp *surjection. Take $x \in X$ and $A \in H(X, \mathcal{S})$. Then $f(p(x, A)) = p(f(x), f[A])$.*

Proof. Since $p(x, A) \in A$ we have that $f(p(x, A)) \in f[A]$. Suppose now that $f(p(x, A)) \neq p(f(x), f[A])$. Then $I_{\mathcal{T}}(f(x), p(f(x), f[A]))$ does not contain $f(p(x, A))$. Therefore $f^{-1}I_{\mathcal{T}}(f(x), p(f(x), f[A]))$ does not contain $p(x, A)$. This is a contradiction, however, since $x \in f^{-1}I_{\mathcal{T}}(f(x), p(f(x), f[A])) \in H(X, \mathcal{S})$ and clearly $f^{-1}I_{\mathcal{T}}(f(x), p(f(x), f[A])) \cap A \neq \emptyset$. ■

2.6. COROLLARY. *Let $f\colon (X, \mathcal{S}) \to (Y, \mathcal{T})$ be a* cp *surjection. Then for all $A \subset X$ we have that $f[I_{\mathcal{S}}(A)] = I_{\mathcal{T}}(f[A])$.*

Proof. Since $f[A] \subset f[I_{\mathcal{S}}(A)]$, by Theorem 2.4 we conclude that

$$I_{\mathcal{T}}(f[A]) \subset f[I_{\mathcal{S}}(A)].$$

On the other hand, $A \subset f^{-1}I_{\mathcal{T}}(f[A])$ and consequently, since f is a cp mapping, $f[I_{\mathcal{S}}(A)] \subset I_{\mathcal{T}}(f[A])$. This proves equality. ■

3. Some remarks concerning λX and μX. This section contains some remarks which are useful in the sequel.

In [27] Verbeek proved that if (X, d) is a compact metric space then there is a canonical metric $\bar{d}$ for λX such that $i\colon (X, d) \hookrightarrow (\lambda X, \bar{d})$ is an isometry (here i is the canonical embedding of X in λX sending $x \in X$ onto the mls $\{A \in 2^X \mid x \in A\}$). This is a very useful result. Before describing Verbeek's [27] metric explicitly, let us first give some definitions.

If (X, d) is a metric space then for all $A \subset X$ and $\varepsilon \geqslant 0$ define

$$B_\varepsilon(A) := \{x \in X \mid d(x, A) \leqslant \varepsilon\},$$
$$U_\varepsilon(A) := \{x \in X \mid d(x, A) < \varepsilon\}.$$

For any $A, B \in 2^X$ the *Hausdorff distance* $d_H(A, B)$ is defined by

$$d_H(A, B) := \inf\{\varepsilon \geqslant 0 \mid A \subset U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A)\}.$$

In case X is compact, d_H is a metric for 2^X.

Now assume that (X, d) is compact. Verbeek [27] has given the following expressions for $\bar{d}$;

(1) $\bar{d}(\mathcal{M}, \mathcal{N}) = \sup_{M \in \mathcal{M}} \min_{N \in \mathcal{N}} d_H(M, N)$

(2) $= \min\{\varepsilon \geqslant 0 \mid \forall M \in \mathcal{M}\colon B_\varepsilon(M) \in \mathcal{N} \text{ and } \forall N \in \mathcal{N}\colon B_\varepsilon(N) \in \mathcal{M}\}$

(3) $= \min\{\varepsilon \geqslant 0 \mid \forall M \in \mathcal{M}\colon B_\varepsilon(M) \in \mathcal{N}\}$

(4) $= \min\{\varepsilon \geqslant 0 \mid \forall N \in \mathcal{N}\colon B_\varepsilon(N) \in \mathcal{M}\}$.

In practical calculations the expressions (3) and (4) are the most useful.

Another very useful result is the following (cf. Verbeek [27]): λX has finitely many components if and only if X has finitely many components. This is proved in the following way: let $C_1, \ldots, C_n$ be the collection of components of a space X. Let

$$\pi\colon X \to \{1, 2, \ldots, n\}$$

be the decomposition. Let

$$\lambda(\pi)\colon \lambda X \to \lambda\{1, 2, \ldots, n\}$$

be the Jensen extension of f (cf. Section 2) defined by

$$\lambda(\pi)(\mathcal{M}) = \{A \subset \{1, 2, \ldots, n\} \mid \pi^{-1}(A) \in \mathcal{M}\}.$$

Verbeek [27] proved that the components of λX coincide with the collection

$$\{\lambda(\pi)^{-1}(\mathcal{M}) \mid \mathcal{M} \in \lambda\{1, 2, \ldots, n\}\}.$$

Since the space $\lambda\{1, 2, \ldots, n\}$ is finite so is the number of components of λX.

This has an important corollary. Let $f = \lambda(\mathrm{id})\colon \lambda X \to \mu X$ be the Jensen mapping (cf. Section 2). Then the function $\varphi\colon \lambda X \to \mu X \times \lambda\{1, 2, \ldots, n\}$ defined by $\varphi(\mathcal{M}) := \langle f(\mathcal{M}), \lambda(\pi)(\mathcal{M})\rangle$ is an embedding. Indeed, it is obvious that φ is continuous and hence it suffices to prove that φ is one to one. Take distinct $\mathcal{M}, \mathcal{N} \in \lambda X$. There are $M \in \mathcal{M}$ and $N \in \mathcal{N}$ such that $M \cap N = \emptyset$. If $M \in \mathcal{T}(X)$ then there is an element of $\mathcal{T}(X)$ disjoint from M and containing N. Hence we may assume that also $N \in \mathcal{T}(X)$. Then $f[N^+] \cap f[M^+] = \emptyset$ since $\mathcal{T}(X)$ is supernormal and consequently $f(\mathcal{M}) \neq f(\mathcal{N})$. If $M \notin \mathcal{T}(X)$ then M is clopen and so N can be taken to be the complement of M. Then clearly $\lambda(\pi)(\mathcal{M}) \neq \lambda(\pi)(\mathcal{N})$. Hence φ is one to one.

We finish this section with a simple but useful result.

3.1. LEMMA. *Let X be a sum of finitely many nondegenerate metrizable continua. Let $F = \{x_1, \ldots, x_n\}$ $(i < j \leqslant n \Rightarrow x_i \neq x_j)$ consisting of at least 3 points. Take a point*

$\mathcal{M} \in \mu X$ such that $\mathcal{M} \in \left(\bigcap_{j=1}^{n} (F-\{x_j\})^+\right) - F$. Then for every $i \leqslant n$ there is an arc $J_i \subset F^+$ connecting $\mathcal{M}$ and x_i while in addition $i<j\leqslant n$ implies that $J_i \cap J_j = \{\mathcal{M}\}$.

Proof. By Corollary 1.3 μX is an AR and hence a Peano continuum. Let d be a convex metric for μX (i.e. a metric for which $B_{\delta_0}(B_{\delta_1}(A)) = B_{\delta_0+\delta_1}(A)$ for every $A \in 2^{\mu X}$; there is a convex metric on every Peano continuum (cf. Bing [0]). It is easy to show that the function $F\colon 2^{\mu X} \times [0, \infty) \to 2^{\mu X}$ defined by $F(\langle A, t\rangle) := B_t(A)$ is continuous (F is sometimes called an expansion homotopy, cf. Curtis and Schori [8]). Without loss of generality we may assume that $d(\mu X \times \mu X) = [0, 1]$. It now is easy to show that the collection

$$\{B_t(\mathcal{M})|\ t \in [0, 1]\} \subset 2^{\mu X}$$

regarded to be a subspace of $2^{\mu X}$ is homeomorphic to $[0, 1]$. This implies that the collection

$$\{I(B_t(\mathcal{M}))|\ t \in [0, 1]\} \subset k(\mu X)$$

($k(\mu X) = \{A \subset \mu X|\ A \neq \emptyset$ and $I(A) = A\}$) is also homeomorphic to $[0, 1]$ since I preserves inclusions (recall that I is continuous, see Theorem 2.1(i)). Let

$$p\colon\ \mu X \times k(\mu X) \to \mu X$$

be the nearest point mapping of μX (cf. Theorem 2.1(ii)). For each $i \leqslant n$ define $H_i \subset \mu X$ by

$$H_i := \{p(x_i, I(B_t(\mathcal{M})))|\ t \in [0, 1]\}.$$

It is clear that H_i is a Peano continuum being a continuous image of $[0, 1]$. In addition it contains both x_i and $\mathcal{M}$ since

$$p(x_i, I(B_0(\mathcal{M}))) = p(x_i, \{\mathcal{M}\}) = \mathcal{M}$$

and

$$p(x_i, I(B_1(\mathcal{M}))) = p(x_i, I(\mu X)) = p(x_i, \mu X) = x_i.$$

In addition, since $\{x_i, \mathcal{M}\} \subset F^+$ we see that $I(x_i, \mathcal{M}) \subset F^+$ and consequently $H_i \subset I(x_i, \mathcal{M}) \subset F^+$. Now take $i<j\leqslant n$. We claim that $H_i \cap H_j = \{\mathcal{M}\}$. To the contrary, assume there exist $s, t \in [0, 1]$ such that

$$p(x_i, I(B_s(\mathcal{M}))) = p(x_j, I(B_t(\mathcal{M}))).$$

Since $H_i \subset I(x_i, \mathcal{M})$ and $x_j \notin I(x_i, \mathcal{M})$ (this is easily seen) we conclude that $x_j \neq p(x_j, I(B_t(\mathcal{M})))$ and consequently $x_j \notin I(B_t(\mathcal{M}))$. In the same way we find that $x_i \notin I(B_s(\mathcal{M}))$. It is clear that not both s and t are 0. Assume that $s = 0$. Then $t \neq 0$ and

$$\mathcal{M} = p(x_j, I(B_t(\mathcal{M}))).$$

This is disproved by the following

FACT. *Suppose that $x \notin C \in k(\mu X)$. Then $p(x, C) \notin \operatorname{int}(C)$.*

Indeed, let $q := p(x, C)$. By the definition of p it is easily seen that

$$I(x, q) \cap C = \{q\}$$

(in fact this property characterizes q). Since $I(x, q)$ is a retract of the connected space μX (cf. Theorem 2.1(ii)) it is connected too. So if $q \in \operatorname{int}(C)$ then $I(x, q) \cap (C-\{q\}) \neq \emptyset$ which is impossible.

We conclude that $\{s, t\} \subset (0, 1)$. Since $\mathcal{M} \in (F-\{x_i\})^+$ and since $(F-\{x_i\})^+$ is nowhere dense in μX (this is easily seen since $F-\{x_i\}$ is nowhere dense) there is a point $\mathcal{L} \in B_s(\mathcal{M}) - (F-\{x_i\})^+$. Then x_i and $\mathcal{L}$ are both not elements of $(F-\{x_i\})^+$; consequently

$$I(x_i, \mathcal{L}) \cap (F-\{x_i\})^+ = \emptyset.$$

Since $\mathcal{L} \in B_s(\mathcal{M}) \subset IB_s(\mathcal{M})$ we see that

$$p(x_i, I(B_s(\mathcal{M}))) \in I(x_i, \mathcal{L}).$$

But $\{x_j, \mathcal{M}\} \subset (F-\{x_i\})^+$ which implies that

$$p(x_j, I(B_t(\mathcal{M}))) \subset I(x_j, \mathcal{M}) \subset (F-\{x_i\})^+.$$

We conclude that $p(x_i, I(B_s(\mathcal{M}))) \neq p(x_j, I(B_t(\mathcal{M})))$; contradiction. Now for each $i \leqslant n$ let $J_i \subset H_i$ be an arc connecting x_i and $\mathcal{M}_i$. Then $\{J_1, \ldots, J_n\}$ is as desired. ■

Remark. It can be shown that the spaces H_i $(1 \leqslant i \leqslant n)$ defined in the proof of the previous lemma are itself all homeomorphic to $[0, 1]$. Hence there is in fact no need for taking subspaces $J_i \subset H_i$.

4. Superextensions of finite sums of [0, 1].

In [14] we showed that the superextension of the closed unit segment $I = [0, 1]$ is homeomorphic to the Hilbert cube Q by showing that it is the inverse limit of a sequence of Hilbert cubes with cellular bonding maps (a continuous surjection $f\colon Q \to Q$ is said to be *cellular* if each point inverse has trivial shape, that is, each point inverse is contractible in each neighborhood of itself; for shape theory we refer to Borsuk [1]). With the same construction we prove here that μX is a Hilbert cube provided that X is a topological sum of finitely many copies of I.

Let $X := [0, 1] \cup [2, 3] \cup \ldots \cup [m-1, m]$. Define $E := \{-2\cdot 3^k|\ k = 0, 1, 2, \ldots\}$ and for each $n \in E$ let X be embedded in $I \times [0, m]$, preserving arc-length, as indicated in Figure 1.

All angles are $\frac{1}{2}\pi$ except the one at $\langle \frac{1}{2}, 0\rangle$ which is $\frac{1}{4}\pi$. Let

$$\mathcal{S} = \{A \subset [0, 1] \times [0, m]|\ (A = \pi_0^{-1}[0, x] \vee A = \pi_0^{-1}[x, 1] \text{ for some } x \in [0, 1] \vee (A = \pi_0^{-1}[0, x] \vee A = \pi_0^{-1}[x, m] \text{ for some } x \in [0, m])\}$$

be the canonical binary normal subbase for $[0, 1]\times[0, m]$. With the same technique as in [14] it follows that $\lambda(X, \mathscr{A}_n)$, where $\mathscr{A}_n = \{S\cap X|\ S\in\mathscr{S}\}$ (here X refers to the embedded copy of X in $[0, 1]\times[0, m]$), is canonically homeomorphic to the convex-hull of X in $[0, 1]\times[0, m]$. Also, using the same technique as in [14], it follows that $\lambda(X, \bigcup_{n\in E}\mathscr{A}_n)$ can be embedded in $\prod_{n\in E}\lambda(X, \mathscr{A}_n)$ as an infinite dimensional compact linearly convex set. Hence $\lambda(X, \bigcup_{n\in E}\mathscr{A}_n)$ is a Hilbert cube, by Keller's [12] theorem.

Now let $\mathscr{T}$ be a countable closed basis for X which is closed under finite intersections and finite unions. The compactness of X now implies that $\mathscr{T}$ separates the

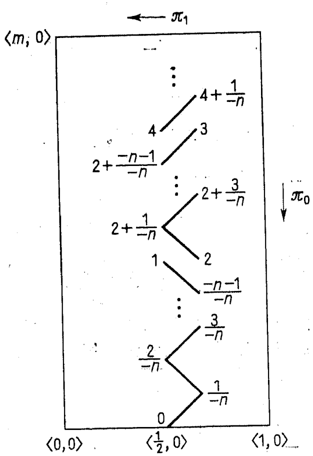

Fig. 1

closed subsets of X, i.e. for all disjoint $A_0, A_1\in 2^X$ there exist disjoint $T_0, T_1\in\mathscr{T}$ such that $A_i\subset T_i$. Define

$$\mathscr{T}' := \mathscr{T}\cap\mathscr{T}(X).$$

It is straightforward to show that $\mathscr{T}'$ separates $\mathscr{T}(X)$, which, among others, shows that $\mathscr{T}'$ is a closed subbase. Define

$$\mathscr{F} := \{(T_0, T_1)|\ T_0, T_1\in\mathscr{T}' \text{ and } T_0\cap T_1 = \emptyset\}$$

and let $\{(T_0^i, T_1^i)|\ i\in N-\{1\}\}$ be an enumeration of $\mathscr{F}$. For each $i\in N-\{1\}$ it is geometrically clear that there is an embedding of X in $[0, 1]\times[0, m]$ of the type sketched in Figure 2 such that there is a $t\in[0, 1]$ with the property that either $(\pi_0(T_0^i)\subset[0, t))$ and $(\pi_0(T_1^i)\subset(t, 1])$ or $(\pi_0(T_1^i)\subset[0, t))$ and $(\pi_0(T_0^i)\subset(t, 1])$.

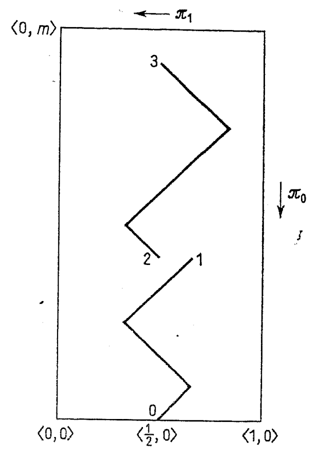

Fig. 2

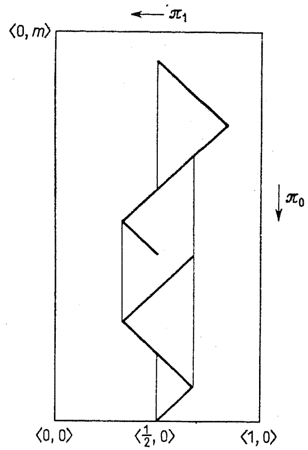

Fig. 3

For sake of simplicity we have only indicated such an embedding for the case $m = 3$. Define $\mathscr{S}_n := \{S\cap X|\ S\in\mathscr{S}\}$ (here X refers again to the embedded copy of X in $[0, 1]\times[0, m]$). As in [14] it is easy to show that $\lambda(X, \mathscr{S}_n)$ is canonically homeomorphic to the space indicated in Figure 3.

Notice that $\lambda(X, \mathcal{S}_n)$ is connected.

Define $\mathcal{S}_1 := \bigcup_{n\in E} \mathcal{A}_n$, where the $\mathcal{A}_n$'s are as defined above. Using precisely the same technique as in [14] Lemma 8, it can be shown that for each $n \in N$ the space $\lambda(X, \bigcup_{i=1}^{n} \mathcal{S}_i)$ is a *Q-manifold*, i.e. a separable metric space which admits an open covering by sets homeomorphic to open subsets of the Hilbert cube Q. In addition, $\lambda(X, \bigcup_{i=1}^{n} \mathcal{S}_n)$ is connected by Theorem 1.2, and has a binary normal subbase since it is easily seen that $\bigcup_{i=1}^{n} \mathcal{S}_i$ is normal. Therefore $\lambda(X, \bigcup_{i=1}^{n} \mathcal{S}_i)$ is an AR by a theorem of van Mill [14]. This implies that $\lambda(X, \bigcup_{i=1}^{n} \mathcal{S}_i)$ is a Hilbert cube since each compact contractible Q-manifold is a Hilbert cube (cf. Chapman [3]).

It now is straightforward to prove that the inverse limit of the inverse sequence

$$\lambda(X, \mathcal{S}_1) \xrightarrow{g_1} \lambda(X, \mathcal{S}_2) \xrightarrow{g_2} \lambda(X, \mathcal{S}_3) \leftarrow \ldots,$$

with bonding maps g_i: $\lambda(X, \mathcal{S}_{i+1}) \to \lambda(X, \mathcal{S}_i)$ defined by

$$g_i(\mathcal{M}) := \mathcal{M} \cap \mathcal{S}_i$$

(cf. Section 2) is homemorphic to μX. (Define for each $i \in N$ a mapping ξ_i: $\mu X \to \lambda(X, \mathcal{S}_i)$ by $\xi_i(\mathcal{M}) := \mathcal{M} \cap \mathcal{S}_i$. It is easily seen that $\xi_i = g_i \circ \xi_{i+1}$ for all $i \in N$. Hence the mapping e: $\mu X \to \varprojlim(\lambda(X, \mathcal{S}_i), g_i)$ defined by $e(\mathcal{M})_i = \xi_i(\mathcal{M})$ is continuous and onto. In addition e is easily seen to be one to one, showing that e is a homeomorphism).

In addition the mappings g_i are cp mappings by Theorem 2.3. This implies that each point inverse of g_i is an AR, being a retract of $\lambda(X, \mathcal{S}_{i+1})$ (cf. Theorem 2.1(ii)). Consequently, g_i is cellular (even in a strong way). It is a deep result of Chapman [4], [5] (cf. also Chapman [6]) that each cellular mapping between Hilbert cubes is a uniform limit of homeomorphisms (a so called *near-homeomorphism*). Hence we find that μX is (homeomorphic to) the inverse limit of a sequence of Hilbert cubes with near-homeomorphisms as bonding maps. Applying results of Brown [2] (cf. also Mioduszewski [21]) we find that μX is a Hilbert cube. This completes the proof of Theorem 4.1.

4.1. THEOREM. *Let X be a finite sum of closed unit segments. Then μX is a Hilbert cube.* ■

5. Superextensions of finite graphs are Hilbert cubes. In this section we prove that if X is a sum of finitely many nondegenerate finite connected graphs the space μX is a Hilbert cube. This result is used in Section 6 to obtain the main result in this paper.

Let us recall the following theorem (cf. van Mill [16], Theorem 2.2.5).

5.1. THEOREM. *Let $\mathcal{S}$ be a binary subbase for the topological space X. Let Y be a subspace of X such that for all $S_0, S_1 \in \mathcal{S}$ with $S_0 \cap S_1 \neq \emptyset$ we have that also $S_0 \cap S_1 \cap Y \neq \emptyset$. Then X is homeomorphic to $\lambda(Y, \mathcal{S}\restriction Y)$, where $\mathcal{S}\restriction Y = \{S \cap Y \mid S \in \mathcal{S}\}$.* ■

The proof of this theorem is very simple; indeed define a function φ: $X \to \lambda(Y, \mathcal{S}\restriction Y)$ by $\varphi(x) := \{S \cap Y \mid S \in \mathcal{S}$ and $x \in S\}$. It is straightforward to prove that φ is a homeomorphism.

We are now prepared to prove the main result in this section.

5.2. THEOREM. *Let X be a sum of finitely many nondegenerate finite connected graphs. Then μX is a Hilbert cube.*

Proof. Let $F = \{x_1, \ldots, x_n\}$ be a finite subset of X with the property that the components of $X - F$ are either homeomorphic to $(0, 1)$ or to $[0, 1)$. We denote the collection of components by $C_1, \ldots, C_m$. We may assume that F does not contain any endpoint of some component of X. For each $i \leqslant m$ let I_i^k $(k \in N)$ be a sequence of arcs such that

(i) $k < l \Rightarrow I_i^k \subset I_i^l$;

(ii) $\bigcup_{k=1}^{\infty} I_i^k = C_i$.

For each $k \in N$ define $X_k := \bigcup_{i\leqslant m} I_i^k$. Clearly $\bigcup_{k=1}^{\infty} X_k$ is dense in X. We will show that $\{\mu X_k \mid k \in N\}$ with appropriate bonding mappings approximates μX. For each $k \in N$ and $i \leqslant m$ let a_i^k and b_i^k denote the endpoints of I_i^k. Define $\mathring{I}_i^k := I^k - \{a_i^k, b_i^k\}$. For all $j \leqslant n$ let D_j^k be the component of $X - \bigcup_{i\leqslant m} \mathring{I}_i^k$ containing x_j. Then D_j^k either is an interval, in case x_j is a cutpoint D_j^k, or is a finite acyclic tree with precisely one branch point, namely x_j. In case D_j^k is an interval, say with endpoints c and d, it is obvious that there is an embedding φ; $D_j^k \to \{c, d\}^+ \subset \mu X_k$ such that $\varphi(c) = c$ and $\varphi(d) = d$ (notice that $\mu X_k \approx Q$ by Theorem 4.1 and that $\{c, d\}^+$ is an AR by Theorem 2.1(ii) and hence that there is an arc in $\{c, d\}^+$ connecting c and d). In case D_j^k is not an interval let $G \subset D_j^k$ be the (finite) set of endpoints of D_j^k. Then $|G| \geqslant 3$. Take a point $\mathcal{M} \in \mu X_k$ such that $\mathcal{M} \in \bigcap_{g\in G} (G - \{g\})^+$. There is such a point since $\{G - \{g\} \mid g \in G\}$ is linked. Now Lemma 3.2 implies that there is an embedding φ: $D_j^k \to G^+ \subset \mu X_k$ such that φ restricted to G is the identity.

This procedure shows that for each $k \in N$ there is an embedding ψ_k: $X \to \mu X_k$ satisfying the following conditions:

(i) ψ_k restricted to X_k is the identity;

(ii) for all $i \leqslant n$ we have that $\psi_k(D_i^k) \subset (D_i^k \cap X_k)^+ \cap \mu X_k$.

We now define other mappings from X to μX_k. Indeed, fix $j \leqslant n$ and consider D_j^k and D_j^{k+1}. Without loss of generality we may assume that $(D_j^{k+1} \cap X_{k+1}) \cap (D_j^k \cap X_k) = \emptyset$ (for all $k \in N$ and $j \leqslant n$). Now, define a mapping f_k: $X \to \mu X_k$ by:

$f_k(x) = x$ for all $x \in X_k$,

f_k restricted to $D_j^k - D_j^{k+1}$ is a homeomorphism of $D_j^k - D_j^{k+1}$ onto $\psi_k[D_j^k] - \{\psi_k(x_j)\}$,

f_k maps D_j^{k+1} onto $\psi_k(x_j)$.

It is clear that f_k defined in this way is well defined and continuous. In addition, let $\xi_k\colon X_{k+1} \to \mu X_k$ be the restriction of f_k to X_{k+1}. Then ξ_k restricted to X_k is the identity and consequently $X_k \subset \xi_k(X_{k+1}) \subset \mu X_k$. Define

$$\mathcal{S}_k := \{T^+ \cap \xi_k(X_{k+1}) |\ T \in \mathcal{T}(X_k)\}.$$

By Theorem 5.1 we conclude that $\lambda(\xi_k(X_{k+1}), \mathcal{S}_k)$ is canonically homeomorphic to μX_k. Also, it is easy to show that $\mathcal{S}_k \subset \mathcal{T}(\xi_k(X_{k+1}))$ and that it is a super-normal subbase. Hence Theorem 2.3 shows that ξ_k can be extended to a cp surjection $\mu(\xi_k)\colon \mu X_{k+1} \to \lambda(\xi_k(X_{k+1}), \mathcal{S}_k)$. Let us identify μX_k and $\lambda(\xi_k(X_{k+1}), \mathcal{S}_k)$. We then have the following diagram:

$$(*)\qquad \begin{tikzcd} & X \arrow[dl, "f_k"'] \arrow[dr, "f_{k+1}"] & \\ \mu X_k & & \mu X_{k+1} \arrow[ll, "\mu(\xi_k)"] \end{tikzcd}$$

CLAIM. *Diagram* $(*)$ *commutes.*

Indeed, take $x \in X$. If $x \in X_{k+1}$ then

$$\mu(\xi_k) f_{k+1}(x) = \mu(\xi_k)(x) = \xi_k(x) = f_k(x),$$

hence there is no problem. Let us assume that $x \notin X_{k+1}$; then $x \in D_j^{k+1}$ for some $j \leqslant n$. Then $f_{k+1}(x) \in \psi_{k+1}(X) - X_{k+1}$. Hence, by construction of the embeddings ψ_k, we have that

$$f_{k+1}(x) \in \psi_{k+1}(D_j^{k+1}) \subset (D_j^{k+1} \cap X_{k+1})^+ \subset \mu X_{k+1}$$

and consequently, since $I(D_j^{k+1} \cap X_{k+1}) = (D_j^{k+1} \cap X_{k+1})^+$, by Corollary 2.6,

$$\mu(\xi_k) f_{k+1}(x) \in I(\xi_k(D_j^{k+1} \cap X_{k+1})) = I(\{\psi_k(x_j)\}) = \{\psi_k(x_j)\}.$$

This implies that $\mu(\xi_k) f_{k+1}(x) = f_k(x)$.

By a similar construction as above for each $k \in N$ we can extend the function $f_k\colon X \to \mu X_k$ to a cp surjection $\mu(f_k)\colon \mu X \to \mu X_k$. Since diagram $(*)$ commutes it is easy to verify that $\mu(f_k) = \mu(\xi_k) \circ \mu(f_{k+1})$ for all $k \in N$ (this also follows from Theorem 2.3 since the composition of cp mappings is a cp mapping).

We conclude that the mapping $e\colon \mu X \to \varprojlim(\mu X_k, \mu(\xi_k))$ defined by $e(\mathcal{M})_k := \mu(f_k)(\mathcal{M})$ is a continuous surjection. We claim that e is one to one, proving that it is a homeomorphism between μX and $\varprojlim(\mu X_k, \mu(\xi_k))$. Indeed, take distinct $\mathcal{M}, \mathcal{N} \in \mu X$ and take $M \in \mathcal{M}$ and $N \in \mathcal{N}$ such that $M \cap N = \emptyset$. Then $(M \cap F) \cap (N \cap F) = \emptyset$. Let $M \cap F = \{x_1, \ldots, x_i\}$ and $N \cap F = \{x_{i+1}, \ldots, x_n\}$ (if $M \cup N \not\supset F$ then enlarge M with the set $F - (M \cup N)$, then M still belongs to $\mathcal{M}$ and also M does not intersect N). Let U_i $(i \leqslant n)$ be open subsets of X such that $x_i \in U_i$ and $i < j$ implies that $U_i \cap U_j = \emptyset$ while in addition $\bigcup_{j \leqslant i} U_j \subset X - N$ and $\bigcup_{j=i+1}^{n} U_i \subset X - M$. Choose $k \in N$ such that $D_i^k \subset U_i$ for all $i \leqslant n$. Then

$$\mu(f_k)(\mathcal{M}) \in \Big((M \cap X_k) \cup (X_k \cap \bigcup_{j=1}^{i} D_j^k)\Big)^+$$

and

$$\mu(f_k)(\mathcal{N}) \in \Big((N \cap X_k) \cup (X_k \cap \bigcup_{j=i+1}^{n} D_j^k)\Big)^+,$$

and as these sets are disjoint, we conclude that $\mu(f_k)(\mathcal{M}) \neq \mu(f_k)(\mathcal{N})$. This shows that $\mu X \approx \varprojlim(\mu X_k, \mu(\xi_k))$. By Theorem 4.1 and by the cellularity of the mappings $\mu(\xi_k)$ (cf. Theorem 2.3) we conclude, with similar arguments as in Section 4, that $\mu X \approx Q$. ■

6. Superextensions of metrizable continua are Hilbert cubes. In this final section we prove that μX is a Hilbert cube provided that X is a finite sum of nondegenerate metrizable continua. We heavily rely on results of Curtis and Schori [9], [10].

As noted in Section 2 for compact Hausdorff X we may regard λX to be a subspace of 2^{2^X}. Now let (X, d) be a compact metric space and let d_H (resp. d_{HH}) be the Hausdorff metric for 2^X (resp. 2^{2^X}). Let $\mathcal{L}(X) \subset 2^{2^X}$ denote the space of all closed families of nonvoid closed subsets of X, which are linked (cf. van Mill and Van de Vel [20]). In van Mill and Van de Vel [20] the following results were established.

6.1. THEOREM. (i) *The inclusion* $(\lambda X, \bar{d}) \to (2^{2^X}, d_{HH})$ *is an isometry (here* $\bar{d}$ *is* Verbeek's [27] *metric for* λX, *see Section* 3);

(ii) *For each* $x \in X$ *the mapping* $h_x\colon \mathcal{L}(X) \to \lambda X$ *defined by* $h_x(\mathcal{L}) := p(x, \bigcap_{L \in \mathcal{L}} L^+)$ *is a contractive retraction (with respect to the metrics* d_{HH} *and* $\bar{d}$).

Before proving the main result in this section we mention a highly nontrivial result of Curtis and Schori [10] (in fact Curtis and Schori proved a stronger assertion than stated below, see [10]).

6.2. THEOREM. *Let X be a nondegenerate Peano continuum. Then there is collection finite connected graphs* $\{\Gamma_i |\ i \in N\}$ *in* X *and for each* $i \in N$ *there is a mapping* $\varphi_i\colon \Gamma_{i+1} \to 2^{\Gamma_i}$ *such that*

(i) $\Gamma_i \to X$ *(in* 2^X);

(ii) *if* $f_i\colon 2^{\Gamma_{i+1}} \to 2^{\Gamma_i}$ *is the mapping induced by* φ_i, *i.e.* $f_i(A) = \bigcup \{\varphi_i(a) |\ a \in A\}$, *then* $\sum_{i=1}^{\infty} d_H(f_i, \mathrm{id}) < \infty$;

(iii) $\{f_i \circ \ldots \circ f_j |\ j \geqslant i\}$ *is an equi-uniformly continuous family for each* i;

(iv) *each* f_i *is onto.* ■

Notice that since the f_i's are onto for every $y \in \Gamma_i$ there is an $x \in \Gamma_{i+1}$ such that $\varphi_i(x) = y$.

Actually, Curtis and Schori, proved Theorem 6.2 in [9] for compact connected polyhedra and in [10] they proved the general case. In our construction we need the result for polyhedra only. It must be noticed that the proof of 6.2 for polyhedra is easier than the proof of the general case.

6.3. PROPOSITION. *Let X be a finite sum of nondegenerate compact connected polyhedra. Then μX is a Hilbert cube.*

Proof. Let $P_1, \dots, P_n$ denote the collection of components of X. For each $i \leqslant n$ let Γ_j^i and φ_j^i $(j \in N)$ be a collection of finite graphs and mappings with properties as in Theorem 6.2. For each $j \in N$ define $X_j := \bigcup_{i \leqslant n} \Gamma_j^i$; in addition define $\varphi_j^*\colon X_{j+1} \to 2^{X_j}$ by $\varphi_j^*(x) = \varphi_j^i(x)$ if $x \in \Gamma_j^i$. It is clear that the φ_j^* defined in this way are continuous. Let $f_j^*\colon 2^{X_{j+1}} \to 2^{X_j}$ be the mapping induced by φ_j^* (i.e. $f_j^*(A) = \bigcup \varphi_j^*[A]$). In addition, let $F_j^*\colon 2^{2^{X_{j+1}}} \to 2^{2^{X_j}}$ be the mapping induced by f_j^* (i.e. $F_j^*(\mathscr{A}) = \{f_j^*(A) |\ A \in \mathscr{A}\}$). Fix $j \in N$. Take $\mathscr{A} \in \mathscr{L}(X_{j+1})$. We claim that $F_j^*(\mathscr{A}) \in \mathscr{L}(X_j)$. Indeed, take $A_0, A_1 \in \mathscr{A}$. Then $A_0 \cap A_1 \neq \emptyset$ implies that $\varphi_j^*[A_0] \cap \varphi_j^*[A_1] \neq \emptyset$. Take $C \in \varphi_j^*[A_0] \cap \varphi_j^*[A_1]$. Then $\emptyset \neq C \subset \bigcup \varphi_j^*[A_0] \cap \bigcup \varphi_j^*[A_1]$. We conclude that $f_j^*(A_0) \cap f_j^*(A_1) \neq \emptyset$ and consequently $F_j^*(\mathscr{A}) \in \mathscr{L}(X_j)$. By induction, choose for every $j \in N$ a point $x_j \in X_j$ such that $\varphi_j^*(x_{j+1}) = x_j$. This can be done (cf. the remark following Theorem 6.2). Now consider the following infinite diagram.

$$\begin{tikzcd}
2^{2^{X_1}} & 2^{2^{X_2}} \arrow[l, "F_1^*"'] & 2^{2^{X_3}} \arrow[l, "F_2^*"'] & \dots \arrow[l, "F_3^*"'] \\
\mathscr{L}(X_1) \arrow[u, hook, "i"] \arrow[d, "h_{x_1}"] & \mathscr{L}(X_2) \arrow[l, "L_1^*"'] \arrow[u, hook, "i"] \arrow[d, shift left, "h_{x_2}"] & \mathscr{L}(X_3) \arrow[l, "L_2^*"'] \arrow[u, hook, "i"] \arrow[d, shift left, "h_{x_3}"] & \dots \arrow[l, "L_3^*"'] \\
\lambda X_1 & \lambda X_2 \arrow[l, "\xi_{x_1}"] \arrow[u, hook, shift left, "i"] & \lambda X_3 \arrow[l, "\xi_{x_2}"] \arrow[u, hook, shift left, "i"] & \dots \arrow[l, "\xi_{x_3}"]
\end{tikzcd}$$

where L_j^* is the restriction of F_j^* to $\mathscr{L}(X_{j+1})$ and i always denotes inclusion and h_{x_j} is the mapping of Theorem 6.1(ii) (i.e. $h_{x_j}(\mathscr{M}) = p(x_j, \bigcap_{M \in \mathscr{M}} M^+)$) and ξ_{x_j} is the mapping of Theorem 2.2 (i.e. $\xi_{x_j}(\mathscr{M}) = p(x_j, \bigcap_{M \in \mathscr{M}} (\bigcup \varphi_j^*[M])^+)$).

CLAIM 1. *Each ξ_{x_j} is onto.*

Indeed, take $\mathscr{M} \in \lambda X_j$ and consider the collection $\{\varphi_j^{*-1}[M] |\ M \in \mathscr{M}\}$. This collection, which is a linked system of course, (cf. the remark following Theorem 6.2), can be extended to a point $\mathscr{N} \in \lambda X_{j+1}$. We claim that $\xi_{x_j}(\mathscr{N}) = \mathscr{M}$. Indeed, since $\{\varphi^{*-1}[M] |\ M \in \mathscr{M}] \subset \mathscr{N}$ we have that $\mathscr{M} \subset \{\bigcup \varphi_j^*[N] |\ N \in \mathscr{N}\}$ and as $\mathscr{M}$ is a maximal linked system we conclude that $\mathscr{M} = \{\bigcup \varphi_j^*[N] |\ N \in \mathscr{N}\}$. Hence

$$\begin{aligned} \xi_j(\mathscr{N}) &= p(x_j, \bigcap_{N \in \mathscr{N}} (\bigcup \varphi_j^*[N])^+) = p(x_j, \bigcap_{M \in \mathscr{M}} M^+) \\ &= p(x_j, \{\mathscr{M}\}) = \mathscr{M} . \end{aligned}$$

This proves the claim.

The following claim is more difficult to prove.

CLAIM 2. *Suppose that $k < l$. Then*

$$h_{x_k} \circ L_k^* \circ L_{k+1}^* \circ \dots \circ L_{l-1}^* \circ i = \xi_{x_k} \circ \xi_{x_{k+1}} \circ \dots \circ \xi_{x_{l-1}} .$$

We will only prove this for the case $k = 1$ and $l = 3$. The proof of the general case is identical (notice that the case $k = 1$ and $l = 2$ is trivial, because of the definitions of L_1^*, h_{x_1} and ξ_{x_1}). Indeed, take $\mathscr{M} \in \lambda X_3$. It is easily verified that $L_2^* L_1^*(\mathscr{M})$ equals the collection $\{\bigcup \varphi_1^*[\bigcup \varphi_2^*[M]] |\ M \in \mathscr{M}\}$. Let this collection be denoted by $\mathscr{A}$. Notice that $\mathscr{A} \subset 2^X$ is closed. We claim that $p(x_1, \bigcap_{A \in \mathscr{A}} A^+)$, which equals $h_{x_1} L_2^* L_1^*(\mathscr{M})$, is the collection

$$\{B \in 2^X |\ x_1 \in B \text{ and } B \cap A \neq \emptyset\ (\forall A \in \mathscr{A})\} \cup \{B \in 2^X |\ \exists A \in \mathscr{A}\colon A \subset B\} .$$

This follows from the following general lemma which is of independent interest.

6.4. LEMMA. *Let X be a compact Hausdorff space, let $x \in X$ and let $\mathscr{A} \subset 2^X$ be a closed (in 2^X) and linked system. Then $p(x, \bigcap_{A \in \mathscr{A}} A^+)$ equals the collection*

$$\{B \in 2^X |\ x \in B \textit{ and } B \cap A \neq \emptyset\ (\forall A \in \mathscr{A})\} \cup \{B \in 2^X |\ \exists A \in \mathscr{A}\colon A \subset B\} .$$

Proof. In van Mill [15] Theorem 2.2 it was shown that the above collection is a pre-mls for $p(x, \bigcap_{A \in \mathscr{A}} A^+)$, that is, a linked subfamily of $p(x, \bigcap_{A \in \mathscr{A}} A^+)$ which is contained in precisely one mls, namely $p(x, \bigcap_{A \in \mathscr{A}} A^+)$.

For each nonvoid collection $\mathscr{C} \subset 2^X$ define

$$C^\dagger := \{B \in 2^X |\ \exists C \in \mathscr{C}\colon C \subset B\}$$

and

$$\mathscr{C}^\perp := \{B \in 2^X |\ \forall C \in \mathscr{C}\colon C \cap B \neq \emptyset\}$$

respectively. In van Mill and Van de Vel [20] $\mathscr{C}^\perp$ was called the collection of transversal sets of $\mathscr{C}$. It is easily seen that $\mathscr{C}^\perp$ is always closed in 2^X and that $\mathscr{C}^\dagger$ is closed provided that $\mathscr{C}$ is closed (use the compactness of 2^X).

We conclude that the collection

$$\{B \in 2^X |\ x \in B \text{ and } B \cap A \neq \emptyset \text{ for all } A \in \mathscr{A}\} \cup \{B \in 2^X |\ \exists A \in \mathscr{A}\colon A \subset B\}$$

equals $\mathscr{A}^\dagger \cup (\mathscr{A} \cup \{x\})^\perp$ and hence is closed in 2^X.

Now assume that there is a point $C \in p(x, \bigcap_{A \in \mathscr{A}} A^+) - (\mathscr{A}^\dagger \cup (\mathscr{A} \cup \{x\})^\perp)$. Let $U_1, \dots, U_n \subset X$ be open such that $C \subset \bigcup_{i \leqslant n} U_i$ and $C \cap U_i \neq \emptyset$ for all $i \leqslant n$ while in addition $\langle \mathrm{cl}_X(U_1), \dots, \mathrm{cl}_X(U_n) \rangle \cap (\mathscr{A}^\dagger \cup (\mathscr{A} \cup \{x\})^\perp) = \emptyset$. If $\bigcup_{i \leqslant n} U_i$ does not contain a member of $(\mathscr{A}^\dagger \cup (\mathscr{A} \cup \{x\})^\perp)$ we conclude that $X - \bigcup_{i \leqslant n} U_i \in p(x, \bigcap_{A \in \mathscr{A}} A^+)$ since $(\mathscr{A}^\dagger \cup (\mathscr{A} \cup \{x\})^\perp)$ is a pre-mls for $p(x, \bigcap_{A \in \mathscr{A}} A^+)$. This contradicts

$C \cap (X - \bigcup_{i\leq n} U_i) = \emptyset$. Therefore $\bigcup_{i\leq n} U_i$ contains an element of $\mathscr{A}^{\uparrow} \cup (\mathscr{A} \cup \{x\})^{\perp}$; consequently, since $(\mathscr{A}^{\uparrow} \cup (\mathscr{A} \cup \{x\})^{\perp})^{\uparrow} = \mathscr{A}^{\uparrow} \cup (\mathscr{A} \cup \{x\})^{\perp}$, we find that $\bigcup_{i\leq n} \mathrm{cl}_X(U_i) \in \mathscr{A}^{\uparrow} \cup (\mathscr{A} \cup \{x\})^{\perp}$ which is a contradiction since $\bigcup_{i\leq n} \mathrm{cl}_X(U_i) \in \langle \mathrm{cl}_X(U_1), \ldots, \mathrm{cl}_X(U_n)\rangle$. ■

This proves that $p(x_1, \bigcap_{A\in\mathscr{A}} A^{+}) = \mathscr{A}^{\uparrow} \cup (\mathscr{A} \cup \{x\})^{\perp}$ (the notation is as in the proof of Lemma 6.4).

We will now calculate $\xi_{x_1}\xi_{x_2}(\mathscr{M})$. Define $\mathscr{B} := \{\bigcup \varphi_2^*[M] \mid M \in \mathscr{M}\}$. Then, by Lemma 6.4, $\xi_{x_2}(\mathscr{M})$ equals $\mathscr{B}^{\uparrow} \cup (\mathscr{B} \cup \{x_2\})^{\perp}$. Clearly $\{\bigcup \varphi_1^*[B] \mid B \in \mathscr{B}\}$ equals $\mathscr{A}$. In addition, if $E \in (\mathscr{B} \cup \{x_2\})^{\perp}$ then, since $\xi_{x_1}(x_2) = x_1$, we find that $\bigcup \varphi_1^*[E]$ contains x_1 and intersects all members from $\mathscr{A}$. Therefore $\xi_{x_1}\xi_{x_2}(\mathscr{M})$ and $p(x, \bigcap_{A\in\mathscr{A}} A^{+})$ are both the union of $\mathscr{A}^{\uparrow}$ and a set consisting of closed sets which all contain x_1 and which all meet every member from $\mathscr{A}$. As $\xi_{x_1}\xi_{x_2}(\mathscr{M})$ and $p(x, \bigcap_{A\in\mathscr{A}} A^{+})$ are maximal linked systems they are equal. This proves Claim 2.

Claim 3. *$\{\xi_{x_i} \circ \ldots \circ \xi_{x_j} \mid j \geq i\}$ is an equi-uniformly continuous family for each i.*

It is clear that there is a metric d for X such that $\{f_i^* \circ \ldots \circ f_j^* \mid j \geq i\}$ is an equi-uniformly continuous family for each i. For this metric we also have that $\sum_{i=1}^{\infty} d_H(f_i^*, \mathrm{id}) < \infty$. A straightforward check now shows that $\{F_i^* \circ \ldots \circ F_j^* \mid j \geq i\}$ is an equi-uniformly continuous family for each i. This implies that $\{L_i^* \circ \ldots \circ L_j^* \mid j \geq i\}$ is an equi-uniformly continuous family for each i. By Claim 2 and by the fact that the h_{x_i}'s are metric contractions (cf. Theorem 6.1(ii)) it follows that $\{\xi_{x_i} \circ \ldots \circ \xi_{x_j} \mid j \geq i\}$ is equi-uniformly continuous for each i.

For each $i \in N$ the set $\lambda X_i \subset 2^{2^{X_i}} \subset 2^{2^X}$. However λX_i is *not* a subspace of λX, although it is easy to define a canonical embedding η_i: $\lambda X_i \to \lambda X$. Indeed, define $\eta_i(\mathscr{M}) := \{A \in 2^X \mid A \cap X_i \in \mathscr{M}\}$. Notice that $\mathscr{M} \subset \eta_i(\mathscr{M})$. It is easily seen that this defines an embedding of λX_i into λX which in addition does not change distance, i.e.

$$d_{HH}(\mathscr{M}, \mathscr{N}) = d_{HH}(\eta_i(\mathscr{M}), \eta_i(\mathscr{N})) \quad \text{for all } \mathscr{M}, \mathscr{N} \in \lambda X_i .$$

The set $\eta_i[\lambda X_i]$ will be denoted by $\lambda^* X_i$. Define g_i: $\lambda^* X_{i+1} \to \lambda^* X_i$ as the composition $\eta_i \circ \xi_{x_i} \circ \eta_{i+1}^{-1}$. Since the η_i's do not change distances Claim 3 implies that $\{g_i \circ \ldots \circ g_j \mid j \geq i\}$ is an equi-uniformly continuous family for each $i \in N$.

Claim 4. *$\bar{d}(g_i, \mathrm{id}) \leq d_H(f_i^*, \mathrm{id})$ for each i.*

Indeed take $\mathscr{M} \in \lambda^* X_{i+1}$ and let $\mathscr{A} := \{M \cap X_{i+1} \mid M \in \mathscr{M}\}$. Then by the definition ξ_{x_i}, we have that $g_i(\mathscr{M})$ contains the family $\{\bigcup \varphi_i^*[A] \mid A \in \mathscr{A}\}$. Let $\delta := d_H(f_i^*, \mathrm{id})$. Now take $M \in \mathscr{M}$ and let $A := M \cap X_{i+1}$. Then $d_H(A, \bigcup \varphi_i^*[A]) \leq \delta$ which implies that $\bigcup \varphi_i^*[A] \subset B_\delta(A) \subset B_\delta(M)$. Therefore, since $\bigcup \varphi_i^*[A] \in g_i(\mathscr{M})$ we conclude that $B_\delta(M) \in \mathscr{M}$. Consequently, $\bar{d}(\mathscr{M}, g_i(\mathscr{M})) \leq \delta$ by expression 3 of Section 3. Hence $\bar{d}(g_i, \mathrm{id}) \leq d_H(f_i^*, \mathrm{id})$.

This implies that $\sum_{i=1}^{\infty} \bar{d}(g_i, \mathrm{id}) \leq \sum_{i=1}^{\infty} d_H(f_i^*, \mathrm{id}) < \infty$. In addition, since $X_i \to X$ (in 2^X) it is easily seen that $\lambda^* X_i \to \lambda X$ (in $2^{\lambda X}$). We conclude that $\lambda X \approx \varprojlim(\lambda^* X_i, g_i)$. This can be seen as follows; denote $\varprojlim(\lambda^* X_i, g_i)$ by Y. For each $\langle y_i\rangle_i \in Y$ the sequence $\langle y_i\rangle_i$ is Cauchy in λX by the fact that $\sum_{i=1}^{\infty} \bar{d}(g_i, \mathrm{id}) < \infty$, and hence converges to some point $y \in \lambda X$. The mapping e: $Y \to \lambda X$ defined by $e(\langle y_i\rangle_i) = \lim_{i\to\infty} y_i$ is easily seen to be continuous, is one to one since $\{g_i \circ \ldots \circ g_j \mid j \geq i\}$ is an equi-uniformly continuous family for each i, and is onto since $\lambda^* X_i \to \lambda X$ $(i \to \infty)$. Of course this kind of argumentation is known (cf. Brown [2], Mioduszewski [21], Curtis and Schori [9]). Identify $\varprojlim(\lambda^* X_i, g_i)$ and λX and let π_i: $\lambda X \to \lambda^* X_i$ be the natural projection of λX onto $\lambda^* X_i$ $(i \in N)$. Identify $\lambda^* X_i$ and λX_i $(i \in N)$. For each $i \in N$ let ϱ_i: $\lambda X_i \to \mu X_i$ be the natural Jensen surjection of λX_i onto μX_i (cf. Section 2 and in particular Theorem 2.3).

Claim 5. *The function β_i: $\mu X_{i+1} \to \mu X_i$ defined by $\beta_i := \varrho_i \circ g_i \circ \varrho_{i+1}^{-1}$ is a well defined* cp *surjection.*

Indeed, take $\mathscr{M} \in \mu X_{i+1}$ and assume that $\beta_i(\mathscr{M})$ contains two distinct points $\mathscr{P}_0$ and $\mathscr{P}_1$. Take $T_0, T_1 \in \mathscr{T}(X_{i+1})$ such that $T_i \in \mathscr{P}_i$ and $T_0 \cap T_1 = \emptyset$. It is clear that we may choose the T_i's in such a way that for each component C of X_{i+1} we have that C is either contained in one of the T_i's or intersects both T_0 and T_1. Let D be a component of λX_i such that $D \cap \varrho_i^{-1}(T_0^+) = \emptyset$. By the remarks in Section 3 we may assume that there is a finite family of clopen subsets $\mathscr{E}$ in X_i such that $D = \bigcap_{E\in\mathscr{E}} E^+$. By the fact that ϱ_i is a cp mapping (cf. Theorem 2.3) and the binarity of $\{A^+ \mid A \in 2^{X_i}\}$ we may assume that there is a $E_0 \in \mathscr{E}$ such that $E_0^+ \cap \varrho_i^{-1}(T_0^+) = \emptyset$ (notice that the first "plus" is in λX, while the second is in μX_{i+1}). Hence $E_0 \subset \varrho_i[E_0^+]$ does not intersect T_0. Consequently $E_0 \subset T_1$ which implies that $E_0 \subset \varrho_i^{-1}[T_1^+]$ and by the fact that ϱ_i is a cp mapping we see that $E_0^+ = I(E_0) \subset \varrho_i^{-1}[T_1^+]$. This procedure shows that every component of λX_{i+1} is either contained in one of $\{\varrho_i^{-1}[T_0^+], \varrho_i^{-1}[T_1^+]\}$ or intersects both $\varrho_i^{-1}[T_0^+]$ and $\varrho_i^{-1}[T_1^+]$. Now, since each component of λX_{i+1} is mapped by g_i in a component of λX_i this implies that each component of λX_{i+1} is either contained in one of $\{g_i^{-1}\varrho_i^{-1}[T_0^+], g_i^{-1}\varrho_i^{-1}[T_1^+]\}$ or intersects both $g_i^{-1}\varrho_i^{-1}[T_0^+]$ and $g_i^{-1}\varrho_i^{-1}[T_1^+]$. For simplicity of notation write $V_j = g_i^{-1}\varrho_i^{-1}[T_j^+]$ $(j \in \{0, 1\})$.

By the binarity of $\{A^+ \mid A \in 2^{X_{i+1}}\}$ and by the fact that $\varrho_i \circ g_i$ is a cp mapping, there are disjoint $A_j \in 2^{X_{i+1}}$ $(j \in \{0, 1\})$ such that $V_j \subset A_j^+$. If A_0 is clopen then so is A_0^+ and consequently $V_0 = A_0^+$ since each component of λX_{i+1} either is contained in one of $\{V_0, V_1\}$ or intersects both V_0 and V_1. Therefore T_0^+ is a proper nonvoid clopen subset of μX_i, since $\varrho_i \circ g_i$ is an identification. This contradicts the connectedness of μX_i (cf. Corollary 1.3). In the same way we have that A_1 is not a clopen subset of X_{i+1}. Hence $\{A_0, A_1\} \subset \mathscr{T}(X_{i+1})$. Since $(\varrho_i \circ g_i)^{-1}(\mathscr{P}_j) \subset A_j^+$ $(j \in \{0, 1\})$ we see that A_0 and A_1 must be both elements of $\mathscr{M}$ (cf. Section 2); this contradicts the linkedness of $\mathscr{M}$.

This proves that β_i is well defined. Also, clearly β_i is continuous. That β_i is a cp mapping now follows from the fact that ϱ_{i+1}, g_i and ϱ_i are cp mappings and from Theorem 2.4.

One moment one might think that this finishes the proof of the proposition since it is intuitively clear that the inverse sequence

$$\mu X_1 \overset{\beta_1}{\leftarrow} \mu X_2 \overset{\beta_2}{\leftarrow} \mu X_3 \leftarrow \ldots$$

approximates μX. We will prove below that this is indeed the case, but since there is *no* canonical embedding of μX_i in μX we cannot apply similar arguments then in the λX case above. This causes a few technical problems.

CLAIM 6. *For each $i \in N$ the projection π_i: $\lambda X \to \lambda X_i$ is a* cp *surjection.*

For each $j > i$ let g_{ji}: $\lambda X_j \to \lambda X_i$ be the composition $g_i \circ \ldots \circ g_{j-1}$. In addition, let g_{ii} be the identity mapping on λX_i. Notice that we have identified the λX_i's with subspaces of λX. Now using the properties of the functions g_i (namely $\sum_{i=1}^{\infty} \bar{d}(g_i, \mathrm{id}) < \infty$ and the equi-uniform continuity) it is easy to calculate that for each closed set $A \subset \lambda X_i$ we have that

$$\pi_i^{-1}(A) = \mathrm{cl}_{\lambda X}\big(\bigcup_{j \geqslant i} g_{ji}^{-1}(A)\big).$$

Now let T be a closed subset of X_i. For each $j \geqslant i$ define $E_j := g_{ji}^{-1}(T^+)$. Notice that $E_i = T^+$. In addition, for each $j \geqslant i$ let p_j be the nearest point mapping of λX_j (cf. Theorem 2.1(ii)). Now take a point $\mathscr{M} \in \lambda X - \pi_i^{-1}(T^+)$. Let $\mathscr{M}_j := \pi_j(\mathscr{M})$ and $\mathscr{N}_j := p_j(\mathscr{M}_j, E_j)$ (notice that the g_{ji}'s are cp mappings). By Corollary 2.5 we have that $\{\mathscr{N}_j |\ j \geqslant i\}$ determines a thread in

$$\lambda X_i \overset{g_i}{\leftarrow} \lambda X_{i+1} \overset{g_{i+1}}{\leftarrow} \ldots$$

and hence a point, say $\mathscr{N}$, in λX. By the fact that $\{g_j \circ \ldots \circ g_i |\ j \geqslant i\}$ is an equi-uniformly continuous family for all $j \geqslant i$ we have that $\delta = \inf\{\bar{d}(\mathscr{M}_j, \mathscr{N}_j) |\ j \geqslant i\} > 0$. This implies that $\bar{d}(\mathscr{M}, \mathscr{N}) \geqslant \delta$. Hence we may take $M \in \mathscr{M}$ such that $B_{\delta/2}(M) \notin \mathscr{N}$ (cf. Section 3, Expression 3). Choose $k \geqslant i$ such that $\mathscr{M}_l \in B_{\delta/2}(M)$ for all $l \geqslant k$. Assume that there exists an $l \geqslant k$ such that $E_l \cap (U_{\delta/2}(M))^+ \neq \emptyset$. Take a point $\mathscr{A} \in E_l \cap (U_{\delta/2}(M))^+$. Choose an index $l_0 \geqslant l$ such that $\pi_{l_1}(\mathscr{A}) = \mathscr{A}_{l_1} \in (U_{\delta/2}(M))^+$ for all $l_1 \geqslant l_0$. Now consider l_0. We have that $\mathscr{A}_{l_0} \in E_{l_0}$, $\mathscr{N}_{l_0} \in E_{l_0}$, $p_{l_0}(\mathscr{M}_{l_0}, E_{l_0}) = \mathscr{N}_{l_0}$ and $\bar{d}(\mathscr{A}_{l_0}, \mathscr{M}_{l_0}) \leqslant \frac{1}{2}\delta < \delta = \bar{d}(\mathscr{N}_{l_0}, \mathscr{M}_{l_0})$. This contradicts a theorem in van Mill and Van de Vel [19]; there it was shown that for each compact metric space (X, d) the nearest point retraction p: $\lambda X \to A$, where A is $(2^X)^+$-closed, is a metric nearest point map, in the sense that $\bar{d}(\mathscr{M}, A) = \bar{d}(\mathscr{M}, p(\mathscr{M}, A))$ for all $\mathscr{M} \in \lambda X$.

We conclude that $E_l \cap (U_{\delta/2}(M))^+ = \emptyset$ for all $l \geqslant k$. Therefore we have that $E_l \subset (X - U_{\delta/2}(M))^+$ for all $l \geqslant k$, and consequently,

$$\pi_i^{-1}(T^+) = \pi_k^{-1}(E_k) = \mathrm{cl}_{\lambda X}\big(\bigcup_{l \geqslant k} E_l\big) \subset (X - U_{\delta/2}(M))^+ .$$

For each $\mathscr{M} \in \lambda X - \pi_i^{-1}(T^+)$ we have constructed a closed set $B \subset X$ such that $\pi_i^{-1}(T^+) \subset B^+$ and $\mathscr{M} \notin B^+$. This implies that $\pi_i^{-1}(T^+)$ is $(2^X)^+$-closed. It now follows that π_i is a cp mapping.

Let h: $\lambda X \to \mu X$ be the Jensen surjection.

CLAIM 7. *For each $i \in N$ the mapping h_i: $\mu X \to \mu X_i$ defined by $h_i := \varrho_i \circ \pi_i \circ h^{-1}$ is a well defined cp surjection.*

Since the mappings π_i and ϱ_i are cp mappings (Claim 5 and Claim 6) this can be proved using the same technique as in the proof of Claim 5.

We now have the following diagram.

$$\begin{tikzcd}
 & \mu X \arrow[dl, "h_1"'] \arrow[d, "h_2"] \arrow[dr, "h_3"] & & \\
\mu X_1 & \mu X_2 \arrow[l, "\beta_1"] & \mu X_3 \arrow[l, "\beta_2"] & \ldots \arrow[l]
\end{tikzcd}$$

Since $\beta_i \circ h_{i+1} = \beta_i \circ \varrho_{i+1} \circ \pi_{i+1} \circ h^{-1} = \varrho_i \circ g_i \circ \varrho_{i+1}^{-1} \circ \varrho_{i+1} \circ \pi_{i+1} \circ h^{-1} = \varrho_i \circ g_i \circ \pi_{i+1} \circ h^{-1} = \varrho_i \circ \pi_i \circ h^{-1} = h_i$ for each $i \in N$ the mapping e: $\mu X \to \varprojlim(\mu X_i, \beta_i)$ defined by $e(\mathscr{M})_i := h_i(\mathscr{M})$ is a well defined continuous surjection. We claim that e is one to one proving that e is a homeomorphism. Indeed, take distinct $\mathscr{M}_0, \mathscr{M}_1 \in \mu X$ and take disjoint $M_0 \in \mathscr{M}_0$ and $M_1 \in \mathscr{M}_1$. We may assume, without loss of generality that for each component C of X we have that C is either contained in one of the M_i's or intersects both T_0 and T_1. As in the proof of Claim 5 one can derive that each component of λX is either contained in one of the $h^{-1}(M_i^+)$'s or intersects both $h^{-1}(M_0^+)$ and $h^{-1}(M_1^+)$. There is an index $l \in N$ such that $\pi_l h^{-1}(M_0^+) \cap \pi_l^{-1} h^{-1}(M_1^+) = \emptyset$. Take two disjoint closed subsets $A_0, A_1 \subset X_l$ such that $\pi_l h^{-1}(M_i^+) \subset A_i^+$ $(i \in \{0, 1\})$. First suppose that A_0 is clopen; then so is A_0^+ and consequently $\pi_l h^{-1}(M_0^+) = A_0^+$ and $\pi_l h^{-1}(M_1^+) = (X_l - A_0)^+$ (use the same technique as in the proof of Claim 5). In addition, it is easily seen that also $h^{-1}(M_0^+) \cup h^{-1}(M_1^+) = \lambda X$. Since h extends the identity on X we have that

$$\begin{aligned}(h^{-1}(M_0^+) \cap X) \cup (h^{-1}(M_1^+) \cap X) &= (M_0^+ \cap X) \cup (M_1^+ \cap X)\\ &= M_0 \cup M_1 = X\end{aligned}$$

which is a contradiction since M_0 and M_1 both belong to $\mathscr{T}(X)$. Therefore, we may assume that neither A_0 nor A_1 are clopen. Hence A_0 and A_1 both belong to $\mathscr{T}(X_l)$. It now follows that $\varrho_l \pi_l h^{-1}(M_0^+) \cap \varrho_l \pi_l h^{-1}(M_1^+) = \emptyset$. We conclude that $h_l(\mathscr{M}_0) \neq h_l(\mathscr{M}_1)$.

Therefore $\mu X \approx \varprojlim(\mu X_i, \beta_i)$. By Theorem 5.2 the spaces μX_i are Hilbert cubes. Since the mappings β_i are cp mappings, they are cellular and applying Chapman's [4], [5] result again we conclude that $\mu X \approx Q$. ■

6.5. COROLLARY. *Let X be a nondegenerate compact connected polyhedron. Then λX is a Hilbert cube.* ■

6.6. COROLLARY TO COROLLARY. *λX is a Hilbert cube if and only if X is a nondegenerate metrizable continuum.*

Proof. By a result in van Mill and Van de Vel [20], Corollary 6.5 suffices to prove 6.6. ■

We now prove the main result in this paper.

6.7. THEOREM. *Let X be a finite sum of nondegenerate metrizable continua. Then μX is a Hilbert cube.*

Proof. Let $C_1, \ldots, C_n$ be the collection of components of X. By Freudenthal's [11] expansion theorem, for each $i \leqslant n$ there is an inverse sequence

$$P_1^i \xleftarrow{\beta_1^i} P_2^i \xleftarrow{\beta_2^i} P_3^i \xleftarrow{\beta_3^i} \ldots$$

of compact connected nondegenerate polyhedra with onto bonding mappings β_j^i ($j \in N$) such that $\varprojlim(P_j^i, \beta_j^i) \approx C_i$. For each $j \in N$ let X_j be the disjoint topological sum of the P_j^i's ($i \leqslant n$). It is now straightforward to prove that

$$X_1 \xleftarrow{\varrho_1} X_2 \xleftarrow{\varrho_2} X_3 \xleftarrow{\varrho_3} \ldots,$$

with bonding mappings defined in the obvious way, approximates X. This implies, by an obvious argument, that $\mu X \approx \varprojlim(\mu X_i, \mu(\varrho_i))$, where $\mu(\varrho_i)$ is the Jensen extension of ϱ_i. Using Proposition 6.3 and the cellularity of the $\mu(\varrho_i)$'s yields $\mu X \approx \varprojlim(\mu X_i, \mu(\varrho_i)) \approx Q$. ■

For each $n \in N$ let $\lambda(n)$ be the cardinality of $\lambda\{1, 2, \ldots, n\}$. Using the same technique as in Section 3 one can prove that if X is a sum of n copies of $[0, 1]$ then λX is homeomorphic to a sum of $\lambda(n)$ copies of Q. Going through the whole process again, one then gets the following result.

6.8. THEOREM. *Let X be a sum of n nondegenerate metrizable continua. Then λX is a sum of $\lambda(n)$ Hilbert cubes.* ■

Details are left to the reader.

6.9. COROLLARY. *Let X be a compact metric space. The following statements are equivalent:*

(i) *λX is a Q-manifold;*

(ii) *X is a sum of finitely many nondegenerate continua.* ■

The number $\lambda(n)$ is only known for $n \leqslant 7$ (cf. Verbeek [27]). It is an intriguing combinatorial problem to calculate the numbers $\lambda(n)$; the only information we can give is that

$$^2\log\lambda(n) \sim \binom{n}{[n/2]}\frac{2^n}{\sqrt{2\pi n}}$$

(cf. Verbeek [27]). Since $\lambda(3) = 4$ and $\lambda(4) = 12$ (cf. Verbeek [27]) there is no compact metric space X for which λX is a sum of i Hilbert cubes, where $4 < i < 12$.

Added in proof. The main result in this paper that $\lambda X \approx Q$ can also be derived by using a recent characterization of the Hilbert cube due to H. Toruńczyk. This was observed independently by C. Bruce Hughes and the author.

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Some combinatorial properties of ultrafilters

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Abstract. Three unrelated combinatorial results are proved: (1) A result relating non-regular ultrafilters to weakly normal ultrafilters; (2) A partitioning property for indecomposable ultrafilters over singular cardinals and (3) A large cardinal-type result for inaccessible cardinals carrying indecomposable ultrafilters.

0. Introduction. Our notation and terminology follows that of the more recent set-theoretic literature. In particular $\alpha, \beta, \gamma, \ldots$ are variables for ordinals while $\varkappa, \lambda, \mu, \ldots$ are reserved for cardinals. The notation $|x|$ refers to the cardinality of the set x and so on. An ultrafilter over a cardinal is always assumed to be uniform.

0.1. Definition. An ultrafilter D over $\varkappa$ is (λ, μ)-*regular* if $\lambda \leqslant \mu$ and there is a set $S \subseteq D$ of power μ such that

$$T \subseteq S \quad \text{and} \quad \lambda \leqslant |T| \rightarrow \bigcap T = 0\,.$$

D is μ-*regular* if it is (ω, μ)-regular. D is *regular* if it is $\varkappa$-regular.

This concept is due to Keisler. It measures the "width" of an ultrafilter. It is diametrically opposite to the notion of completeness of ultrafilters. It is a well-known fact that the existence of suitably complete ultrafilters implies the existence of normal ultrafilters. In the case of simply non-regular ultrafilters we have to replace the condition of normality by a weaker one:

0.2. Definition. An ultrafilter D over $\varkappa$ is weakly normal if every pressing down function is bounded by a constant $<\varkappa$, i.e. if $f\colon \varkappa \rightarrow \varkappa$ s.t. $f < \mathrm{id} \pmod D$, then there is a $\xi < \varkappa$ s.t. $f \leqslant \xi \pmod D$.

Kanamori [3] was the first to show that suitably non-regular ultrafilters have weakly normal ultrafilters below them in the Rudin–Keisler order.

0.3. Definition. Given two ultrafilters D, U over $\varkappa$ say $D \leqslant_{\mathrm{RK}} U$ if there is a function $f\colon \varkappa \rightarrow \varkappa$ s.t. $f_*(U) = D$; i.e. for all $x \subseteq \varkappa$:

$$x \in D \leftrightarrow f^{-1}(x) \in U\,.$$

Given $f, g\colon \varkappa \rightarrow \varkappa$ say $f \leqslant_{\mathrm{RK}} g \pmod D$ if there is a function $\varphi\colon \varkappa \rightarrow \varkappa$ s.t. $f = \varphi \circ g$