fixed, then there are only finitely many different k and f, hence $\mathrm{LO}(Q_0, \ldots, Q_m)$ is decidable. $\mathrm{LO}(Q_i\colon i<\omega)$ is decidable, because the decision methods for $\mathrm{LO}(Q_0, \ldots, Q_m)$ are uniform in m.

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Prime and coprime modules

by

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Abstract. The results of this paper generalize the notion of prime ideal. Consequently, there is defined prime module and its dual, coprime module. Similarly as the Jacobson radical is defined, we introduce the notion of prime radical. Besides the essentials of the calculus of prime and coprime modules, the main purpose of the paper is to show how these notions are related to the general theory of preradicals as a tool for structural investigation of rings and modules.

1. Introduction. In the following, R is an associative ring with unit and R-mod stands for the category of unital left R-modules. R is called *left V-ring* if all simple modules are injective. Further, R is said to be *left* (*right*) *duo-ring* if all left (right) ideals are two-sided. As usually, $E(M)$ will denote the injective hull of a module M. A submodule N of a modu!: M is called *characteristic* if $f(N)\subseteq N$ for each $f\in \mathrm{Hom}(M, M)$.

Recall that a preradical r for R-mod is a subfunctor of the identity functor. We shall say that r is

idempotent if $r(r(M)) = r(M)$ for every $M\in R$-mod,

a *radical* if $r(M/r(M)) = 0$ for every $M\in R$-mod,

hereditary if $r(N) = N\cap r(M)$ for every $N, M\in R$-mod, $N\subseteq M$,

superhereditary if it is hereditary and the class $\mathscr{T}_r$ of all r-torsion modules is closed under direct products,

cohereditary if $r(M/N) = (r(M)+N)/N$ for all N, $M\in R$-mod such that $N\subseteq M$ (in this case, $r(M) = r(R)M$ for all $M\in R$-mod),

splitting if $r(M)$ is a direct summand for each $M\in R$-mod.

Let r be an arbitrary preradical. For each $M\in R$-mod we define $\bar{r}(M) = \sum N$, $N\subseteq M$ and $r(N) = N$, and $\tilde{r}(M) = \bigcap P$, $P\subseteq M$ and $r(M/P) = 0$. It is easy to see that $\bar{r}$ is the largest idempotent preradical contained in r and $\tilde{r}$ is the least radical containing r. The definition of inclusion, sum and intersection of preradical is obvious. Further, we define $r^1 = r_1 = r$ and $r^{n+1}(M) = r(r^n(M))$, $r_{n+1}(M)/r_n(M) = r(M/r_n(M))$ for every module M.

If I is a two-sided ideal then we define a cohereditary radical r and a superhereditary preradical s corresponding to I by $r(M) = IM$ and

$$s(M) = \{m\in M \mid Im = 0\},$$

for each $M \in R$-mod. In this case, $I = r(R) = \bigcap K$, $s(R/K) = R/K$. Further, we define preradicals id and zer by $\mathrm{id} M = M$ and $\mathrm{zer} M = 0$ for all $M \in R$-mod.

Let $\mathscr{A}$ be a non-empty class of modules. We define the idempotent preradical $p_{\mathscr{A}}$ and the radical $p^{\mathscr{A}}$ by $p_{\mathscr{A}}(M) = \sum \mathrm{Im} f$, $f \in \mathrm{Hom}(A, M)$, $A \in \mathscr{A}$, and $p^{\mathscr{A}}(M) = \bigcap \mathrm{Ker} f$, $f \in \mathrm{Hom}(M, A)$, $A \in \mathscr{A}$, respectively. If $\mathscr{S}$ is a representative set of simple modules then we put $\mathrm{Soc} = p_{\mathscr{S}}$ and $\mathscr{J} = p^{\mathscr{S}}$. Finally, we define $\mathscr{Y} = p^{\mathscr{B}}$, where $\mathscr{B}$ is the class of all modules which are small in its injective hull (i.e., $M + K = E(M)$ implies $K = E(M)$) and $\mathscr{Z}$ denotes the singular submodule. Obviously, for every module M, $\mathrm{Soc} M$ is the intersection of all essential submodules of M and $\mathscr{J}(M)$ is the sum of all small submodules of M.

For further details concerning preradicals, the reader is referred e.g. to [1]–[5]. Some other concepts of prime submodule (ideal) can be found e.g. in [6] and [7].

2. Prime modules. Let A, B be two submodules of a module M. Put $A *_M B = \sum f(A)$, $f \in \mathrm{Hom}(M, B)$.

2.1. Lemma. *Let A, B, C be three submodules of a module M. Then*

(i) *$A *_M B$ is a submodule of M and $A *_M B \subseteq B$,*

(ii) $0 *_M A = A *_M 0 = 0$,

(iii) *$A *_M M$ is the least characteristic submodule of M containing A,*

(iv) *$A *_M B$ is a characteristic submodule of M provided that B is so,*

(v) $(A *_M B) *_M C \subseteq A *_M (B *_M C)$,

(vi) *$(A *_M B) *_M C = A *_M (B *_M C)$, provided M is projective.*

Proof. The assertions (i)–(v) are obvious.

(vi) Let $D = \coprod_{\mathrm{Hom}(M,C)} B$ and let for every $g \in \mathrm{Hom}(M, C)$, p_g be the canonical projection of D onto B. Then $h\colon D \to B *_M C$ defined via $h(x) = \sum g(p_g(x))$, $g \in \mathrm{Hom}(M, C)$, is an epimorphism. If $f\colon M \to B *_M C$ and $a \in A$ are arbitrary then, M being projective, there is $k\colon M \to D$ with $hk = f$ and $f(a) = \sum g(p_g(k(a)))$. However $p_g k \in \mathrm{Hom}(M, B)$, so that $p_g(k(a)) \in A *_M B$ and consequently $f(a) \in (A *_M B) *_M C$. Thus $A *_M (B *_M C) \subseteq (A *_M B) *_M C$.

The equality (vi) does not hold in general. For example, if Z is the additive group of integers and Q that of rational numbers then $0 = (Z *_Q Z) *_Q Q \neq Z *_Q (Z *_Q Q) = Q$.

2.2. Lemma. *Let I, K be left ideals of R. Then $I *_R K = IK$.*

Proof. Obvious.

A module M is called *prime* if $p^M = p^N$ for every non-zero submodule N of M. Clearly, the class of all prime modules is closed under submodules.

2.3. Proposition. *The following are equivalent for a module M:*

(i) *$A *_M B \neq 0$ for all non-zero submodules $A, B \subseteq M$,*

(ii) *$p^N(M) = 0$ for every non-zero submodule $N \subseteq M$,*

(iii) *If $0 \neq N \subseteq M$ then M is isomorphic to a submodule of a direct product of copies of N,*

(iv) *M is prime.*

Proof. (i) implies (ii). Suppose that $p^B(M) = A \neq 0$ for some $0 \neq B \subseteq M$. Then $f(A) = 0$ for every $f \in \mathrm{Hom}(M, B)$, and so $A *_M B = 0$.

(ii) implies (iii). Let $0 \neq N \subseteq M$ be a submodule. Since $p^N(M) = 0$, there is a set f_i, $i \in I$, of homomorphisms of M into N such that $0 = \bigcap_{i \in I} \mathrm{Ker} f_i$. Hence M can be imbedded into the direct product of $\mathrm{Im} f_i$. Consequently, M is isomorphic to a submodule of a direct product of copies of N.

(iii) implies (iv). Let $0 \neq N \subseteq M$ be a submodule. Obviously, $p^M \subseteq p^N$ and $p^N(N) = 0$. Since M is isomorphic to a submodule of a direct product of copies of N, we have $p^N(M) = 0$. Consequently, p^M being a radical, $p^N \subseteq p^M$.

(iv) implies (i). If $0 \neq A, B \subseteq M$ and $A *_M B = 0$ then $f(A) = 0$ for all $f \in \mathrm{Hom}(M, B)$, and so $0 \neq A \subseteq p^B(M) = p^M(M) = 0$, a contradiction.

2.4. Proposition. (i) *A module M is prime iff $p^M = p^C$ for every non-zero cyclic submodule C of M.*

(ii) *Every direct sum of copies of a simple module is a prime module.*

Proof. Obvious.

In contrast to 2.4(ii), direct products of copies of a simple module need not be prime. It follows from the fact that such direct products may contain non-zero submodule with zero socles.

A submodule N of a module M is called 1-*prime* if M/N is a prime module. It is 2-*prime* if $A *_M B \not\subseteq N$, whenever A, B are submodules of M and $A \not\subseteq N$, $B \not\subseteq N$.

2.5. Lemma. *The following are equivalent for a module M:*

(i) *M is prime.*

(ii) *0 is a 1-prime submodule of M.*

(iii) *0 is a 2-prime submodule of M.*

Proof. It follows immediately from 2.3.

2.6. Proposition. *Every characteristic 2-prime submodule is 1-prime.*

Proof. Let N be a characteristic 2-prime submodule of a module M. Suppose that $A/N *_{M/N} B/N = 0$ for some submodules A, B of M containing N. If $f\colon M \to B$ is a homomorphism then f induces a homomorphism g of M/N into B/N, since $f(N) \subseteq N$. According to the hypothesis, $g(A/N) = 0$. Hence $f(A) \subseteq N$ and we see that $A *_M B \subseteq N$. However N is 2-prime, and therefore either $A \subseteq N$ or $B \subseteq N$. Thus either $A/N = 0$ or $B/N = 0$. According to 2.3, M/N is a prime module, i.e. N is a 1-prime submodule of M.

2.7. Proposition. *Let N be a 1-prime submodule of a projective module M. Then N is a 2-prime submodule.*

Proof. Suppose $A \nsubseteq N$ and $B \nsubseteq N$. Since M/N is prime, there is $f\colon M/N \to (B+N)/N$ with $0 \neq f((A+N)/N)$. However M is projective and hence there is $g\colon M \to B$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r] \arrow[d, "g"'] & M/N \arrow[d, "f"] \\ B \arrow[r] & B+N/N \end{tikzcd}$$

Since $0 \neq f((A+N)/N)$, $g(A) \nsubseteq B \cap N$, and so $A *_M B \nsubseteq N$.

2.8. Proposition. *Let N be a 2-prime submodule of a projective module M. Denote by C the largest characteristic submodule of M contained in N. Then C is a 1-prime submodule of M.*

Proof. With respect to 2.6, it is sufficient to show that C is a 2-prime submodule. Let $A, B \subseteq M$ and $A *_M B \subseteq C$. Then

$$\begin{aligned}(A *_M M) *_M (B *_M M) &= A *_M (M *_M (B *_M M)) = A *_M ((M *_M B) *_M M) \\ &\subseteq A *_M (B *_M M) = (A *_M B) *_M M \subseteq C *_M M = C \subseteq N.\end{aligned}$$

Since N is 2-prime, either $A *_M M \subseteq N$ or $B *_M M \subseteq N$. However, both $A *_M M$ and $B *_M M$ are characteristic.

2.9. Corollary. *Let M be a projective module without non-trivial characteristic submodules. Then M is prime.*

Proof. Let $0 \neq M$. Then M contains a proper maximal submodule N. Since M/N is simple, N is a 1-prime submodule of M, and hence M is prime by 2.5, 2.7 and 2.8.

2.10. Proposition. *The following are equivalent for a left ideal I:*

(i) *I is a 1-prime submodule of R.*

(ii) *For every left ideal K with $I \subsetneqq K$ and every $x \in R \setminus I$ there is $y \in K$ such that $Iy \subseteq I$ and $xy \notin I$.*

(iii) *For all $x, y \in R \setminus I$ there is $z \in R$ such that $Izy \subseteq I$ and $xzy \notin I$.*

Proof. (i) implies (ii). Let K be a left ideal with $I \subsetneqq K$ and $x \in R \setminus I$. Since $p^{K/I}(R/I) = 0$, there is a homomorphism $f\colon R/I \to K/I$ with $f(x+I) \neq 0$. The element y defined by $f(1+I) = y+I$ has the desired property.

(ii) implies (i). Let K be an arbitrary left ideal with $I \subsetneqq K$. If $x \in R \setminus I$ is arbitrary and $y \in K$ is such that $Iy \subseteq I$ and $xy \notin I$ then the mapping $f\colon R/I \to K/I$ defined via $f(r+I) = ry+I$ is a homomorphism and $f(x+I) \neq 0$. Consequently, $p^{K/I}(R/I) = 0$ and I is 1-prime.

The equivalence of (ii) and (iii) is obvious.

2.11. Proposition. *The following are equivalent for a left ideal I:*

(i) *I is a 2-prime submodule of R.*

(ii) *If K, L are left ideals and $KL \subseteq I$ then either $K \subseteq I$ or $L \subseteq I$.*

(iii) *If K is a two-sided ideal, L is a left ideal and $KL \subseteq I$ then either $K \subseteq I$ or $L \subseteq I$.*

(iv) *If $a, b \in R$ and $aRb \subseteq I$ then either $a \in I$ or $b \in I$.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are obvious.

(iv) implies (i). Let A, B be left ideals such that $A \nsubseteq I$, $B \nsubseteq I$. Suppose that $A *_R B = AB \subseteq I$. If $a \in A \setminus I$, $b \in B \setminus I$, then $aRb \subseteq AB \subseteq I$, and hence either $a \in I$ or $b \in I$, a contradiction. Thus $A *_R B \nsubseteq I$ and I is a 2-prime submodule of R.

2.12. Proposition. *The following are equivalent for a two-sided ideal I:*

(i) *I is a 1-prime submodule of R.*

(ii) *I is a 2-prime submodule of R.*

(iii) *If K, L are two-sided ideals with $KL \subseteq I$ then either $K \subseteq I$ or $L \subseteq I$.*

Proof. (i) is equivalent to (ii) by 2.6, 2.7 and (ii) is equivalent to (iii) by 2.11.

A two-sided ideal satisfying the equivalent conditions of 2.12 will be called *prime ideal*. This coincides with the usual notion of prime ideal. Thus, for a left ideal I, the expressions "I is a prime module" and "I is a prime ideal" indicate two different situations, the latter denoting simply the fact that I is a two-sided ideal satisfying the conditions of 2.12.

2.13. Proposition. *Let M be a prime module and $I = (0{:}M)$. Then*

(i) *if $P \xrightarrow{f} M$ is a projective presentation of M then P/IP is a prime module,*

(ii) *I is a prime ideal.*

Proof. (i) Let C be the largest characteristic submodule of P contained in $\operatorname{Ker} f$. Clearly, $C = KP$ for some two-sided ideal K. However, IP is a characteristic submodule of P contained in $\operatorname{Ker} f$ and $KM = 0$. Thus $IP = KP$ and we can use 2.8.

(ii) There is a free presentation $F \to M$. By (i), F/IF is prime. However, R/I is a submodule of F/IF. Thus I is 1-prime in R and, being two-sided, I is a prime ideal.

2.14. Corollary. *The following are equivalent for a ring R:*

(i) *R is a prime ring* (i.e., *0 is a prime ideal*).

(ii) *R is a prime module.*

(iii) *R has a faithful prime module.*

(iv) *Every submodule of a projective module is a prime module.*

2.15. Proposition. *Let R be a left duo-ring. Then the following are equivalent for a module M:*

(i) *M is prime.*

(ii) *There is a prime ideal I such that M is isomorphic to a submodule of a direct product of copies of R/I.*

Proof. Apply 2.13 and 2.4(i).

3. Prime radical. Let $\mathscr{M}$ be the class of all prime modules. We define the prime radical $\mathscr{P}$ on R-mod by $\mathscr{P} = p^{\mathscr{M}}$.

3.1. PROPOSITION. (i) *$\mathscr{P}$ is a radical, $\mathscr{Y} \cap \mathscr{J} \subseteq \mathscr{P}$ and $\widetilde{\mathrm{Soc}} \cap \mathscr{J} \subseteq \mathscr{P} \subseteq \mathscr{J}$.*

(ii) *For every module M, $\mathscr{P}(M)$ coincides with the intersection of all 1-prime submodules of M.*

(iii) *For every projective module M, $\mathscr{P}(M)$ coincides with the intersection of all characteristic 1-prime submodules.*

(iv) *For every projective module M, $\mathscr{P}(M)$ coincides with the intersection of all 2-prime submodules.*

Proof. (i) Obviously, $\mathscr{P}$ is a radical and the inclusion $\mathscr{P} \subseteq \mathscr{J}$ follows immediately from the fact that every simple module is prime. If M is a prime module and $\widetilde{\mathrm{Soc}}\, M \neq 0$ then M has a non-zero simple submodule S and M is isomorphic to a submodule of a direct product of copies of S. Since $\mathscr{J}(S) = 0$, we immediately have $\mathscr{J}(M) = 0$. Thus $\widetilde{\mathrm{Soc}}\, N \cap \mathscr{J}(N) = 0$ for every $N \in \mathscr{M}$ and consequently, $\widetilde{\mathrm{Soc}} \cap \mathscr{J}$ being a radical, $\widetilde{\mathrm{Soc}} \cap \mathscr{J} \subseteq \mathscr{P}$. Finally, suppose that M is prime and $\mathscr{J}(M) \neq 0$. Since $\mathscr{J}(M)$ is the sum of all small submodules, there is a non-zero submodule N small in M. Then obviously $N \in \mathscr{B}$ and consequently $\mathscr{Y}(M) = 0$, M being isomorphic to a submodule of a direct product of copies of N. We have proved that for every $M \in \mathscr{M}$, $\mathscr{J}(M) \cap \mathscr{Y}(M) = 0$, and hence $\mathscr{J} \cap \mathscr{Y} \subseteq \mathscr{P}$.

The remaining assertions follow immediately from 2.6, 2.7, 2.8 and the fact that $\mathscr{M}$ is closed under submodules.

3.2. PROPOSITION. (i) *$\mathscr{P}(R)$ is the intersection of all prime ideals.*

(ii) *$\mathscr{P}(R)$ is a nil-ideal and $\mathscr{P}(R) = \bigcap (0{:}M)$, $M \in \mathscr{M}$.*

(iii) *$\mathscr{P}(R)$ contains all left T-nilpotent left (right) ideals as well as all right T-nilpotent left (right) ideals.*

Proof. (i) and (ii) are obvious.

(iii) First, let I be a right T-nilpotent left ideal and K be a prime ideal. Suppose that there is $x_0 \in I \setminus K$. Then, for some $a_0 \in R$, $x_1 = x_0 a_0 x_0 \notin K$. Proceeding in this way, we get the elements $x_{i+1} = x_i a_i x_i$ that are not in K. On the other hand, observing the sequence $a_0 x_0, a_1 x_1, \ldots$ and using the right T-nilpotency of I we see that $x_n = 0$ for some n, a contradiction. We have proved that $I \subseteq \mathscr{P}(R)$. If I is a right T-nilpotent right ideal then Ra is right T-nilpotent for all $a \in I$. The rest is similar.

3.3. PROPOSITION. *Let r be a preradical such that $\bar{r} =$ zer. Then*

(i) *$r(M)$ is small in M for every $M \in R$-mod,*

(ii) *$r \subseteq \mathscr{J}$,*

(iii) *$r(R)$ is left T-nilpotent and $r(R) \subseteq \mathscr{P}(R)$,*

(iv) *if r is cohereditary then $r \subseteq \mathscr{P}$.*

Proof. (i) Let $r(M)+N = M$ and $f\colon M \to M/N$ be canonical. Then $M/N = f(r(M)) \subseteq r(M/N)$. Thus M/N is r-torsion, and so $N = M$.

(ii) Obviously, $\mathscr{J}(M)$ coincides with the sum of all small submodules.

(iii) Let $a_1, a_2, \ldots \in r(R)$ be arbitrary, F be a free module with countable infinite basis $x_1, x_2, \ldots$, $y_i = x_i - a_i x_{i+1}$, $i = 1, 2, \ldots$, A_i be the submodule of F generated by $y_1, \ldots, y_i$ and $A = \bigcup_{i=1}^{\infty} A_i$. Obviously, $A + r(F) = F$ and (i) yields $A = F$. Hence $x_1 = b_1 x_1 - b_1 a_1 x_2 + b_2 x_2 - b_2 a_2 x_3 + \ldots + b_n x_n - b_n a_n x_{n+1}$ for some $b_1, \ldots, b_n \in R$. Consequently $b_1 = 1$, $b_2 = a_1$, $b_3 = a_1 a_2, \ldots, b_n = a_1 a_2 \ldots a_{n-1}$ and $a_1 a_2 \ldots a_n = 0$. Hence $r(R)$ is left T-nilpotent and $r(R) \subseteq \mathscr{P}(R)$ by 3.2.

(iv) This follows immediately from (iii).

3.4. PROPOSITION. *Let r be a preradical such that $r^n =$ zer for some n. Then $r \subseteq \mathscr{P}$.*

Proof. Let M be a prime module and $N = r^k(M) \neq 0$ for some $k \geqslant 1$. Then $p^N(M) = 0$, i.e. there is $f\colon M \to N$ with $f(r(M)) \neq 0$. Consequently, $0 \neq r(N) = r^{k+1}(M)$. However $r^n(M) = 0$, and hence $r(M) = 0$ and $r \subseteq \mathscr{P}$.

3.5. PROPOSITION. *Let M be a projective module with $\mathscr{P}(M) = 0$. Then*

(i) *$A *_M A \neq 0$ for every $0 \neq A \subseteq M$,*

(ii) *$A \cap B = 0$, provided $A, B \subseteq M$ and $A *_M B = 0$,*

(iii) *M is prime, provided that every non-zero submodule of M is essential.*

Proof. (i) Let $0 \neq A \subseteq M$. There are $P \in \mathscr{M}$ and $f\colon M \to P$ such that $f(A) \neq 0$. Since P is prime, there is $g\colon P \to f(A)$ with $g(f(A)) \neq 0$. However M is projective, and consequently there is $h\colon M \to A$ such that $f(h(m)) = g(f(m))$ for each $m \in M$. Now $0 \neq h(A) \subseteq A *_M A$.

(ii) We have $(A \cap B) *_M (A \cap B) \subseteq A *_M B = 0$. By (i), $A \cap B = 0$.

(iii) Apply (ii) and 2.5.

3.6. PROPOSITION. *The following are equivalent for a ring R:*

(i) *R is a left V-ring.*

(ii) *$\mathscr{J}$ is hereditary.*

(iii) *$\mathscr{J} =$ zer.*

(iv) *$\mathscr{P} =$ zer.*

(v) *$\mathscr{P}$ is hereditary.*

Proof. (i) implies (ii). Since $\mathscr{J}$ is a radical, it suffices to show that $\mathscr{J}(M) = 0$ implies $\mathscr{J}(E(M)) = 0$. However $\mathscr{J} = p^{\mathscr{S}}$ and every simple module is injective.

(ii) implies (iii). If $x \in \mathscr{J}(M)$ then $\mathscr{J}(R/(0{:}x)) = R/(0{:}x)$, $\mathscr{J}$ being hereditary, and so $(0{:}x) = R$ and $x = 0$.

(iii) implies (iv) and (iv) implies (v) trivially.

(v) implies (i). Let M be a non-zero simple module. If $E(M)$ is not prime then $M \subseteq \mathscr{P}(E(M))$, and hence $M = \mathscr{P}(M) \subseteq \mathscr{J}(M) = 0$, a contradiction. Thus $E(M)$ is prime, hence $p^M(E(M)) = 0$ and there is $f\colon E(M) \to M$ with $f(M) \neq 0$. Clearly, f is an isomorphism.

3.7. PROPOSITION. *The following are equivalent for a ring R:*

(i) *$\mathscr{P} = \mathscr{J}$ is cohereditary.*

(ii) *$\mathscr{P}$ is cohereditary.*

(iii) *$R/\mathscr{P}(R)$ is a left V-ring.*

Proof. (i) implies (ii) trivially.

(ii) implies (iii). Clearly, $\mathscr{P}$ is cohereditary on $R/\mathscr{P}(R)$-mod and $\mathscr{P}(R/\mathscr{P}(R)) = 0$. Thus $\mathscr{P} =$ zer on $R/\mathscr{P}(R)$-mod and we can apply 3.6.

(iii) implies (i). Since $\mathscr{J}(R/\mathscr{P}(R)) = 0$, $\mathscr{J}(R) = \mathscr{P}(R)$. Further, $\mathscr{J} = \mathscr{P} =$ zer on $R/\mathscr{P}(R)$-mod. If $M \in R$-mod and $\mathscr{P}(M) = 0$, $\mathscr{P}(R)M = 0$ and M is an $R/\mathscr{P}(R)$-module. Now, we see that the class of all $\mathscr{P}$-torsionfree modules and the class of all $\mathscr{J}$-torsionfree modules are closed under factormodules. Since $\mathscr{P}$ and $\mathscr{J}$ are radicals and $\mathscr{P}(R) = \mathscr{J}(R)$, $\mathscr{P} = \mathscr{J}$ is cohereditary.

The following lemma is clear.

3.8. Lemma. *Let r be a splitting radical such that 0 is the only cyclic r-torsion module. Then every r-torsion module is injective.*

3.9. Corollary. *The following are equivalent:*

(i) *$\mathscr{P}$ is splitting.*

(ii) *$\mathscr{P}$ is idempotent and every $\mathscr{P}$-torsion module is injective.*

3.10 Corollary. *The following are equivalent:*

(i) *$\overline{\mathscr{P}}$ is splitting.*

(ii) *Every $\mathscr{P}$-torsion module is injective.*

3.11. Proposition. *Let R be a ring such that every module is prime. Then R is isomorphic to a matrix ring over a skew-field.*

Proof. First, R is a left V-ring by 3.6. Further, $p^M =$ zer for every non-zero module M. In particular, for every simple module S, $p^R(S) = 0$ and S is isomorphic to a left ideal. Thus every simple module is projective and R is completely reducible. Finally, R is a simple ring e.g. by 2.11.

4. Coprime modules. Let A, B be two submodules of a module M. Put $A\square_M B = \bigcap f^{-1}(A)$, $f \in \operatorname{Hom}(M, M)$, $f(B) = 0$.

4.1. Lemma. *Let A, B, C be submodules of a module M. Then*

(i) *$A\square_M B$ is a submodule of M and $B \subseteq A\square_M B$,*

(ii) *$M\square_M A = M = A\square_M M$,*

(iii) *$A\square_M 0$ is the largest characteristic submodule of M contained in A,*

(iv) *$A\square_M(B\square_M C) \subseteq (A\square_M B)\square_M C$,*

(v) *$A\square_M(B\square_M C) = (A\square_M B)\square_M C$, provided M is injective and artinian.*

Proof. (i), (ii) and (iii) are obvious.

(iv) Let h_i, $i \in I$, be the system of all endomorphism of M such that $h_i(C) = 0$. Put $Y = \bigcap_{i\in I} h_i^{-1}(B)$. Clearly, $x \in (A\square_M B)\square_M C$ iff $fh_i(x) \in A$ for all $i \in I$ and all $f \in \operatorname{Hom}(M, M)$ with $f(B) = 0$. Similarly, $x \in A\square_M(B\square_M C)$ iff $k(x) \in A$ for every $k \in \operatorname{Hom}(M, M)$ with $k(Y) = 0$. Hence $A\square_M(B\square_M C) \subseteq (A\square_M B)\square_M C$.

(v) Since M is artinian, there is a finite subsystem $h_1, \dots, h_n$ such that

$Y = \bigcap_{j=1}^{n} h_j^{-1}(B)$. Let $x \in (A\square_M B)\square_M C$, $k\colon M \to M$ be such that $k(Y) = 0$, $g\colon M \to M/Y$ be canonical and $t\colon M/Y \to \prod_1^n M/B$ be defined by $t(a+Y) = \langle h_1(a)+B, \dots, h_n(a)+B\rangle$. Clearly, t is a monomorphism and $k = qtg$ for some $q\colon \prod M/B \to M$. Finally, let $u_1, \dots, u_n$ be canonical homomorphisms from M into $\prod_1^n M/B$. Thus $k(x) = qtg(x) = qu_1h_1(x)+\dots+qu_nh_n(x)$. However, $qu_j \in \operatorname{Hom}(M, M)$, $qu_j(B) = 0$, $h_j(C) = 0$, and hence $k(x) \in A$.

4.2. Lemma. *Let I, K be left ideals. Then $I\square_R K = (I\colon (0:K)_r)_l$.*

Proof. Obvious.

A module M is called *coprime* if $p_{M/N} = p_M$ for every submodule $N \subsetneqq M$.

4.3. Proposition. *The following are equivalent for a module M:*

(i) *$A\square_M B \neq M$ for all submodules $A, B \subsetneqq M$.*

(ii) *$p_{M/N}(M) = M$ for every submodule $N \subsetneqq M$.*

(iii) *If $N \subsetneqq M$ then M is a homomorphic image of a direct sum of copies of M/N.*

(iv) *M is coprime.*

Proof. (i) implies (ii). Let $N \subsetneqq M$ be such that $p_{M/N}(M) = K \neq M$. Then $K\square_M N \neq M$, and consequently there are $x \in M$ and $f\colon M \to M$ such that $f(N) = 0$ and $f(x) \notin K$. Hence $g\colon M/N \to M$ defined via $g(m+N) = f(m)$ is a homomorphism and $g(x) \notin K$, a contradiction.

(ii) implies (iii). Since $p_{M/N}(M) = M$, $M = \sum f(M/N)$, $f \in \operatorname{Hom}(M/N, M)$, and consequently M is a homomorphic image of a direct sum of copies of M/N.

(iii) implies (iv). Obviously, $p_{M/N} \subseteq p_M$ and $p_{M/N}(M/N) = M/N$. Thus $p_{M/N}(M) = M$ and consequently $p_M \subseteq p_{M/N}$.

(iv) implies (i). Let $A, B \subsetneqq M$ and $A\square_M B = M$. If $g\colon M \to M/B$ is canonical and $f \in \operatorname{Hom}(M/B, M)$ is arbitrary then $fg(M) \subseteq A$. Hence $p_{M/B}(M) \subseteq A$, a contradiction.

4.4. Proposition. (i) *A module M is coprime iff $p_M = p_{M/N}$ for every non-zero cocyclic factormodule M/N of M.*

(ii) *Every direct sum of copies of a simple module is a coprime module.*

Proof. Obvious.

4.5. Proposition. *Let N be a characteristic submodule of a module M. Suppose that $N \subsetneqq A\square_M B$, for all submodules $A, B \subsetneqq N$. Then N is a coprime module.*

Proof. Let $K, L \subsetneqq N$ be arbitrary and $f\colon M \to M$ be such that $f(K) = 0$ and $f(N) \nsubseteq L$. Then f induces a homomorphism $g\colon N/K \to N$ such that $\operatorname{Im} g \nsubseteq L$, and consequently $p_{N/K}(N) = N$.

4.6. Proposition. *Let N be a coprime module and M be an injective module containing N. Then $N \nsubseteq A\square_M B$, whenever $A, B \subseteq M$, $N \nsubseteq A$ and $N \nsubseteq B$.*

Proof. Suppose, on the contrary, that $N \nsubseteq A$, $N \nsubseteq B$ and $N \subseteq A\square_M B$. Then there is $f\colon N/N\cap B\to N$ with $f(N/N\cap B)\nsubseteq N\cap A$. If we put $h(x+y)=f(x+(N\cap B))$ for all $x\in N$, $y\in B$, then h is a well-defined homomorphism of $N+B$ into M such that $h(B)=0$ and $h(N)\nsubseteq A$. Now the injectivity of M gives rise to $g\in \mathrm{Hom}(M, M)$ extending h, a contradiction.

4.7. PROPOSITION. *Let N be a coprime module contained in an artinian injective module M and C be the least characteristic submodule of M containing N. Then C is a coprime module.*

Proof. Dual to that of 2.8.

4.8. LEMMA. *Let M be a coprime module with $\mathscr{J}(M)\neq M$. Then M is completely reducible.*

Proof. If $\mathscr{J}(M)\neq M$ then there is a submodule $N\subsetneqq M$ such that M/N is simple. Hence $M=p_M(M)=p_{M/N}(M)\subseteq \mathrm{Soc}\, M$.

4.9. LEMMA. *Let M be a non-zero coprime module and I be a left ideal. Then $IM\neq 0$ iff $IM=M$.*

Proof. Let $IM\neq M$. Then $p_{M/IM}(M)=p_M(M)=M$, so that $IM=0$.

4.10. COROLLARY. *Every coprime module is completely reducible, provided at least one of the following conditions holds:*

(i) *Every non-zero module has a proper maximal submodule.*

(ii) *Every non-zero module has a non-zero minimal submodule.*

Proof. (i) Apply 4.8.

(ii) Let M be a coprime module and $I=(0:M)$. Then M is coprime as an R/I-module and $\mathrm{Soc}(R/I)M\neq 0$, so that Lemma 4.9 yields that M is completely reducible.

4.11. PROPOSITION. *Every module is coprime iff R is isomorphic to a matrix ring over a skew-field.*

Proof. If every non-zero module is coprime then obviously $p_M=\mathrm{id}$ for each $0\neq M\in R$-mod, and consequently R is completely reducible. Finally, if $I\neq R$ is a two-sided ideal then $p_{R/I}(R)=R$, and so $I=0$. The converse implication is obvious.

5. Preradical $\mathscr{R}$. Let $\mathscr{N}$ be the class of all coprime modules. Put $\mathscr{R}=p_{\mathscr{N}}$.

5.1. PROPOSITION. (i) *$\mathscr{R}$ is an idempotent preradical, $\mathrm{Soc}\subseteq\mathscr{R}\subseteq\mathscr{J}+\mathrm{Soc}$ and $\mathscr{R}\subseteq\mathscr{Z}+\mathrm{Soc}$.*

(ii) *For every module M, $\mathscr{R}(M)$ coincides with the sum of all coprime submodules of M.*

Proof. (i) Obviously, $\mathscr{R}$ is an idempotent preradical and $\mathrm{Soc}\subseteq\mathscr{R}$. Further, the inclusion $\mathscr{R}\subseteq\bar{\mathscr{J}}+\mathrm{Soc}$ follows immediately from 4.8. Finally, if M is a coprime module with $\mathrm{Soc}\, M\neq M$ then there is an essential submodule $N\subsetneqq M$. However $\mathscr{Z}(M/N)=M/N$ and consequently, M being coprime, $\mathscr{Z}(M)=M$.

(ii) It follows immediately from the fact that $\mathscr{N}$ is closed under factormodules.

5.2. COROLLARY. *Let M be a finitely generated module with $M=\mathscr{R}(M)$. Then M is completely reducible.*

Proof. We have $M=\bar{\mathscr{J}}(M)+\mathrm{Soc}\, M$ by Proposition 5.1(i). However, $\bar{\mathscr{J}}(M)\subseteq\mathscr{J}(M)$ is a small submodule of M.

5.3. LEMMA. *Let r be a preradical and $K_r=\bigcap(0:m)$, $m\in r(M)$, $M\in R$-mod. Then*

(i) *K_r is a two-sided ideal and $K\subseteq\bigcap I$, $r(R/I)=R/I$,*

(ii) *if r is hereditary then $K_r=\bigcap I$, $r(R/I)=R/I$,*

(iii) *if we denote by $\mathrm{sh}\, r$ the superhereditary preradical corresponding to K_r then $r\subseteq\mathrm{sh}\, r$ and $\mathrm{sh}\, r$ is the smallest superhereditary preradical containing r.*

Proof. Obvious.

5.4. PROPOSITION. (i) *$K_{\mathscr{R}}$ is equal to the intersection of $(0:M)$, where M runs through all coprime modules.*

(ii) *$K_{\mathscr{R}}\subseteq\mathscr{J}(R)$.*

(iii) *$K_{\mathscr{R}}$ contains every left T-nilpotent left (right) ideal.*

Proof. (i) and (ii) are obvious.

(iii) Let I be a left T-nilpotent left ideal. If $I\nsubseteq K_{\mathscr{R}}$ then, with respect to 4.9, $IM=M$ for some non-zero coprime module M. On the other hand, IM is small in M, a contradiction. If I is a left T-nilpotent right ideal then Ra is left T-nilpotent for each $a\in I$.

5.5. PROPOSITION. *Let r be a preradical such that $\bar{r}=\mathrm{id}$. Then*

(i) *$r(M)$ is essential in M for all $M\in R$-mod,*

(ii) *$\mathrm{Soc}\subseteq r$,*

(iii) *K_r is right T-nilpotent.*

Proof. (i) If $M\in R$-mod, $N\subseteq M$ and $r(M)\cap N=0$ then $r(N)=0$, hence $\bar{r}(N)=0$ and so $N=0$.

(ii) It follows from (i), using the fact that for each $M\in R$-mod, $\mathrm{Soc}\, M$ is the intersection of all essential submodules.

(iii) Let $I=\{x\in R\mid$ for all $a_1, a_2, \ldots\in K_r$ there is $n\geqslant 1$ with $a_n\ldots a_2a_1x=0\}$. Obviously $r(R/I)=0$, hence $I=R$ and $1\in I$.

5.6. PROPOSITION. *Let r be a preradical such that $r_n=\mathrm{id}$ for some n. Then $\mathscr{R}\subseteq r$.*

Proof. Let M be a coprime module and $N=r_k(M)\neq M$ for some $k\geqslant 1$. Then $p_{M/N}(M)=M$ yields $f(M/N)\nsubseteq r(M)$ for some $f\colon M/N\to M$, and consequently $r(M/N)\neq M/N$. Thus $r_{k+1}(M)\neq M$, which is a contradiction.

5.7. PROPOSITION. *Soc is superhereditary iff $R/\mathscr{J}(R)$ is completely reducible.*

Proof. If Soc is superhereditary then $R/\mathscr{J}(R)=R/\bigcap I\subseteq\prod R/I$, I runs over all maximal left ideals, is completely reducible. Conversely, if $R/\mathscr{J}(R)$ is completely reducible then every $R/\mathscr{J}(R)$-module is completely reducible, however every direct product of completely reducible modules is an $R/\mathscr{J}(R)$-module.

5.8. Proposition. *The following are equivalent*:

(i) $\mathscr{R}$ *is cohereditary.*

(ii) $\mathscr{R} = \mathrm{id}$.

(iii) R *is completely reducible.*

Proof. (i) implies (ii). Suppose $\mathscr{R} \neq \mathrm{id}$. Then $I = \mathscr{R}(R) \neq R$. Since $\mathscr{R}$ is cohereditary, every non-zero factormodule of R/I is $\mathscr{R}$-torsionfree, a contradiction.

(ii) implies (iii) by Corollary 5.2.

(iii) implies (i) trivially.

5.9. Proposition. *The following are equivalent*:

(i) $\mathscr{R}$ *is superhereditary.*

(ii) $R/K_{\mathscr{R}}$ *is completely reducible.*

(iii) $\mathscr{R} = \mathrm{Soc}$ *is superhereditary.*

In this case, $K_{\mathscr{R}} = \mathscr{J}(R)$.

Proof. (i) implies (ii). By Lemma 5.3, the two-sided ideal corresponding to $\mathscr{R}$ is just $I = K_{\mathscr{R}}$. Consequently, $\mathscr{R}(R/I) = R/I$, $\mathscr{R} = \mathrm{id}$ for R/I-modules and we can apply Proposition 5.8.

(ii) implies (iii). Clearly, $K_{\mathscr{R}} = \mathscr{J}(R)$. By Proposition 5.7, Soc is superhereditary and the corresponding ideal is $\mathscr{J}(R) = K_{\mathscr{R}}$. Hence $\mathscr{R} \subseteq \mathrm{Soc}$, while the converse inclusion always holds.

5.10. Proposition. *The following are equivalent*:

(i) $\mathscr{R}$ *is hereditary.*

(ii) $\mathscr{R} = \mathrm{Soc}$.

(iii) *Every coprime module is completely reducible.*

Proof. Use 5.2.

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