Čech extensions and localization of homotopy functors

by

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Abstract. The localized Kan extensions of Deleanu and Hilton are shown to be "β-representable". That is, they are of the form $[\beta—, Y]$, where β denotes Stone–Čech compactification. The classifying spaces are computed. Finally, the general theory of β-representable cofunctors is discussed.

Introduction. Let h be a cohomology theory defined on the category of finite CW-complexes. In [7], [8] Deleanu and Hilton consider "localized" (at a class of primes) Kan extensions of h to a larger category $\mathscr{T}_1$. When $\mathscr{T}_1$ is the category of 1-connected finite dimensional CW-complexes and h has finitely generated coefficients, the extensions are well behaved. Here, they are shown to be prorepresentable though not necessarily representable ([7] 4.1–9).

In [3], [4], [5] Allan Calder and this author explored the relationship between uniform homotopy and homotopy under hypothesis similar to those mentioned above. It is the purpose of the present paper to make exact the relationship between this work and that of Deleanu and Hilton. This will be seen to clarify the nature of such localized Kan extensions.

More specifically, suppose we are given a category of topological spaces $\mathscr{T}$, a full subcategory $\mathscr{P}$ and a set valued homotopy confunctor F on $\mathscr{P}$. Let $\tilde{\mathscr{T}}$ (resp. $\tilde{\mathscr{P}}$) denote the quotient categories of spaces and homotopy classes of maps. F may be considered as a functor on $\tilde{\mathscr{P}}$ and its (right) Kan extension, $F^{\tilde{\mathscr{P}}}$ to $\tilde{\mathscr{T}}$ taken [9]. It is this construction that is usually called the Kan extension of F ([7], [9]).

Alternatively, one extends directly from $\mathscr{P}$ to $\mathscr{T}$. We will denote this extension by $F^{\mathscr{P}}$ and call it the Čech extension of F. This name is justified since, as was shown in [3], [5], for the "classical" pairs $(\mathrm{Comp}\,T_2, \mathrm{fPol})$ $(\mathrm{CReg}\,T_2, \mathrm{fPol})$, $(\mathrm{Top}, \mathrm{lfPol})$ etc. $F^{\mathscr{P}}$ coincides with the usual Čech extension by families of covers.

We will be interested in determining the relationship between $F^{\mathscr{P}}$ and $F^{\tilde{\mathscr{P}}}$ for pairs such as those considered by Deleanu and Hilton. It is worth noting in passing that for such pairs a definition of Čech extension by families of covers does not seem appropriate.

As an example, let π be a Serre class of finite abelian groups determined by a set of primes. Let $\mathscr{S}_\pi$ denote the category of 1-connected finite simplicial complexes with homology in π. Finally, let fdNorm denote the category of finite dimensional normal spaces and hfdNorm $\to$ fdNorm, the quotient functor homotopy. We show:

THEOREM 1. *Let F be a homotopy functor on $\mathscr{S}_\pi$. Then for the pair* (fdNorm, $\mathscr{S}_\pi$) *we have*

$$F^{\mathscr{S}_\pi} = F^{\tilde{\mathscr{S}}_\pi} h .$$

In particular $F^{\mathscr{S}_\pi}$ is a homotopy functor.

Next, for a given 1-connected CW-complex, let F_B denote the confunctor $[-, B]$ on $\mathscr{S}_\pi$. Let π' be the complement of π in the set of primes and let $B \xrightarrow{e} B_\pi$, denote the localization of B at π' (see for example [8]). Let B^π denote the 1-connected covering of the fibre of the map e. We show:

THEOREM 2. *For the pair* ($\mathrm{CReg}T_2$, $\mathscr{S}_\pi$) *we have that*

$$F_B^{\mathscr{S}_\pi} = [\beta -, B^\pi] .$$

Where β denotes Stone–Čech compactification.

Finally, combining 1 and 2 we have

THEOREM 3. *For the pair* (fdNorm, $\mathscr{S}_\pi$)

$$F_B^{\tilde{\mathscr{S}}_\pi} = [\beta -, B^\pi] .$$

Moreover, $F_B^{\tilde{\mathscr{S}}_\pi}$ is half-exact carrying Puppe sequences into long exact sequences.

Theorem 3 provided an alternative to pro-representability. We will call cofunctors of the form $[\beta -, Y]$ "β-representable". In the third section of this paper we study the following general problem:

Given a subcategory $\mathscr{P} \subseteq$ fPol and a homotopy cofunctor F on $\mathscr{P}$, when is its Čech extension to $\mathrm{CReg}T_2$ β-representable?

We prove the following:

THEOREM 4. *Let $\mathscr{P} \subseteq$ fPol be closed with respect to homotopy type, finite products and equalizers. Let F satisfy the Mayer–Vietoris axiom and the wedge axiom* [1]. *Then if F is*

(f) *Abelian group valued or*

(2) *countable set valued*

its Čech extension to $\mathrm{CReg}T_2$ *is β-representable.*

An appropriate generalization of Theorem 1 is also presented.

Finally, we would like to thank Allan Calder for his help in clarifying several points raised in this paper.

§ 1. Preliminaries.

1.1. NOTATION. Throughout this paper we will be considering set valued cofunctors on various categories and forming (right) Kan extensions [13] to certain larger categories. For the situations we consider these extensions always exist (see [3] or [5]). It will be necessary to keep track of the category over which these extensions are formed, thus we introduce the following notation.

Let $F\colon \mathscr{P} \to$ Set be as above. Let $\mathscr{P} \subseteq \mathscr{T}$. Denote the (right) Kan extension of F to $\mathscr{T}$ by

$$F^{\mathscr{P}}\colon \mathscr{T} \to \mathrm{Set} .$$

We will be considering situations of the following sort. Let $\mathscr{P}_0 \subseteq \mathscr{P}_1 \subseteq \mathscr{T}$. Let $F\colon \mathscr{P}_1 \to$ Set be as above. Considering F as a cofunctor on $\mathscr{P}_0$ as well, we will want to know when $F^{\mathscr{P}_0} \equiv F^{\mathscr{P}_1}$. The following test will suffice for our applications.

1.2. LEMMA [7]. *Suppose*:

1) *For every $X \in |\mathscr{P}_1|$ (objects of $\mathscr{P}_1$) and $x \in F(X)$, there exists $Y \in |\mathscr{P}_0|$, $y \in F(Y)$ and a map $f\colon X \to Y$ in $\mathscr{P}_1$ such that $F(f)y = x$.*

2) *For every $Y_1, Y_2 \in |\mathscr{P}_0|$, $y_i \in F(Y_i)$ and maps $f_i\colon X \to Y_i$ with $X \in |\mathscr{P}_1|$, and $F(f_i)y_i = F(f_2)y_2$, there exists $Y \in |\mathscr{P}_0|$, $y \in F(y)$ and $g_i \in \mathrm{Mor}_{\mathscr{P}_0}(Y_i, Y)$ with $y_i = F(g_i)y$.*

Then

$$F^{\mathscr{P}_0} = F^{\mathscr{P}_1} .$$

The proof is a simple application of the point-wise definition of Kan extension.

We now wish to review the results of [3], [4], [5]. We do not require F to be set valued.

1.3. DEFINITION.

a) Given a category $\mathscr{A}$, a *congruence* [11] on $\mathscr{A}$ may be thought of as a functor $R\colon \mathscr{A} \to \tilde{\mathscr{A}}$ such that $|\mathscr{A}| = |\tilde{\mathscr{A}}|$, R is the identity on objects, and for each $X, Y \in |\mathscr{A}|$ we have that $R\colon \mathrm{Mor}_{\mathscr{A}}(X, Y) \to \mathrm{Mor}_{\tilde{\mathscr{A}}}(X, Y)$ is onto.

b) Given a congruence R in $\mathscr{A}$, a (co)functor F on $\mathscr{A}$ is called an *R-(co)functor* if $F = \tilde{F}R$ for some $\tilde{F}$ on $\tilde{\mathscr{A}}$.

c) Given categories $\mathscr{P} \subseteq \mathscr{T}$ and a congruence R in $\mathscr{P}$, we defined the *codeterminate extension* of R to $\mathscr{T}$ ([5] 1.5). This is a congruence $R^{\mathscr{P}}$ in $\mathscr{T}$. When $\mathscr{P}$ is a full subcategory of $\mathscr{T}$, $R^{\mathscr{P}}$ may be defined as follows.

Let $f, g \in \mathrm{Mor}_{\mathscr{T}}(X, Y)$. Set $R^{\mathscr{P}}(f) = R^{\mathscr{P}}(g)$ if and only if for every R-functor F on $\mathscr{P}$ we have that $F^{\mathscr{P}}(f) = F^{\mathscr{P}}(g)$ ([5] 1.12). Note that again we assume that the Kan extensions exist.

We will make use of the following application of the notion of codeterminate extension.

Let F be an R-functor on a category $\mathscr{P} \subseteq \mathscr{T}$. Let $\tilde{\mathscr{P}}$ and $\tilde{\mathscr{T}}$ be the quotient categories under a congruence $\tilde{R} \subseteq R^{\mathscr{P}}$ ($\tilde{R}(f) = \tilde{R}(g) \Rightarrow R^{\mathscr{P}}(f) = R^{\mathscr{P}}(g)$). We have

1.4. THEOREM ([5] 1.13). $F^{\mathscr{P}} = \tilde{F}^{\tilde{\mathscr{P}}} \circ \tilde{R}$. *That is, $F^{\mathscr{P}}$ may be computed on the quotient category.*

We will usually write $F^{\tilde{\mathscr{P}}}$ for $\tilde{F}^{\tilde{\mathscr{P}}}$ since the meaning is unambiguous.

In [3], [5] it was shown how one might compute the codeterminate extension of various congruences. The following theorem was the basis for those computations.

1.5. THEOREM ([5] 1.7). *Given a congruence R in $\mathcal{P}\subseteq\mathcal{T}$ and $f, g\in \mathrm{Mor}_{\mathcal{T}}(X, Y)$, we have that $R^{\mathcal{P}}(f) = R^{\mathcal{P}}(g)$ if and only if for every $P\in|\mathcal{P}|$ and $\pi\colon Y\to P$ there exists $Q_1, Q_2, \ldots, Q_n\in|\mathcal{P}|$, $\varphi_i\in \mathrm{Mor}_{\mathcal{T}}(X, Q_i)$ and $\pi_{i,j}\in \mathrm{Mor}_{\mathcal{P}}(Q_i, P)$ $(j = 0, 1)$ such that:*

1) $\pi f = \pi_{1,0}\varphi_1, \pi g = \pi_{n,1}\varphi_n$,

2) $\pi_{i,1}\varphi_i = \pi_{i+1,0}\varphi_{i+1}$, *all* i,

3) $R(\pi_{i,0}) = R(\pi_{i,1})$, all i.

The following definition and theorem is typical of the applications in [3], [5].

1.6. DEFINITION. Let $\mathcal{P}$ be a full subcategory of the category of finite polyhedra. Let $\mathrm{Comp}T_2$ denote the category of compact Hausdorff spaces. Let h denote the congruence homotopy in $\mathcal{P}$. We define a congruence on $\mathrm{Comp}T_2$ called *homotopy over* $\mathcal{P}$ and denoted by $h_{\mathcal{P}}$ by letting $h_{\mathcal{P}}(f) = h_{\mathcal{P}}(g)$ if and only if for every $P\in|\mathcal{P}|$ and $\pi\colon Y\to P$ we have that πf and πg are homotopic.

1.7. THEOREM. *Suppose $\mathcal{P}$ is closed under homotopy type then*

$$h^{\mathcal{P}} = h_{\mathcal{P}}.$$

Proof. By "closed" we mean that if $P\in|\mathcal{P}|$ and if P' is a finite polyhedron homotopy equivalent to P then $P'\in|\mathcal{P}|$.

First, suppose $h^{\mathcal{P}}(f) = h^{\mathcal{P}}(g)$ and $\pi\colon Y\to P$ then 1.5 gives a prescription for a homotopy of πf to πg.

Next, let $F\colon X\times I\to P$ be a homotopy of πf to πg. Let $\hat{P}$ be a closed regular neighborhood of the diagonal in $P\times P$ [11]. Let $\{\mathcal{U}_\alpha\}$ be a finite cover of P such that $\mathcal{U}_\alpha\times\mathcal{U}_\alpha\subseteq\hat{P}$. Using the cover $\{\mathcal{U}_\alpha\}$ and since X is compact we may choose $0 = t_1 < t_2 < \ldots < t_{n+1} = 1$ such that $F_{t_i}\times F_{t_{i+1}}\colon X\to\hat{P}\subseteq P\times P$. Also since the diagonal of $P\times P$ is a deformation retract of $\hat{P}$ we have that the projections p_1 and $p_2\colon \hat{P}\to P$ onto the first and second factors are homotopic.

We now must only list the data required by 1.5. $Q_i = \hat{P}$, $i = 1, \ldots, n$. Since $\hat{P}$ has the homotopy type of P it is in $\mathcal{P}$. Finally, $\varphi_i = F_{t_i}\times F_{t_{i+1}}$ and $\pi_{i,0} = p_1$, $\pi_{i,1} = p_2$, all i.

One verifies that this data satisfies the requirements of 1.5.

As in [3], [5], we may extend 1.7 to $\mathrm{CReg}T_2$ (Completely Regular T_2) as follows. For $X\in|\mathrm{CReg}T_2|$, $P\in|\mathcal{P}|$ and $f\colon X\to P$, let $\beta f\colon \beta X\to P$ be the unique extension of f to its Stone–Čech compactification. Define maps $f, g\colon X\to Y$ in $\mathrm{CReg}T_2$ to be *uniformly homotopic over* $\mathcal{P}$ if and only if for every $P\in|\mathcal{P}|$ and $\pi\colon Y\to P$ we have $\beta(\pi f)\sim\beta(\pi g)$ (see [4]). We denote this relation by $h^{\beta}_{\mathcal{P}}$.

1.8. THEOREM. *Let $\mathcal{P}$ be closed under homotopy type then considering $\mathcal{P}\subseteq\mathrm{CReg}T_2$ we have that*

$$h^{\mathcal{P}} = h^{\beta}_{\mathcal{P}}.$$

In the next section we will find that it is sometimes more convenient to compute our extensions over categories of finite CW-complexes 1.4–8 imply that this is possible as follows.

1.9. Let $\mathcal{P}$ be as in 1.6. Again, let $h\colon \mathcal{P}\to\tilde{\mathcal{P}}$ denote the relation homotopy. Let $\mathcal{P}^c\subseteq\tilde{\mathcal{P}}$ be the category of finite CW-complexes homotopy equivalent to simplicial complexes in $\mathcal{P}$ (homotopy classes of maps). Finally, let F_1 be a functor on $\mathcal{P}^c$ and $F = (F_1|\tilde{\mathcal{P}})\circ h$.

1.10. THEOREM. *On* $\mathrm{CReg}T_2$ *we have that*

$$F_1^{\mathcal{P}^c}\circ h^{\beta}_{\mathcal{P}} = F^{\mathcal{P}}.$$

Proof. Applying 1.8 to 1.4 we have that $F^{\tilde{\mathcal{P}}}\circ h^{\beta}_{\mathcal{P}} = F^{\mathcal{P}}$. Also, for compact spaces homotopy and uniform homotopy coincide, and homotopy equivalences are homotopy over $\mathcal{P}$ equivalences. Thus, every object in $\mathcal{P}^c$ is equivalent to an object in $\tilde{\mathcal{P}}$. By elementary considerations about Kan extensions we have that $F_1^{\mathcal{P}^c} = F^{\mathcal{P}}$.

Finally, we will make use of certain results on representable functors. We introduce the following notation.

For B a space having the homotopy type of a CW-complex, let $F_B = [-, B]$ (homotopy classes of maps into B). Let $f\mathcal{P}$ denote the category of finite polyhedra. We have the following extension of ([9] 3.14 appendix).

1.11. THEOREM ([5] 3.2). *On* $\mathrm{CReg}T_2$ *we have that*

$$F_B^{f\mathcal{P}} = [\beta-, B].$$

In fact, 1.11 also generalizes the classical result which states that the Čech cohomology (finite covers) of a space and its Stone-Čech compactification agree ([11] 9.12). In general, $[\beta-, B]$ is not a homotopy functor. Indeed, Dowker [10] shows that $[\beta R^1, S^1]$ is an uncountable set. However, one has the following result.

1.12. THEOREM ([4] 4.2). *Let $X\in|\mathrm{fdNorm}|$ (finite dimensional normal) and let B be of finite type finite fundamental group. Let $i\colon X\to\beta X$ be the inclusion of X into its Stone–Čech compactification then $i^*\colon [\beta X, B]\simeq[X, B]$ is a bijection.*

§ 2. Examples of β-representable functors.

2.1. NOTATION. In this section we will compute $F_B^{\mathcal{P}}$ for certain subcategories $\mathcal{P}\subseteq f\mathcal{P}$. We will also wish to study the relationship between $F_B^{\mathcal{P}}$ and $F_B^{\tilde{\mathcal{P}}}$ in these situations. For the reasons discussed in the introduction and for clarity, we will call $F_B^{\mathcal{P}}$ the *Čech extension* and call $F_B^{\tilde{\mathcal{P}}}$ the *Kan extension*.

In all our examples $\mathcal{P}\subseteq f\mathcal{P}$ will be a full subcategory all of who is objects are simply connected. We will call such categories *simply connected* at the risk of confusion with other uses of the term.

Finally, again, functors of the form $[\beta-, B]$ will be called *β-representable*.

We begin with a simple observation on 1.12.

2.1. THEOREM. *Let $\mathcal{P}\subseteq f\mathcal{P}$ be simply connected and closed under homotopy type, then on $\mathrm{fdNorm}\, h_{\mathcal{P}} = h^{\beta}_{\mathcal{P}}$. Hence, for any homotopy functor F, $F^{\mathcal{P}}$ is also a homotopy functor and $F^{\tilde{\mathcal{P}}} h_{\mathcal{P}} = F^{\mathcal{P}}$.*

Proof. Let $P \in |\mathscr{P}|$ and $X \in |\mathrm{fdNorm}|$ then by 1.12 we have that maps $f, g\colon X \to P$ are homotopic if and only if βf and βg are homotopic and hence $h_{\mathscr{P}} = h_{\mathscr{P}}^{\beta}$. The result now follows from 1.8 and 1.4.

2.1. essentially says that on fdNorm the Čech and Kan extension from simply connected categories agree. We now give two specific examples of such categories.

Firstly, given B, let $B_n \xrightarrow{p} B$ be the $(n-1)$-connected covering of B. That is, $\pi_i(B_n) = 0$ for $i<n$ and $p_*\colon \pi_i(B_n) \cong \pi_i(B)$ for $i \geqslant n$. Let $\mathscr{P}_n \subseteq f\mathscr{P}$ be the full subcategory of $(n-1)$-connected finite complexes.

2.2. THEOREM. *On* $\mathrm{CReg}T_2$ *we have that*

$$F_B^{\mathscr{P}_n} = F_{B_n}^{f\mathscr{P}} = [\beta -, B_n]\,.$$

Proof. By elementary homotopy theoretic considerations we know that $p_*\colon [P, B_n] \cong [P, B]$ for $P \in |\mathscr{P}_n|$. Hence we have that $F_B^{\mathscr{P}_n} = F_{B_n}^{\mathscr{P}_n}$.

Next, since $h_{\mathscr{P}}^{\beta} \supseteq h^{\beta}$ for any $\mathscr{P} \subseteq f\mathscr{P}$ we may apply 1.4 to conclude $F_{B_n}^{\mathscr{P}_n} = F_{B_n}^{\tilde{\mathscr{P}}_n} h^{\beta}$. Thus, if $\mathscr{P}_n^c$ and $f\mathscr{P}^c$ are the corresponding categories of finite CW-complexes, by 1.10 and 1.11 it suffices to show that $F_{B_n}^{\mathscr{P}_n^c} = F_{B_n}^{f\mathscr{P}^c}$.

The remainder of the proof is a straight forward application of 1.2. Let C be a finite CW-complex. By attaching a finite number of cells in dimensions $\leqslant n$ we may embed C in an $(n-1)$-connected complex C' such that any map $f\colon C \to B_n$ may be factored through the diagram

$$\begin{tikzcd} & C' \arrow[d] \\ C \arrow[ur, "i"] \arrow[r, "f"'] & B_n \end{tikzcd}$$

This is essentially 1) of 1.2.2) follows by applying the above construction to the equalizers of the appropriate maps.

The above proof is obviously an example of a more general procedure. We now review another example, that of Deleanu and Hilton [)].

Let π be a family of primes. Let A_π be the Serre class of π-torsion abelian groups. Let $\mathscr{C}_\pi$ be the category of 1-connected CW-complexes whose homotopy groups belong to A_π. Finally, let $\mathscr{P}_\pi \subseteq f\mathscr{P}$ be the category of 1-connected finite polyhedra in $\mathscr{C}_\pi$.

2.3. LEMMA. *Let* $B \in |C_\pi|$. *Then on* $\mathrm{CReg}T_2$ *we have that*

$$F_B^{\mathscr{P}_\pi} = F_B^{f\mathscr{P}} = [\beta -, B]\,.$$

Proof. As in 2.2, the proof reduces to showing that $F_B^{\tilde{\mathscr{P}}_\pi} = F_B^{f\mathscr{P}^c}$.

Since B is 1-connected we know (2.2) that $F_B^{f\mathscr{P}^c} = F_B^{\mathscr{P}_1^c}$. Thus suffices to show that $F_B^{\mathscr{P}_1^c} = F_B^{\tilde{\mathscr{P}}_\pi}$. This is exactly what is done in [7] 4.15–18 by Deleanu and Hilton.

In order to complete the program of 2.2 we must associate a suitable space in $\mathscr{C}_\pi$ to each 1-connected CW-complex.

Let $\pi \cup \pi'$ be a decomposition of the set of primes into disjoint subsets. For a 1-connected CW-complex B, let $e\colon B \to B_{\pi'}$ be its localization at π' (see for example [8]). Let $\hat{B}^\pi$ be the fibre of the map e and B^π be the 1-connected covering space of $\hat{B}^\pi$.

2.4. LEMMA. $B^\pi \in |\mathscr{C}_\pi|$ *and for* $P \in |\mathscr{P}_\pi|$ *we have that*

$$[P, B^\pi] \cong [P, B]\,.$$

Proof. Since $B_{\pi'}$ has the homotopy of B localized at π' and P has homology in π one has that $H^n(P, \pi_n(B_{\pi'})) = 0$ for all n. Hence all the obstructions for a given map $f\colon P \to B_{\pi'}$ being homotopic to a constant map vanish ([12] 11.3) and thus $[P, B_{\pi'}] = 0$.

Also, $P \in |\mathscr{P}_\pi|$ implies that its suspension $S(P) \in |\mathscr{P}_\pi|$ so $[S(P), B_{\pi'}] = 0$.

We may now apply these two facts to the Eckmann–Hilton homotopy sequence of the fibration $\hat{B}^\pi \xrightarrow{i} B \xrightarrow{e} B_{\pi'}$ to conclude that $i_*\colon [P, \hat{B}^\pi] \simeq [P, B]$.

To check that $\hat{B}^\pi$ has homotopy in A_π one again notes that $B_{\pi'}$ has the homotopy of B localized at π'. An examination of the ordinary homotopy sequence of the fibration shows that $\pi_n(\hat{B}^\pi)$ is in A_n.

Unfortunately, $\hat{B}^\pi$ may not be 1-connected. But since the objects of $\mathscr{P}_\pi$ are one connected we have $[P, B^\pi] \simeq [P, \hat{B}^\pi]$.

Combining 2.3 and 2.4 we have

2.5. THEOREM. *Let* B *a* 1-*connected* CW-*complex. Then on* $\mathrm{CReg}T_2$ *we have*

$$F_B^{\mathscr{P}_\pi} = F_{B^\pi}^{f\mathscr{P}} = [\beta -, B^\pi]\,.$$

We complete this section by reviewing the work of Deleanu and Hilton [7] in the above setting.

By 2.1 in conjunction with 1.10 we have that on fdNorm $F_B^{\mathscr{P}_\pi} = F_B^{\mathscr{P}_\pi^c} h_{\mathscr{P}_\pi}$. Hence, the study of $F^{\mathscr{P}_\pi^c}$ is essentially the study of the properties of $F_B^{\mathscr{P}_\pi} = [\beta -, B^\pi]$.

We first observe that $F_B^{\mathscr{P}_\pi}$ is half exact ([7] 2.14) by establishing the following more general results.

2.6. LEMMA. *Suppose* $[\beta -, Y]$ *is a homotopy cofunctor on* fdNorm. *Let* $A \xrightarrow{i} X \xrightarrow{j} X \cup_i CA$ *be a cofibration in* fdNorm *then*

$$[\beta A, Y] \xleftarrow{i^*} [\beta X, Y] \xleftarrow{j^*} [\beta(X \cup_i CA), Y]$$

is an exact sequence of sets.

Proof. Let $i*[f]=0$. Let $F\colon \beta A\times I\to Y$ be a contraction of f. Using f/X and $F/A\times I$ as data we may construct $g\colon X\cup_i CA\to Y$. But f and F take values in a compact subset of Y hence g extends uniquely to $\beta g\colon \beta(X\cup_i CA)\to Y$ and $j\times[\beta g]=[f]$.

Next let $g\colon \beta(X\cup_i CA)\to Y$. Since $[\beta-, Y]$ is assumed to be a homotopy functor and ji is homotopic to the constant map C, we have $i*j*[g]=(ji)*[g] = C*[g]=0$.

One also wishes to know under what conditions $[\beta X, \Omega Y]\cong[\beta(SX), Y]$ (see [7] § 3). Under the assumption that X and $S(X)$ are finite dimensional normal, one may directly apply 1.12 as follows.

2.7. Lemma. *Let Y be 2-connected and of finite type then*

$$[\beta X, \Omega Y]\cong[\beta(SX), Y].$$

Proof. Since ΩY is simply connected and of finite type we have

$$[\beta X, \Omega Y]\cong[X, \Omega Y],$$

$$[\beta(SX), Y]\cong[SX, Y].$$

2.7 may be extended to

2.8. Theorem. *Let $Y=\bigcup Y_k$ in the weak topology, and where the Y_k are 2-connected and of finite type then*

$$[\beta X, \Omega Y]=[\beta(SX), Y].$$

Proof. By a simple point-set topological argument we have

$$[\beta X, \Omega Y]=\varprojlim\,[\beta X, \Omega Y_i],$$

$$[\beta(SX), Y]=\varprojlim\,[\beta(SX), Y_i].$$

One now observes that the term by term equivalence given by 2.7 commutes with taking limits.

Finally, we consider a specific example.

2.9. Example (see [7] 4.1). Let $B=K(Z, n)$ with $n>2$. For a given prime p, let $\pi=\{p\}$, the set containing the single prime p, and let $Z_{p\infty}$ be the p-component of the rationals mod 1. One verifies that $B^{\pi}=K(Z_{p\infty}, n-1)$, hence on fd Norm

$$F^{\mathscr{P}_\pi}_{K(Z,n)}=[\beta-, K(Z_{p\infty}, n-1)]=\lim_{r\to\infty}[\beta-, K(Z_{p^r}, n-1)]=\lim_{r\to\infty}[-, K(Z_{p^r}, n-1)]$$

this last equality by 1.12.

§ 3. β-Representable Čech extensions. In this section we address the general question of when is the Čech extension of a homotopy cofunctor F on a subcategory $\mathscr{P}\subseteq f\mathscr{P}$ is β-representable. The theorem we obtain is that if $\mathscr{P}$ is closed in a suitable sense under products and equalizers and F satisfies the Wedge and Mayer–Vietoris axioms [1] then if F is

(1) countable set valued or

(2) Abelian group valued

its Čech extension is β-representable.

The method of proof is to use the appropriate version of Brown's Theorem [1], [2] to show that the Kan extension of F, considered as a functor on $\mathscr{P}^c$, is representable on $f\mathscr{P}^c$; this in conjunction with 1.6 and 1.11 yields our theorem. We will only present full details of the proof under hypothesis (1). The proof under hypothesis (2) being similar though more tedious.

3.1. Remarks. In the first part of this section $\mathscr{P}^c\subseteq f\mathscr{P}^c$ will be assumed to be closed under equalizers and to contain 1 point spaces. Also, until further noted F will be countable set valued cofunctor on $\mathscr{P}^c$ satisfying the Wedge and Mayer–Vietoris Axiom.

Recall [9] that $F^{\mathscr{P}^c}(X)$ is computed as a suitable direct limit over commutative diagrams of the form

$$\begin{tikzcd} & P_1 \arrow[d, "f"] \\ X \arrow[ur, "\xi_1"] \arrow[r, "\xi_2"'] & P_2 \end{tikzcd}$$

where ξ_1, f, ξ_2 are homotopy classes of maps. We will denote such a diagram by $f\colon (P_1, \xi_1)\to(P_2, \xi_2)$ and the category associated with such diagrams by $(\mathscr{P}^c, X)$. It will also be convenient to set $F(P, \xi)\equiv F(P)$.

Finally, we may assume that $\mathscr{P}^c$ is small since, as usual, it suffices to work with representatives of the equivalence classes of $\mathscr{P}^c$.

3.2. Definition [9]. Given categories $\mathscr{A}\subseteq\mathscr{B}$ we say $\mathscr{A}$ is *weakly cofinal* in $\mathscr{B}$ if for every $B\in|\mathscr{B}|$ there exists on $A\in|\mathscr{A}|$ and a morphism $A\xrightarrow{f}B$.

The following is similar in proof to ([8] 1.11 Appendix).

3.3. Theorem. *Let $\mathscr{W}\subseteq(\mathscr{P}^c, X)$ be a small weakly cofinal subcategory then*

$$F^{\mathscr{P}^c}(X)=\Big(\bigcup_{(P,\xi)\in|\mathscr{W}|}F(P, \xi)\Big)/R$$

where R is the relation generated by all pairs $(f_1^*(\beta), f_2^*(\beta))$ with $(P, \xi)\in|\mathscr{P}^c, X)|$, $\beta\in F(P, \xi)$, $f_i\colon (P_i, \xi_i)\to(P, \xi)$ and $(P_i, \xi_i)\in|\mathscr{W}|$, $i=1, 2$.

We denote the obvious maps $F(P, \xi)\to F^{\mathscr{P}^c}(X)$ by θ_ξ.

Under the hypothesis of this section we have the following alternate description of R.

3.4. Lemma. *Let $(\alpha_1, \alpha_2)\in R$ with $\alpha_i\in F(P_i, \xi_i)$ then there exists $(P, \xi)\in|(\mathscr{P}^c, X)|$, $\beta\in F(P, \xi)$ and $f_i(P_i, \xi_i)\to(P, \xi)$ with $\alpha_i=f_i^*(\beta)$.*

Proof. The proof is by induction using the following special case. Suppose (α_i, α_2) is generated by the following diagram:

$$\begin{tikzcd}
& \overset{\bar\beta_1}{(\bar P_1, \xi_1)} & & \overset{\bar\beta_2}{(\bar P_2, \xi_2)} & \\
\underset{\alpha_1}{(P_1, \xi_1)} \arrow[ur, "f_{1,1}"] & & \underset{\alpha_3}{(P_3, \xi_3)} \arrow[ul, "f_{1,3}"'] \arrow[ur, "f_{2,3}"] & & \underset{\alpha_2}{(P_2, \xi_2)} \arrow[ul, "f_{2,2}"']
\end{tikzcd}$$

with $f_{1,1}^*\bar\beta_1 = \alpha_1, f_{1\cdot 3}^*\bar\beta_1 = f_{2,3}^*\bar\beta_2 = \alpha_3$ and $f_{2,2}^*\bar\beta_2 = \alpha_2$. Since $\mathscr{P}^c$ is closed under equalizers and satisfies the Mayer–Vietoris axiom we may find $(P, \xi) \leftarrow |\mathscr{P}^c|$, $\beta \in F(P, \xi)$ and $g_r(P, \xi) \to (\bar P_r, \xi_r)$, $r = 1, 2$ with $g_r^*\beta = \bar\beta_r$. Set $f_i = g_i f_{i,i}$.

To complete the proof one observes that $(\alpha_1, \alpha_2) \in R$ must be generated out of a diagram of the form

$$\begin{tikzcd}
& \overset{\bar\beta_1}{(\bar P_1, \xi_1)} & & \overset{\bar\beta_2}{(\bar P_2, \bar\xi_2)} & & \overset{\bar\beta_{n-1}}{(\bar P_{n-1}, \bar\xi_{n-1})} & \\
\underset{\alpha_1}{(P_1, \xi_1)} \arrow[ur, "f_{1,1}"] & & \underset{\alpha_3}{(P_3, \xi_3)} \arrow[ul, "f_{1,3}"'] \arrow[ur, "f_{2,3}"] & \cdots & \underset{\alpha_n}{(P_n, \xi_n)} \arrow[ul] \arrow[ur] & & \underset{\alpha_2}{(P_2, \xi_2)} \arrow[ul, "f_{n-1,2}"']
\end{tikzcd}$$

Repeated application of the argument for 3.5 gives the result.

We are now able to state and prove the main theorem of this section.

3.6. THEOREM. *Let $\mathscr{P}^c \subseteq f\mathscr{P}^c$ and F be as in* 3.1 *then $F^{\mathscr{P}^c}$ satisfies the Wedge and Mayer–Vietoris axiom on $f\mathscr{P}^c$ and hence is representable (see* [2]).

Proof. We first show $F^{\mathscr{P}^c}$ satisfies the Mayer–Vietoris axiom.

Suppose we are given $X_i \in F^{\mathscr{P}^c}(X_i)$ $(i = 1, 2)$ and maps f_i: $A \to X_i$ such that $F^{\mathscr{P}^c}(f_1)x_1 = F^{\mathscr{P}^c}(f_2)x_2$. We wish to construct $x \in F^{\mathscr{P}^c}(X_1 \cup_A X_2)$ such that $F^{\mathscr{P}^c}(j_i)x = x_i$ where j_i: $X_i \to X_1 \cup_A X_2$ are the respective inclusions.

By 3.3 we may choose ξ_i: $X_i \to P_i$ and $\alpha_i \in F(P_i, \xi_i)$ such that $x_i = \theta_{\xi_i}(\alpha_i)$. Hence, $\theta_{\xi_1 f_1}(\alpha_1) = \theta_{\xi_2 f_2}(\alpha_2)$. We may now apply 3.4 to find ξ: $A \to P$, $\beta \in F(P, \xi)$ and h_i: $(P_i, \xi_i f_i) \to (P, \xi)$ with $F(h_i)\beta = \alpha_i$.

Using this data we may construct $\bar\xi$: $X_1 \cup_A X_2 \to P$ such that $\theta_{\bar\xi}(\beta) \in F^{\mathscr{P}^c}(X_1 \cup_A X_2)$ is a suitable value for x.

To show that $F^{\mathscr{P}^c}$ satisfies the Wedge axiom, let $X_1 \vee \ldots \vee X_n \in |f\mathscr{P}|$. Let $(w\mathscr{P}^c, X)$ be the wedge category over X. Objects in $(w\mathscr{P}^c, X)$ are diagrams of the form

$$\bigvee_{i=1}^{n} X_i \xrightarrow{\bigvee_{i=1}^{n} \xi_i} \bigvee_{i=1}^{n} P,$$

where $(P_i, \xi_i) \in |\mathscr{P}^c, X_i)|$, and maps are appropriate wedges of maps. Since F satisfies the wedge axiom on $\mathscr{P}^c$ a simple computation shows that $F^{w\mathscr{P}^c}(X) = \mathop{\times}\limits_{i=1}^{n} F^{\mathscr{P}^c}(X_i)$. We wish to show $F^{w\mathscr{P}^c}(X) = F^{\mathscr{P}^c}(X)$.

Firstly, $(w\mathscr{P}^c, X)$ is weakly cofinal in $(\mathscr{P}^c, X)$ since given ξ: $\bigvee_{i=1}^{n} X_i \to P$ we have that

$$\xi = \varphi\Big(\bigvee_{i=1}^{n} (\xi j_i)\Big)$$

where j_i: $X_i \to X$ are the inclusion maps and φ is the folding map of the wedge of n copies of P to P.

We are therefore able to apply 3.3 to $(w\mathscr{P}^c, X) \subseteq (\mathscr{P}^c, X)$ to conclude

$$F^{\mathscr{P}^c}(X) = \Big(\bigcup_{(P,\xi) \in |w\mathscr{P}^c, X|} F(P, \xi)\Big)/R$$

and

$$F^{w\mathscr{P}^c}(X) = \Big(\bigcup_{(P,\xi) \in |w\mathscr{P}^c, X|} F(P, \xi)\Big)/R'$$

where R (resp. R') is generated by all pairs $(f_1^*(\beta), f_2^*(\beta))$ with $\beta \in F(P, \xi)$ and $(P, \xi) \in |(\mathscr{P}^c, X)|$ (resp. $(w\mathscr{P}^c, X)$).

We wish to show $R = R'$. Again, this follows from the observation that for any map f: $\bigvee_{i=1}^{n} P_i \to P$ we have the factorization $f = \varphi\big(\bigvee_{i=1}^{n} (fj_i)\big)$.

Combining 3.6 with 1.11 we have the following theorem.

3.7. THEOREM. *Let F be a countable set valued homotopy cofunctor on $\mathscr{P} \subseteq f\mathscr{P}$ such that F considered as a functor on $\mathscr{P}^c$, and $\mathscr{P}^c$ itself satisfy the hypothesis of* 3.1. *Then the Čech extension $F^{\mathscr{P}}$ to* CRegT_2 *is β-representable. In particular, $F^{\mathscr{P}} = [\beta -, B]$ where B is given by* 3.6.

3.8. Final Remarks. The hypothesis of 3.1 include the example of Section 2. Also, assuming $\mathscr{P}$ to be simply connected we again have that $F^{\mathscr{P}}$ is a homotopy cofunctor on fdNorm. Hence, we have available the technical observations at the end of Section 2.

Under the additional hypothesis that $\mathscr{P}$ is closed under products one can prove 3.6 (as well as 3.7) assuming F to be abelian group valued. One uses the representation of $F^{\mathscr{P}}(X)$ given in ([8] 1.11 Appendix) and proves the appropriate version of 3.4 using the additional hypothesis on $\mathscr{P}$. One also uses Adam's version of Brown's theorem [1].

Finally, one might expect to be able to prove 3.6–7 under the hypothesis that F is abelian monoid valued (using Deleanu's version of Brown's theorem [6]). Unfortunately, we do not know the appropriate version of 3.3–4 that our method seems to require.

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Accepté par la Rédaction le 2. 1. 1978

The number of countable models of a theory of one unary function

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Abstract. If T is a theory in the language of one unary function symbol then T has 1, $\aleph_0$, or $2^{\aleph_0}$ countable models.

§ 1. **Introduction.** Let L^0 denote the language containing equality and one unary function symbol. We prove:

THEOREM 1. *If T is a complete first order theory in L^0, then T has 1, $\aleph_0$, or $2^{\aleph_0}$ countable models.*

The part of the theorem claiming that if T has $>\aleph_0$ countable models then T has $2^{\aleph_0}$ countable models is the first-order Vaught conjecture for L^0. The $L^0_{\omega_1\omega}$ Vaught conjecture was claimed by Burris in [1] but an error was found by Arnold Miller. After writing the first draft of this paper I learned that Miller [5] had already proven Theorem 1 by a different method in a more general setting, and some information about the $L^0_{\omega_1\omega}$ case.

The following theorem of Shelah gives information about the number of uncountable models of a theory in L^0.

THEOREM (Shelah). *If T is a complete first-order theory in L^0 then either T has 2^λ models of power λ for all $\lambda \geqslant \aleph_1$ or T has $\leqslant \beth_n(|\alpha|)$ models of power $\aleph_\alpha$ for some $n<\omega$ and all $\alpha \geqslant \omega$.*

There is a similar theorem for $L^0_{\omega_1\omega}$.

The proof uses general considerations of stability. The problem of the number of countable models of a first-order theory of linear order was solved in Rubin [6].

I am indebted to Mati Rubin for calling my attention to the error in [1], and to him and to Miller for detecting errors in earlier versions of the present paper.

§ 2. **Preliminaries.** We preserve the notation and definitions of [4]. Here is a brief review. (For model-theoretic notation and definitions see [3].) The language contains one unary function symbol f, and equality.

The distance between a and b relative to a set A is $d_A(a, b) = \min\{r$: there are k, l such that $k+l = r$ and there are $x_0, \ldots, x_k, y_0, \ldots, y_l \in A$ such that $a = x_0$, $b = y_0$, $f(x_i) = x_{i+1}$ for $i<k$, $f(y_j) = y_{j+1}$ for $j<l$, and $x_k = y_l\}$. A path from a to b is such a sequence $\langle x_0, \ldots, y_l\rangle$. We say a is *above* b if there is a path from a to b which contains $f(b)$. The set A is *below* b if b is above every element of A. Notice