The theory of abelian p-groups with the quantifier I is decidable

by

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Abstract. Define $\mathfrak{A} \models Ix(\varphi(x), \psi(x))$ iff $\{a: \mathfrak{A} \models \varphi(a)\}$ and $\{a: \mathfrak{A} \models \psi(a)\}$ are of the same power (Härtig). Let $T_p(I)$ be the theory of abelian p-groups with the quantifier I and $L(I)$ the corresponding language. The nonlogical signs are "+", "—", "0", and "$p^n|x$" ($\underset{\mathrm{Df}}{=} \exists y(p^n y = x)$). Extending the set of elementary basic sentences of Szmielew a set of basic sentences is given such that every formula of $L(I)$ is equivalent relative to $T_p(I)$ to a boolean combination of basic sentences and atomic formulas. Using this the decidability of $T_p(I)$ is shown.

1. Introduction. In [3] the decidability problem of the theory $T(I)$ of abelian groups with the quantifier I has been discussed:

$$\mathfrak{A} \models Ix(\varphi(x), \psi(x)) \quad \text{iff} \quad \{a: \mathfrak{A} \models \varphi(a)\} \quad \text{and} \quad \{a: \mathfrak{A} \models \psi(a)\}$$

have the same power. This quantifier was introduced by Härtig [5]. Let L be the elementary language of group theory with the nonlogical symbols "+", "−", "0", and "$q^n|$" where $q^n|x$ is defined by $\exists y(q^n y = x)$ for every prime q and every n. Let $L(I)$ be the language corresponding to $T(I)$.

Extending the set of elementary basic sentences of Szmielew [8] in [3] a set of basic sentences was given such that every formula of $L(I)$ is equivalent relative to $T(I)$ to a boolean combination of basic sentences and atomic formulas.

The problem of decidability of $T(I)$ remained open. The equivalence was shown to the problem of effective solvability of certain systems of equations and unequations in the naturals.

By a *p-group* is meant a group in which the orders of the elements are powers of the prime p. In this paper we use the results and ideas of [3] to prove:

THEOREM. *The theory of abelian p-groups with the quantifier I is decidable.*

The theory of abelian p-groups with the quantifier I we denote by $T_p(I)$. Contrary to [3] $T_p(I)$ is not the I-theory of an EC_Δ-class (in elementary sense).

Q_α denotes the generalized quantifier "there exist ω_α-many" [6]. Let $L(Q_\alpha)$ be the corresponding language. $T(Q_\alpha)$ and $T_p(Q_\alpha)$ we use to denote the theory of abelian groups (resp. of abelian p-groups) with the quantifier Q_α. Let T be the elementary theory of abelian groups and T_p the elementary theory of abelian p-groups.

If $\Sigma \subseteq L(I)$ (resp. $\Sigma \subseteq L(Q_\alpha)$) and $\mathfrak{A}$ is an abelian p-group with $\mathfrak{A} \models \Sigma$ then $\mathfrak{A}$ is called a *p-model of Σ*. $\oplus$ we use to denote the direct sum. $\mathfrak{A}^\lambda$ is the λ-fold direct sum of $\mathfrak{A}$. Let $\mathfrak{Z}(p^i)$ be the cyclic group of order p^i and $\mathfrak{Z}(p^\infty)$ the group of type p^∞.

A p-model $\mathfrak{A}$ is *normal* iff $\mathfrak{A} \overset{\sim}{\rightarrow} \bigoplus_{0<i<\omega} \mathfrak{Z}(p^i)^{\lambda(i)} \oplus \mathfrak{Z}(p^\infty)^{\lambda(\omega)}$ where finitely many $\lambda(\varkappa) \neq 0$ only.

We define a sentence φ *occurs* in a set Σ of sentences iff $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$. If we say a group we mean abelian group. If X is a subset of a group $\mathfrak{A}$ the subgroup $\ulcorner X \urcorner$ is the subgroup of $\mathfrak{A}$ generated by the elements of X. $|X|$ we use to denote the power of X.

2. Basic subgroups. The basic subgroup of p-groups is one of the most fundamental notions for working with abelian p-groups. This notion is due to Kulikov. Let $\mathfrak{A}$, $\mathfrak{B}$ be p-groups.

$\mathfrak{B}$ is a pure subgroup of $\mathfrak{A}$ if for every n, every $b \in \mathfrak{B}$, and every $a \in \mathfrak{A}$ with $p^n a = b$ there is some $c \in \mathfrak{B}$ with $p^n c = b$. $\mathfrak{A}$ is divisible if for every n and every $a \in \mathfrak{A}$ there is some $b \in \mathfrak{A}$ with $p^n b = a$.

A subgroup $\mathfrak{B}$ of $\mathfrak{A}$ is a basic subgroup of $\mathfrak{A}$ iff

(i) $\mathfrak{B}$ is a direct sum of cyclic groups.

(ii) $\mathfrak{B}$ is pure in $\mathfrak{A}$.

(iii) $\mathfrak{A}/\mathfrak{B}$ is divisible.

THEOREM A (Kulikov). *Every p-group $\mathfrak{A}$ contains a basic subgroup $\mathfrak{B}$.*

THEOREM B (Baer). *Assume that $\mathfrak{B}$ is a subgroup of the p-group $\mathfrak{A}$ and $\mathfrak{B} = \bigoplus_{n<\omega_n} \mathfrak{B}_n$ where $\mathfrak{B}_n$ is a direct sum of cyclic groups of order p^n. Then $\mathfrak{B}$ is a basic subgroup of $\mathfrak{A}$ if and only if*

$$\mathfrak{A} = \mathfrak{B}_1 \oplus \ldots \oplus \mathfrak{B}_n \oplus \ulcorner \mathfrak{B}_n^* \cup p^n \mathfrak{A} \urcorner$$

or every n where $\mathfrak{B}_n^* = \mathfrak{B}_{n+1} \oplus \mathfrak{B}_{n+2} \oplus \ldots$

(For the proofs see [4].)

Every p-group $\mathfrak{A}$ is isomorphic to $\mathfrak{Z}(p^\infty)^\lambda \oplus \mathfrak{A}'$ where $\mathfrak{A}'$ does not contain any nontrivial divisible subgroup. $\mathfrak{A}'$ is called reduced.

PROPOSITION 1. *If $\mathfrak{A} = \mathfrak{Z}(p^\infty)^\lambda \oplus \mathfrak{A}'$ where $\mathfrak{A}'$ is reduced and $\mathfrak{B}$ is basic subgroup of $\mathfrak{A}'$ then $\mathfrak{Z}(p^\infty)^\lambda \oplus \mathfrak{B}$ is an elementary subgroup of $\mathfrak{A}$.*

This follows from $\mathfrak{A}' > \mathfrak{B}$ proved in [1] p. 795 using the results of Szmielew. Then you get

COROLLARY 1. *Every sentence of L fulfilled in a p-group is true in a direct sum of finitely many groups of the form $\mathfrak{Z}(p^n)$ and $\mathfrak{Z}(p^\infty)$.*

Let $[p^n]\mathfrak{A}$ *be the subgroup* $\{x\colon p^n x = 0\}$ *of* $\mathfrak{A}$.

PROPOSITION 2. *For every p-group $\mathfrak{A}$ $|\mathfrak{A}| > \omega$ implies $|\mathfrak{A}| = |[p]\mathfrak{A}|$.*

Proof. By Theorems A and B $|[p]\mathfrak{A}| \geqslant \omega$. Therefore the First Theorem of Prüfer (A bounded group is a direct sum of cyclic groups) implies $|[p]\mathfrak{A}| = |[p^n]\mathfrak{A}|$ for $n \geqslant 1$. Then the assertion follows from $\mathfrak{A} = \bigcup_{0<n<\omega} [p^n]\mathfrak{A}$.

3. $T_p(Q_\alpha)$ is decidable. The Szmielew basic sentences for the elementary theory of abelian groups are our starting-point [8]:

$\mathfrak{A} \models \zeta_1(p, n, k)$ means "There is a subgroup of $\mathfrak{A}$ isomorphic to $\mathfrak{Z}(p^n)^k$".

$\mathfrak{A} \models \zeta_2(p, n, k)$ means "$\mathfrak{A}/p^n\mathfrak{A} \models \zeta_1(p, n, k)$".

$\mathfrak{A} \models \zeta_3(p, n, k)$ means "There is a direct summand of $\mathfrak{A}$ isomorphic to $\mathfrak{Z}(p^n)^k$".

$\zeta_4(m) \underset{\mathrm{Df}}{=} \forall x(mx = 0)$.

Thereby $n, m > 0$. Define

$$\zeta_i(p, n, \mathfrak{A}) = \sup(\{k \in \omega\colon \mathfrak{A} \models \zeta_i(p, n, k)\}) \quad \text{for} \quad i \in \{1, 2, 3\}.$$

In [2] the following new basic sentences of the theory $T(Q_\alpha)$ of abelian groups with the quantifier Q_α are added:

$$Q_\alpha x(px = 0 \wedge p^{n-1}|x) \quad \text{and} \quad Q_\alpha x(m|x) \quad \text{where} \quad m, n \geqslant 1.$$

The results of Szmielew are extended by the following:

THEOREM 1. *There is an effective procedure to construct for every formula of $L(Q_\alpha)$ an equivalent relative to $T(Q_\alpha)$ boolean combination of basic sentences and atomic formulas.*

Further we use:

LEMMA 1. *Given an abelian group $\mathfrak{A}$.*

1.1. $\zeta_i(p, n, \mathfrak{A}) = \zeta_i(p, m, \mathfrak{A}) + \sum_{j=n}^{m-1} \zeta_3(p, j, \mathfrak{A})$ *for $m > n$ and $i \in \{1, 2\}$* [8].

1.2. $\mathfrak{A} \models \neg Q_\alpha x(m|x)$ *iff* $\mathfrak{A} = \mathfrak{B} \oplus \mathfrak{L}$ *with* $|\mathfrak{B}| < \omega_\alpha$ *and* $m\mathfrak{L} = 0$ [2].

1.3. *Let $n < m$ and*

$$\mathfrak{A} \models \neg Q_\alpha x(px = 0 \wedge p^{m-1}|x) \wedge \bigwedge_{n \leqslant i < m} \neg \zeta_3(p, i, k_i)$$

for some naturals k_i. Then $\mathfrak{A} \models \neg Q_\alpha x(px = 0 \wedge p^{n-1}|x)$.

In this chapter we consider $T_p(Q_\alpha)$. We prove:

THEOREM 2.

1.1. *$T_p(Q_\alpha)$ is decidable.*

1.2. *Every sentence of $L(Q_\alpha)$ true in an abelian p-group is fulfilled in a normal p-group of power smaller than $\omega_{\alpha+1}$.*

Proof. Using the elimination procedure of Theorem 1 we have to decide whether a finite set Σ of negated and unnegated basic sentences of $L(Q_\alpha)$ has a p-model or not. If such a p-model exists we construct a normal one. Without loss of generality we can suppose that the basic sentences occurring in Σ are of the form $\zeta_i(p, n, k)$ $(1 \leqslant i \leqslant 3)$, $Q_\alpha x(px = 0 \wedge p^n|x)$, and $Q_\alpha x(p^n|x)$ only.

$$\text{(Mention } T_p \models \zeta_4(p^n) \leftrightarrow \neg\zeta_1(p, n+1, 1).)$$

We show that Σ has a normal p-model if it satisfies the following conditions (1)–(6). Then Theorem 2 is proved because such a condition is necessary for consistency of Σ relative to $T_p(Q_\alpha)$ or we can assume it, replacing Σ by a finite set Ω of finite sets Σ' of unnegated and negated basic sentences such that $\mathfrak{A}$ is a p-model of Σ iff $\mathfrak{A}$ is a p-model of some $\Sigma' \in \Omega$.

Remark that we do not destroy a condition if we realize a new one. The condition

(1) $\Sigma \cap L$ has a p-model

is necessary. Further we can confine to sets Σ such that:

(2) If $\alpha > 0$ then formulas $Q_\alpha x(p^n|x)$ and $\neg Q_\alpha x(p^n|x)$ are not in Σ.

To get (2) replace $Q_\alpha x(p^n|x)$ by $Q_\alpha x(px = 0 \wedge p^n|x)$. This is possible because for every p-group $\mathfrak{A}$ with $\mathfrak{A} \models \Sigma$ the premise of (2) implies $\mathfrak{A} \models Q_\alpha(p^n|x)$ iff $\mathfrak{A} \models Q_\alpha x(px = 0 \wedge p^n|x)$ by Proposition 2.

(3) If $\neg Q_0 x(p^m|x) \in \Sigma$ then $Q_0 x(p^n|x) \notin \Sigma$ for any n and further $\zeta_2(p, i, k)$ (resp. $\neg\zeta_2(p, i, k)$) is in Σ iff $\zeta_1(p, i, k)$ (resp. $\neg\zeta_1(p, i, k)$) is in Σ.

By Lemma 1.2 $\mathfrak{A} \models \neg Q_0 x(p^m|x)$ implies that $\mathfrak{A} \xrightarrow{\sim} \mathfrak{B} \oplus \mathfrak{Q}$ with $|\mathfrak{B}| < \omega$ and $p^m\mathfrak{Q} = 0$. Therefore we can assume (3) replacing $Q_0 x(p^n|x)$ by $Q_0 x(px = 0 \wedge p^n|x)$ and adding some $\zeta_j(p, i, k)$ resp. $\neg\zeta_j(p, i, k)$.

(4) If $Q_0 x(p^m|x) \in \Sigma$ and $\neg\zeta_1(p, n, k) \in \Sigma$ then $m \geqslant n$ and $\neg\zeta_2(p, n, j)$, $\zeta_1(p, n, j) \in \Sigma$ for some $j < k$.

If $\mathfrak{A} \models \Sigma$ then by Theorems A and B

$$(*) \qquad \mathfrak{A} \xrightarrow{\sim} \bigoplus_{i<\omega} \mathfrak{Z}(p^i)^{\lambda(i)} \oplus \mathfrak{Z}(p^\infty)^{\lambda(\omega)} \quad \text{with} \sum_{n \leqslant \varkappa \leqslant \omega} \lambda(\varkappa) < \omega\,.$$

We can assume $m \geqslant n$ considering instead of Σ the sets

$$\Sigma_1 = (\Sigma \setminus \{Q_0 x(p^m|x)\}) \cup \{Q_0 x(px = 0 \wedge p^m|x)\}$$

and

$$\Sigma_2 = (\Sigma \setminus \{Q_0 x(p^m|x)\}) \cup \{\neg Q_0 x(px = 0 \wedge p^m|x),\, Q_0 x(p^n|x)\}$$

if $n > m$.

From $m \geqslant n$ then follows $\lambda(\omega) > 0$ in $(*)$. Therefore $\zeta_2(p, n, \mathfrak{A}) < \zeta_1(p, n, \mathfrak{A})$. We get (4) replacing Σ by the set of all

$$\Sigma_j = \Sigma \cup \{\neg\zeta_2(p, n, j),\, \zeta_1(p, n, j)\} \quad \text{for} \quad j < k\,.$$

Using tautologies of $T(Q_\alpha)$ we further suppose:

(5) There is at most one formula of the form $Q_\alpha x(p^n|x)$, one formula of the form $\neg Q_\alpha x(p^n|x)$, one formula of the form $Q_\alpha x(px = 0 \wedge p^n|x)$, and one formula of the form $\neg Q_\alpha x(px = 0 \wedge p^n|x)$ in Σ.

Since $\neg\zeta_1(p, m+1, k)$ or $\neg Q_\alpha x(p^m|x)$ implies $\neg Q_\alpha x(px = 0 \wedge p^m|x)$ by Lemma 1.3 the following condition is necessary for consistency of Σ relative to $T_p(Q_\alpha)$.

(6) There is some j such that for all n and m: If $Q_\alpha x(px = 0 \wedge p^n|x) \in \Sigma$, and $\neg Q_\alpha x(px = 0 \wedge p^m|x) \in \Sigma$ or $\neg\zeta_1(p, m+1, k) \in \Sigma$ or $\neg Q_\alpha x(p^m|x) \in \Sigma$ then $n < j \leqslant m$ and neither $\neg\zeta_3(p, j, k)$ nor $\neg\zeta_2(p, i, k)$ are in Σ for any k and any $i \leqslant j$.

Now let us construct a normal p-model of Σ assuming (1)-(6). By (1) and Corollary 1 there is some direct sum $\mathfrak{A}$ of finitely many groups $\mathfrak{Z}(p^n)$, $\mathfrak{Z}(p^\infty)$ with $\mathfrak{A} \models \Sigma \cap L$.

1. Case $\neg Q_\alpha x(p^m|x) \in \Sigma$ for some m.

By (2) $\alpha = 0$. If you replace in $\mathfrak{A}$ every direct summand $\mathfrak{Z}(p^\infty)$ by $\mathfrak{Z}(p^n)$ for sufficient large n by (3) you get a finite group $\mathfrak{A}'$ with $\mathfrak{A}' \models \Sigma \cap L$.

If $Q_0 x(px = 0 \wedge p^n|x) \notin \Sigma$ for any n $\mathfrak{A}' \models \Sigma$ by (3).

If $Q_0 x(px = 0 \wedge p^n|x)$ is in Σ then by (5) there is no other formula of this form. Take the number j that exists by (6). Then by (2), (5), (6) $\mathfrak{A}' \oplus \mathfrak{Z}(p^j)^{\omega_\alpha} \models \Sigma$.

2. Case neither $\neg Q_\alpha x(p^m|x)$ nor $Q_\alpha x(p^m|x)$ is in Σ for any m.

If $Q_\alpha x(px = 0 \wedge p^n|x) \notin \Sigma$ for any n then $\mathfrak{A} \models \Sigma$.

If $Q_\alpha x(px = 0 \wedge p^n|x) \in \Sigma$ for some n apply (6) and (5) as above whenever the premise of (6) is fulfilled. Otherwise $\mathfrak{A} \oplus \mathfrak{Z}(p^\infty)^{\omega_\alpha}$ is a model of Σ.

3. Case $\neg Q_\alpha x(p^m|x) \notin \Sigma$ for any m but $Q_\alpha x(p^m|x) \in \Sigma$. By (2) $\alpha = 0$. By the second case and (5) there is some normal p-model $\mathfrak{B}$ of $\Sigma \setminus \{Q_0 x(p^m|x)\}$. If $\neg\zeta_1(p, n, k) \notin \Sigma$ for any n, k then $\mathfrak{B} \oplus \mathfrak{Z}(p^\infty)$ is a normal p-model of Σ. Otherwise by (4) there is some $n \leqslant m$ and some k and j such that $j < k$, and $\neg\zeta_1(p, n, k) \in \Sigma$, $\zeta_1(p, n, j) \in \Sigma$, and $\neg\zeta_2(p, n, j) \in \Sigma$. Then $\mathfrak{B}$ must have a direct summand isomorphic to $\mathfrak{Z}(p^\infty)$ and therefore $\mathfrak{B} \models \Sigma$. ■

We need Theorem 2 to prove the main result.

4. p-Systems. If $k_0, \ldots, k_s$ are naturals then a term $p^{\sum k_j z_j + k_0}$ is called a *p-term* in the variables $z_1, \ldots, z_n$. In this chapter finite sets Γ of equations $\Pi = 0$ and unequations $\Pi \neq 0$ are considered where each Π is a linear combination $\sum m_i t_i$ of p-terms t_i with coefficients m_i in the integers. Such a set Γ is called a p-system. By a solution of a p-system Γ we mean a solution in natural numbers. We prove:

THEOREM 3. *There is an effective procedure to decide whether a given p-system has a solution or not.*

At first we consider a single p-equation $\Pi = \sum_{i=1}^{s} m_i p^{\sigma_i} = 0$ with

$$\sigma_i = \sum_{j=1}^{n} k_{ij} z_j + k_{i0}.$$

LEMMA 2. *Assume $p \nmid m_i$ for every i with $1 \leqslant i \leqslant s$. If $c_1, \ldots, c_n$ is a solution of $\Pi = 0$ then there are i and j such that $i \neq j$ and $\sigma_i(c_1, \ldots, c_n) = \sigma_j(c_1, \ldots, c_n)$.*

The proof of the lemma is clear. How can we determine the solutions of $\Pi = 0$?

Assume w.l.o.g. $p \nmid m_i$. Let $\Pi_{ij} = 0$ be the equation you get replacing σ_j by σ_i. Applying Lemma 2 it follows:

$\Pi = 0$ has a solution $c_1, \ldots, c_n$ iff there is a pair $\langle i, j \rangle$ with $i \neq j$ such that $\sigma_i(c_1, \ldots, c_n) = \sigma_j(c_1, \ldots, c_n)$ and $c_1, \ldots, c_n$ is a solution of $\Pi_{ij} = 0$.

Put every $\Pi_{ij} = 0$ in the form $\sum_{i=1}^{s-1} m_i' p^{\sigma_i'}$ with $p \nmid m_i'$ and apply Lemma 2 to every $\Pi_{ij} = 0$ again. After $s-1$ steps we get a finite set of pairs $\langle mp^{\sigma} = 0, \theta \rangle$ where θ is a finite set of linear equations in the variables $z_1, \ldots, z_n$ and coefficients in the integers, and p^{σ} is a p-term. Then $c_1, \ldots, c_n$ is a solution of $\Pi = 0$ iff there is some $\langle mp^{\sigma} = 0, \theta \rangle$ such that $m = 0$ and $c_1, \ldots, c_n$ is a solution of θ. Let Ω be the set of all θ such that $\langle Op^{\sigma} = 0, \theta \rangle$ is obtained in the procedure above. We have proved:

LEMMA 3. *For every p-equation $\Pi = 0$ in the variables $z_1, \ldots, z_n$ a finite set $\Omega(\Pi)$ of finite systems of linear equations in $z_1, \ldots, z_n$ and coefficients in the integers can be constructed effectively such that $c_1, \ldots, c_n$ is a solution of $\Pi = 0$ iff $c_1, \ldots, c_n$ is a solution of some system in $\Omega(\Pi)$.*

Now consider some p-system Γ. Using Lemma 3 we get a finite set $\Omega^*(\Gamma)$ of systems of linear equations and unequations with coefficients in the integers such that $c_1, \ldots, c_n$ is a solution of Γ iff $c_1, \ldots, c_n$ is a solution of some system in $\Omega^*(\Gamma)$. Therefore Theorem 3 follows from

LEMMA 4. *There exists an effective procedure to decide for every system θ^* of linear equations and unequations with coefficients in the integers whether it has a solution in naturals numbers or not.*

Lemma 4 is implied by the fact that we can formulate "θ^* has a solution" in the elementary language of Presburger arithmetic, and this theory is decidable [7].

5. Basic sentences of $T_p(I)$. In [3] the set of Szmielew basic sentences is extended such that there is an effective procedure to construct for every formula φ of $L(I)$ a boolean combination ψ of basic sentences and atomic formulas equivalent to φ relative to $T(I)$. Then φ and ψ are equivalent relative to $T_p(I)$. Now those basic sentences needed for $T_p(I)$ will be described. Consider the following set Z_p of conjunctions of atomic formulas:

$$Z_p \underset{\mathrm{Df}}{=} \{\pi(x)\colon \pi(x) = (vp^m x = 0 \wedge \bigwedge_{i=1}^{t} p^{s_i} | p^{r_i} x) \text{ where}$$

$$v \in \{0, 1\},\ r_t < m,\ s_i > r_i,\ r_j > r_i \text{ and } s_j > s_i + r_j - r_i \text{ if } j > i\}.$$

A new basic sentence of $T_p(I)$ depends on two finite sequences A and B of formulas of Z_p with accentuated subsequences $\varphi_1, \ldots, \varphi_n$ respectively $\psi_1, \ldots, \psi_m$ such that φ_1 is the first element of A and ψ_1 is the first element of B.

We write $\varphi_i \succ \eta$ if φ_i precede η, η is not accentuated, and there is no accentuated φ_j between φ_i and η in A. $\psi_j \succ \eta$ is defined analogiously. We assume

$$T \vDash (\eta \to \varphi_i) \wedge \neg(\varphi_i \to \eta) \quad \text{if} \quad \varphi_i \succ \eta$$

and

$$T \vDash (\eta \to \psi_j) \wedge \neg(\psi_j \to \eta) \quad \text{if} \quad \psi_j \succ \eta.$$

We make the convention that two elements on different places of A respectively B are not identified even if they are the same formula of Z_p.

Furthermore for every new basic sentence we need a set C of subsequences of A and a set D of subsequences of B such that every one-element-sequence is a member of C respectively D. In a formula such a subsequence is to be interpreted as the conjunction of its members.

If μ and ν are subsequences let $\mu \circ \nu$ be the subsequence of all members of μ and ν. Define

$$\Delta'(A, B, C, D) \underset{\mathrm{Df}}{=}$$

$$\bigwedge_{\substack{\mu, \nu, \mu \circ \nu \in C \\ \text{or } \mu, \nu, \mu \circ \nu \in D}} \exists y(\mu(y - z_\mu) \wedge \nu(y - z_\nu)) \wedge \bigwedge_{\substack{\mu, \nu \in C,\, \mu \circ \nu \notin C \\ \text{or } \mu, \nu \in D,\, \mu \circ \nu \notin D}} \neg \exists y(\mu(y - z_\mu) \wedge \nu(y - z_\nu)).$$

Every A, B, C, D as above determine a new basic sentence Δ of $T_p(I)$ if

$$T_p \cup \{\ldots \underset{\mu \in C \cup D}{\exists} z_\mu \ldots \Delta'(A, B, C, D)\}$$

is consistent:

$$\Delta(A, B, C, D) \underset{\mathrm{Df}}{=} \ldots \underset{\mu \in C \cup D}{\exists} z_\mu \ldots [\Delta'(A, B, C, D) \wedge$$

$$\wedge (Iy)(\bigvee_{i=1}^{n} \varphi_i(y - z_{\varphi_i}) \wedge \bigwedge_{\varphi_i \succ \eta} \neg \eta(y - z_\eta),\ \bigvee_{j=1}^{m} \psi_i(y - z_{\psi_j}) \wedge \bigwedge_{\psi_j \succ \eta} \neg \eta(y - z_\eta))].$$

From the results in [3] we get

THEOREM 4. *There is an effective procedure to construct for every formula φ of $L(I)$ a boolean combination of basic sentences and atomic formulas equivalent to φ relative to $T_p(I)$.*

We define $|\varphi(x)|_{\mathfrak{A}} = |\{a \in \mathfrak{A}\colon \mathfrak{A} \vDash \varphi(a)\}|$. Consider some p-group $\mathfrak{A}$, a new basic sentence $\Delta(A, B, C, D)$, and an assignment $\mathfrak{a} = (\ldots, a_\mu, \ldots)$ of the variables $\ldots, z_\mu, \ldots$ in $\Delta'(A, B, C, D)$ such that $(\mathfrak{A}, \mathfrak{a}) \vDash \Delta'(A, B, C, D)$ and $\mathfrak{A} \vDash \neg Q_0 x \chi(x)$ for every accentuated $\chi(x)$ of A or B. As shown in [3] for every $\pi(x)$ in C or D we can compute some naturals w_i $(1 \leqslant i \leqslant f)$ and $v \in \{0, 1\}$ (in dependence of $\pi(x)$

only) with $w_1 \leqslant 1$, $w_i \leqslant w_{i+1} \leqslant w_i + 1$ such that

$$(1) \qquad |\pi(x)|_{\mathfrak{A}} = p^{\sum\limits_{1\leq i<f} w_i\xi_3(p,i,\mathfrak{A}) + w_f\zeta_4(p,f,\mathfrak{A}) + v\sum\limits_{f<i}\zeta_1(p,i,\mathfrak{A})}.$$

Mention that we can choose the same f for every π of C or D using Lemma 1.1.

By $\Delta'(A, B, C, D)$ $|\pi(x-a_\pi)\wedge\eta(x-a_\eta)|_{\mathfrak{A}}$ is coded for every $\pi, \eta \in C$ (resp. D):

$$|\pi(x-a_\pi)\wedge\eta(x-a_\eta)|_{\mathfrak{A}} = \begin{cases} 0 & \text{if } \pi\circ\eta \notin C \text{ (resp. D)}, \\ |\pi\circ\eta(x-a_{\pi\circ\eta})|_{\mathfrak{A}} & \text{otherwise.}\end{cases}$$

Furthermore remark $|\pi(x-b)|_{\mathfrak{A}} = |\pi(x)|_{\mathfrak{A}}$ for every $b \in \mathfrak{A}$.

Therefore we can prove (see [3]):

LEMMA 5. *There is an effective procedure to construct for every new basic sentence $\Delta(A, B, C, D)$ of $T_p(I)$ a linear combination $F_\Delta(x_1, \ldots, x_{f+1})$ of p-terms with coefficients in the integers such that for every p-group $\mathfrak{A}$: $\mathfrak{A}$ is a model of Δ with $\mathfrak{A} \models \neg Q_0 x\eta(x)$ for every accentuated η in A or B and $|[p]|\mathfrak{A}| < \omega$ iff*

$$\mathfrak{A} \models \ldots \exists z_\mu \ldots \Delta'(A, B, C, D)$$

and

$$x_i = \zeta_3(p, i, \mathfrak{A}) \quad \textit{if } 1 \leqslant i < f,$$

$$x_f = \zeta_1(p, f, \mathfrak{A}) \quad \textit{and}$$

$$x_{f+1} = \begin{cases} \sum\limits_{j>f}\zeta_1(p, j, \mathfrak{A}) & \textit{if there is some accentuated } \eta(x) = (p^n|x) \textit{ in } A \textit{ or } B, \\ \textit{any natural otherwise} \end{cases}$$

are naturals with $F_\Delta(x_1, \ldots, x_{f+1}) = 0$. ($f$ can be choosen arbitrary large.)

6. Proof of the main result.

THEOREM 5.

5.1. *$T_p(I)$ is decidable.*

5.2. *Every sentence of $L(T)$ true in an abelian p-group is fulfilled in a normal one.*

Proof. Using Theorem 4 we have to decide only whether a given finite set Σ of unnegated and negated basic sentences of $T_p(I)$ has a p-model or not. If a p-model exists by our decision procedure we get a normal one. To get certain properties of Σ we often replace the set Σ in question by a finite set of sets Σ' such that

Σ has a model iff one of the sets Σ' has a model.

If $\Delta(A, B, C, D)$ is a new basic sentence occurring in Σ then let $\varphi_1, \ldots, \varphi_n$ be the accentuated formulas of A, $\psi_1, \ldots, \psi_m$ the accentuated formulas of B,

Φ_i the formula $\varphi_i(y-z_{\varphi_i}) \wedge \bigwedge\limits_{\varphi_i > \eta} \neg\eta(y-z_\eta)$

and

Ψ_j the formula $\psi_j(y-z_{\psi_j}) \wedge \bigwedge\limits_{\psi_j > \eta} \neg\eta(y-z_\eta)$.

By the argument above we can assume w.o.l.g.

$$(1) \qquad T+\Sigma\cap L \models \ldots \exists z_\mu \ldots [\Delta'(A, B, C, D) \wedge \bigwedge_i \exists y \Phi_i(y) \wedge \bigwedge_j \exists y \Psi_j(y)]$$

for every Δ occurring in Σ.

As in the proof of Theorem 2

(2) we can confine us to elementary basic sentences of the form $\zeta_i(p, n, k)$ for $n, k \geqslant 1$ only.

Let us mention that Q_0 is definable by I. Define

$$(\leqslant x)(\mu(x), \nu(x)) \underset{\text{Df}}{=} Ix(\mu(x) \vee \nu(x), \nu(x))$$

and

$$(<x)(\mu(x), \nu(x)) \underset{\text{Df}}{=} (\leqslant x)(\mu(x), \nu(x)) \wedge \neg(\leqslant x)(\nu(x), \mu(x)).$$

Then for every $\mathfrak{A}$ with $\mathfrak{A} \models Q_0 x\varphi(x) \vee Q_0 x\psi(x)$

$$\mathfrak{A} \models (\leqslant x)(\varphi(x), \psi(x)) \quad \text{iff} \quad |\varphi(x)|_{\mathfrak{A}} \leqslant |\psi(x)|_{\mathfrak{A}}.$$

Furthermore

$$T(I) \models Q_0 x\mu(x) \vee Q_0 x\nu(x)$$
$$\rightarrow \big(Ix(\mu(x), \nu(x)) \leftrightarrow (\leqslant x)(\mu(x), \nu(x)) \wedge (\leqslant x)(\nu(x), \mu(x))\big).$$

Let Y be $\{(px = 0 \wedge p^n|x), (p^n|x) : n \geqslant 1\}$. For our investigations it is useful to admit certain negated and unnegated sentences $Q_0 x\eta(x)$ and $(\leqslant x)(\eta(x), \pi(x))$ for $\eta, \pi \in Y$ in Σ.

If $\pi(x) = (vp^m x = 0 \wedge \bigwedge\limits_{i=1}^{t} p^{r_i}|p^{s_i}x) \in Z_p$ where $v \in \{0, 1\}$ then

(i) $\quad T(Q_0) \models Q_0 x\pi(x) \leftrightarrow Q_0 x(vpx = 0 \wedge p^w|x)$ where

$$w = \begin{cases} r_1, & \text{if } s_1 = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Let $\pi^*(x)$ be $(vpx = 0 \wedge p^w|x)$. Define $\Sigma \cap L(Q_\upsilon) = \Sigma(Q_0)$. Let $\Sigma(Q_0)^-$ be $\Sigma(Q_0)$ without any $Q_0 x\eta(x)$ for $\eta(x) \in Y$.

For every new basic sentence $\Delta = \Delta(A, B, C, D)$ we can show using (1) (see [3]):

$$\text{(ii)} \qquad T(I) \models \Delta \leftrightarrow \bigwedge_i \neg(Q_0 x\varphi_i(x) \wedge \bigwedge_j \neg Q_0 x\psi_j(x) \wedge \Delta \vee \exists \ldots z_\mu \ldots \Delta' \wedge$$
$$\wedge \Big(\bigvee_{i,j} (Q_0 x\varphi_i(x) \wedge Ix(\varphi_i(x), \psi_j(x)) \wedge$$
$$\wedge \bigwedge_{k\neq i} (\leqslant x)(\varphi_k(x), \varphi_i(x)) \wedge \bigwedge_{k\neq j} (\leqslant x)(\psi_k(x), \psi_j(x))\Big)\Big).$$

By the two equivalences (i) and (ii) we can suppose for every $\Delta(A, B, C, D)$ occurring in Σ (use (1)):

(3) Either for every accentuated $\pi(x)$ in A or B $\neg Q_0 x\pi^*(x) \in \Sigma$ or Δ is of the form $(\leqslant x)(\mu(x), \nu(x))$ for $\mu, \nu \in Y$ and $Q_0 x\mu(x)$, $Q_0 x\nu(x) \in \Sigma$.

Now we define Σ_0 and Σ_1 as in [3]:

$$\Sigma_0 = \Sigma(Q_0)^- \cup \{\Delta\colon \Delta \text{ occurs in } \Sigma \text{ and for every accentuated } \mu(x) \text{ in } A \text{ or } B\ \neg Q_0 x\mu^*(x) \in \Sigma\}.$$

$$\Sigma_1 = (\Sigma \setminus \Sigma_0) \cup \Sigma(Q_0).$$

Clearly $|(px = 0 \wedge p^m|x)|_{\mathfrak{A}} \leqslant |(p^m|x)|_{\mathfrak{A}}$ for every $\mathfrak{A}$. If $Q_0 x(p^m|x) \in \Sigma$ therefore we can consider instead of Σ the sets

$$\Sigma \cup \{\neg Q_0 x(px = 0 \wedge p^m|x)\} \quad \text{and} \quad (\Sigma' \cup \{Q_0 x(px = 0 \wedge p^m|x)\}) \setminus \{Q_0 x(p^m|x)\}.$$

Σ' you get replacing $(p^m|x)$ by $(px = 0 \wedge p^m|x)$ in all formulas $(\leqslant x)(p^m|x, \eta(x))$ and $(\leqslant x)(\eta(x), p^m|x)$ occurring in Σ.

Therefore it is possible to assume

(4) If $Q_0 x(p^m|x) \in \Sigma$ then $\neg Q_0 x(px = 0 \wedge p^m|x) \in \Sigma$.

If $Q_0 x(p^m|x)$ and $\neg Q_0 x(px = 0 \wedge p^m|x)$ are in Σ and $\mathfrak{A}$ is a p-model of Σ then

$$\mathfrak{A} \leftrightarrows \bigoplus_{0<i<\omega} \mathfrak{Z}(p^i)^{\lambda(i)} \oplus \mathfrak{Z}(p^\infty)^{\lambda(\omega)}$$

where $\sum_{m<\varkappa\leqslant\omega} \lambda(\varkappa)$ is finite and $\lambda(\omega) \geqslant 1$. This follows from Theorems A and B. Then $\zeta_2(p, m, \mathfrak{A}) < \zeta_1(p, m, \mathfrak{A})$. Therefore we are only interested in p-models of Σ_0 with $\zeta_2(p, m, \mathfrak{A}) < \zeta_1(p, m, \mathfrak{A})$ if $Q_0 x(p^m|x) \in \Sigma$. We call them p-models of Σ_0 with AP (additional property).

LEMMA 6. *If Σ_0 has a p-model with* AP *it has a normal p-model $\mathfrak{A}$ with* AP *and* $|[p]\mathfrak{A}| < \omega$.

Proof. If $\Sigma_0 \cap L = \Sigma_0$ then the assertion is clear. Otherwise there exists some $\neg Q_0 x\eta(x) \in \Sigma_0$ with $\eta(x) = (px = 0 \wedge p^m|x)$ or $\eta(x) = (p^m|x)$.

If $\mathfrak{A}$ is a p-model of Σ_0 with AP by Theorems A and B $\neg Q_0 x\eta(x) \in \Sigma_0$ implies

$$\mathfrak{A} \leftrightarrows \bigoplus_{0<i<\omega} \mathfrak{Z}(p^i)^{\lambda(i)} \oplus \mathfrak{Z}(p^\infty)^{\lambda(\omega)} \quad \text{with} \quad \sum_{m<\varkappa\leqslant\omega} \lambda(\varkappa) < \omega.$$

Define $\lambda'(\varkappa) = \lambda(\varkappa)$ if $\lambda(\varkappa) < \omega$. Then there are natural numbers $\lambda'(\varkappa)$ for $\varkappa$ with $\lambda(\varkappa) \geqslant \omega$ such that

$$\mathfrak{B} = \bigoplus_{0<i<\omega} \mathfrak{Z}(p^i)^{\lambda'(i)} \oplus \mathfrak{Z}(p^\infty)^{\lambda'(\omega)} \models \Sigma_0.$$

$\mathfrak{B}$ has AP. ■

First we reduce our problem to the search for p-models of Σ_0 with AP. We use the ideas of [3].

Let $Y(\Sigma_1)$ be the set of all $\mu(x), \nu(x) \in Y$ with $(\leqslant x)(\mu(x), \nu(x)) \in \Sigma_1$. Working with Σ_1 we use $(<x)$ and Ix only as abbreviations. Without loss of generality we can assume

(5) For every $\mu(x), \nu(x) \in Y(\Sigma_1)$
either $(<x)(\mu(x), \nu(x)) \in \Sigma_1$,
or $Ix(\mu(x), \nu(x)) \in \Sigma_1$,
or $(<x)(\nu(x), \mu(x)) \in \Sigma_1$.

Further suppose

(6) $(\leqslant x)(p^m|x, \mu(x)) \in \Sigma_1$ for every $(p^m|x), \mu(x) \in Y(\Sigma_1)$.

This is possible by (5) because $(<x)(\mu(x), p^m|x) \in \Sigma_1$ would imply $|(p^m|x)|_{\mathfrak{A}} > \omega$ for every p-model $\mathfrak{A}$ of Σ_1. Then $|(p^m|x)|_{\mathfrak{A}} = |(px = 0 \wedge p^m|x)|_{\mathfrak{A}}$ by Proposition 2. We could replace $(p^m|x)$ by $(px = 0 \wedge p^m|x)$ in every formula of $\Sigma_1 \setminus \Sigma_0$. Assuming (5) necessary conditions are:

(7) If $T(Q_0) + \Sigma(Q_0) \models \neg Q_0 x(\neg\mu(x) \wedge \nu(x))$ then $(\leqslant x)(\nu(x), \mu(x)) \in \Sigma_1$.

(8) There is a function τ from $Y(\Sigma_1)$ in the infinite cardinals such that $\tau(\mu) \leqslant \tau(\nu)$ iff $(\leqslant x)(\mu(x), \nu(x)) \in \Sigma_1$, and $\tau(\mu) = \omega$ for some $\mu \in Y(\Sigma_1)$.

In [3] is shown:

LEMMA 7. *Let Σ^* be a finite set of unnegated and negated basic sentences of $T(Q_0)$. Let $W \cup \{\pi\}$ be a finite subset of Y such that $Q_0 x\eta(x) \in \Sigma^*$ for every $\eta \in W \cup \{\pi\}$ and $\pi(x) = (px = 0 \wedge p^n|x)$. Assume that for every $\eta \in W$*

$$T(Q_0) + \Sigma^* \models \neg Q_0 x(\neg\eta(x) \wedge \pi(x))$$

is not true. If $\mathfrak{A} \models \Sigma^*$ and $\omega \leqslant |\mathfrak{A}| \leqslant \lambda$ then there exists some $\varkappa$ such that $\mathfrak{B} = \mathfrak{A} \oplus \mathfrak{Z}(p^\varkappa)^\lambda \models \Sigma^*$, $|\eta|_{\mathfrak{A}} = |\eta|_{\mathfrak{B}}$ for $\eta \in W$ and $|\pi|_{\mathfrak{B}} = \lambda$.

LEMMA 8. *Σ has a normal p-model iff $\Sigma(Q_0)$ has a p-model and Σ_0 has a p-model with* AP.

Proof. We prove the nontrivial direction. Let $\mathfrak{C}_0$ be a countable normal p-model of Σ_0 with AP. This exists by Lemma 6.

$$\mathfrak{C}_0 \leftrightarrows \bigoplus_{0<i<\omega} \mathfrak{Z}(p^i)^{\lambda_0(i)} \oplus \mathfrak{Z}(p^\infty)^{\lambda_0(\omega)}.$$

By Theorem 2 there is a countable normal p-model $\mathfrak{C}_1$ of $\Sigma(Q_0)$

$$\mathfrak{C}_1 \leftrightarrows \bigoplus_{0<i<\omega} \mathfrak{Z}(p^i)^{\lambda_1(i)} \oplus \mathfrak{Z}(p^\infty)^{\lambda_1(\omega)}.$$

Define

$$\mathfrak{A}_0 \leftrightarrows \bigoplus_{0<i<\omega} \mathfrak{Z}(p^i)^{\lambda(i)} \oplus \mathfrak{Z}(p^\infty)^{\lambda(\omega)}$$

where

$$\lambda(\varkappa) = \begin{cases} \lambda_1(\varkappa) & \text{if } \lambda_1(\varkappa) = \omega, \\ \lambda_0(\varkappa) & \text{otherwise}. \end{cases}$$

Then $\mathfrak{A}_0$ is a countable normal p-model of $\Sigma(Q_0)+\Sigma_0$. Mention that $|\eta|_{\mathfrak{A}_0} = |\eta|_{\mathfrak{C}_0}$ if $\neg Q_0 x\eta(x) \in \Sigma_0$ because in this case $\lambda_1(i)<\omega$ if $\mathfrak{Z}(p^i) \models \exists x\eta(x)$ and $\lambda_1(\omega)<\omega$ if $\mathfrak{Z}(p^\infty) \models \exists x\eta(x)$.

Then $\mathfrak{A}_0 \models \Sigma_0$ is shown easily. If $Q_0 x(px = 0 \wedge p^n|x) \in \Sigma(Q_0)$ then $\mathfrak{A}_0 \models Q_0 x(px = 0 \wedge p^n|x)$ because by normality of $\mathfrak{C}_1$ there is some $\varkappa>n$ with $\lambda_1(\varkappa) = \lambda(\varkappa) = \omega$. If $Q_0 x(p^m|x) \in \Sigma(Q_0)$ then by (3) and AP $\lambda(\omega) \neq 0$ and therefore $\mathfrak{A}_0 \models Q_0 x(p^m|x)$. It follows $\mathfrak{A}_0 \models \Sigma_0+\Sigma(Q_0)$.

By (5) and (8) there is an enumeration $\mu_0, \ldots, \mu_i, \ldots$ of $Y(\Sigma_1)$ such that $i<j$ iff $\tau(\mu_i)\leqslant\tau(\mu_j)$ iff $(\leqslant x)(\mu_i(x), \mu_j(x)) \in \Sigma_1$. By (6) and (8) $\tau(\mu_i) = \omega = |\mu_i|_{\mathfrak{A}_0}$ if $\mu_i = (p^m|x)$ for some m. Using (7) we can apply Lemma 7 step by step to get some normal p-model $\mathfrak{B}$ of Σ with $|\mu_i|_{\mathfrak{B}} = \tau(\mu_i)$. ■

Lemma 8 implies Theorem 5.2. Effectively every Σ was replaced by a finite set of sets $\Sigma_0 \cup \Sigma_1$ such that (1)-(8) were fulfilled. To prove Theorem 5.1 by Lemma 8 and Theorem 2 we have to decide only whether Σ_0 has a p-model with AP or not. This will be done by the following

LEMMA 9. *There is an effective method to construct for every Σ_0 a set Ω of p-systems $\Gamma(\Sigma_0)$ such that Σ_0 has a p-model with AP iff some $\Gamma(\Sigma_0)$ of Ω has a solution.*

Then Lemma 9 and Theorem 3 imply the main result Theorem 5.1.

Proof of Lemma 9. For every $\Delta(A, B, C, D)$ occurring in Σ_0 fix

$$F_\Delta(x_1, \ldots, x_{f+1}) = 0$$

as constructed by Lemma 5. As remarked in Lemma 5 the number f can be choosen arbitrary large. Therefore and by Lemma 1.1 we can assume w.l.o.g. that there exists some natural f such that

(9) 1. Every $F_\Delta = 0$ for some Δ occurring in Σ_0 is constructed in the variables $x_1, \ldots, x_{f+1}$.

2. If $\zeta_3(p, i, k)$ occurs in Σ_0 then $i<f$.

3. If $\zeta_j(p, i, k)$ occurs in Σ_0 for $j \in \{1, 2\}$ then $i = f$.

4. If $\neg Q_0 x(px = 0 \wedge p^m|x) \in \Sigma_0$ or $\neg Q_0 x(p^m|x) \in \Sigma_0$ then $m<f$.

Let $\Gamma'(\Sigma_0)$ be the following system of equations and unequations in the variables $x_1, \ldots, x_{f+1}, y_f$:

(a) $F_\Delta = 0$ if $\Delta(A, B, C, D) \in \Sigma_0$,

$F_\Delta \neq 0$ if $\neg\Delta(A, B, C, D) \in \Sigma_0$,

b) $x_i \geqslant k$ if $\zeta_3(p, i, k) \in \Sigma_0$,

$x_i < k$ if $\neg\zeta_3(p, i, k) \in \Sigma_0$,

$x_f \geqslant k$ if $\zeta_1(p, f, k) \in \Sigma_0$,

$x_f < k$ if $\neg\zeta_1(p, f, k) \in \Sigma_0$,

$y_f \geqslant k$ if $\zeta_2(p, f, k) \in \Sigma_0$,

$y_f < k$ if $\neg\zeta_2(p, f, k) \in \Sigma_0$,

c) $x_{f+1} = 0$ if $x_f = 0$ or $y_f < x_f$,

$x_f \geqslant y_f$,

$x_f = y_f$ if $\neg Q_0 x(p^m|x) \in \Sigma_0$ for some m,

$x_f > y_f$ if $Q_0 x(p^m|x) \in \Sigma$.

If Σ_0 has a p-model with AP then there is a normal p-model $\mathfrak{A}$ of Σ_0 with AP and $|[p]\mathfrak{A}|<\omega$ by Lemma 6. Then

$$x_i = \zeta_3(p, i, \mathfrak{A}) \quad \text{for} \quad i<f,$$
$$x_f = \zeta_1(p, f, \mathfrak{A}),$$
$$y_f = \zeta_2(p, f, \mathfrak{A})$$

and

$$x_{f+1} = \begin{cases} \sum\limits_{j>f} \zeta_1(p, j, \mathfrak{A}) & \text{if this is finite}, \\ 0 & \text{otherwise}, \end{cases}$$

is a solution of $\Gamma'(\Sigma_0)$ by (1), (3), (4), Lemma 5, and (9). On the other hand if $x_1, \ldots, x_{f+1}, y_f$ is a solution of $\Gamma'(\Sigma_0)$ then

$$\mathfrak{B} = \bigoplus_{0<i<f} \mathfrak{Z}(p^i)^{x_i} \oplus \mathfrak{Z}(p^f)^{(y_f-1)\varepsilon} \oplus \mathfrak{Z}(p^{f+x_{f+1}})^\varepsilon \oplus \mathfrak{Z}(p^\infty)^{x_f-y_f}$$

with

$$\varepsilon = \begin{cases} 1 & \text{if } y_f \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

is a countable normal model of Σ_0 with AP.

You see

$$\zeta_3(p, i, \mathfrak{B}) = x_i \quad \text{if} \quad i<f,$$
$$\zeta_1(p, f, \mathfrak{B}) = x_f,$$
$$\zeta_2(p, f, \mathfrak{B}) = y_f$$

and

$$\sum_{i>f} \zeta_1(p, i, \mathfrak{B}) = x_{f+1} \quad \text{if the left side is finite.}$$

By construction of $\Gamma'(\Sigma_0)$ and (2), (9) $\mathfrak{B} \models \Sigma_0 \cap L(Q_0)$. Since we have assumed (1), $\mathfrak{B} \models \ldots \exists z_\mu \ldots \Delta'(A, B, C, D)$ for every Δ occurring in Σ_0. Furthermore $|[p]\mathfrak{B}|<\omega$.

Therefore Lemma 5 implies

$$\mathfrak{B} \models \Delta(A, B, C, D) \quad \text{if} \quad \Delta \in \Sigma_0$$

and

$$\mathfrak{B} \models \neg\Delta(A, B, C, D) \quad \text{if} \quad \neg\Delta \in \Sigma_0 .$$

Since $x_f > y_f$ if $Q_0 x(p^m|x) \in \Sigma$ for some m, $\mathfrak{B}$ has AP. It is shown that $\Omega_0 = \{\Gamma'(\Sigma_0)\}$ has the property desired in the lemma. Now we have to replace Ω_0 by a finite set of p-systems. To do this we construct effectively a sequence of finite sets Ω_i $(0 \leqslant i \leqslant f)$ of systems built up like $\Gamma'(\Sigma_0)$ such that:

(*) Some $\Gamma_i \in \Omega_i$ has a solution iff some Γ_{i+1} of Ω_{i+1} has a solution.

Step by step we cancel all conditions b) and c) of $\Sigma'(\Gamma_0)$. Assume this is done for $x_1, \ldots, x_{j-1}$ $(j<f)$ in Ω_{j-1}. Let $\Gamma_{j-1} \in \Omega_{j-1}$. Since $T_p(Q_0)$ is decidable we assume w.l.o.g. $\Sigma_0 \cap L(Q_0)$ is consistent. Therefore it is possible to replace effectively all conditions b) for x_j by a condition $k_1 \leqslant x_j$ or a condition $k_1 \leqslant x_j < k_2$ where $0 \leqslant k_1 < k_2$. In the case $k_1 \leqslant x_j < k_2$ we omit this condition varying about all possibilities of substitutions $x_j = k$ with $k_1 \leqslant k < k_2$. If there is $k_1 \leqslant x_j$ only we substitute $x_j = z_j + k_1$. Then the systems Γ_{f-1} of Ω_{f-1} are constructed in the variables $z_1, \ldots, z_{f-1}, x_f, y_f, x_{f+1}$ and the conditions b) and c) of Γ_{f-1} contain the variables x_f, y_f, x_{f+1} only. W.l.o.g. assume that the condition b) is of the form $k_1 \leqslant x_f$ $(<k_2)$, $k_1' \leqslant y_f$ $(<k_2')$.

If the conditions b) and c) of Γ_{f-1} are consistent with $x_f = y_f = x_{f-1} = 0$ then let Ω_{f1} be the set of all $\Gamma_{f-1} \in \Omega_{f-1}$ restricted to part a) where x_f, y_f, x_{f+1} are replaced by 0. Ω_{f2} you get from Ω_{f-1} substituting $x_f = k$ for all $k>0$ with $\max\{k_1, k_1'\} \leqslant k < \min\{k_2, k_2'\}$ if $x_f < k_2$ or $y_f < k_2'$ comes true or

$$x_f = \max\{1, k_1, k_1'\} + z_f$$

otherwise, and omitting b) and c), if Γ_{f-1} b), c) is consistent with $x_f = y_f > 0$. Ω_{f3} you get similarly by the substitution $x_{f+1} = 0$ and $x_f = k$ for all k with $k_1 \leqslant k < k_2$ and $k_1 < k$ if $x_f < k_2$ comes true or $x_{f+1} = 0$ and $x_f = \max\{k_1, k_1'+1\} + z_f$ otherwise, if b) and c) are consistent with $x_f > y_f$. Then $\Omega = \Omega_f = \Omega_{f1} \cup \Omega_{f2} \cup \Omega_{f3}$ has the desired property. The construction was effective. ■

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