As $\underset{\alpha<\lambda}{\triangle} t' \circ \pi(\alpha)$ is unbounded, take $P \in \underset{\alpha<\lambda}{\triangle} t' \circ \pi(\alpha)$ such that $\pi^{-1}(\beta) \in P$, then $P \in t'(\beta) = F(A_\beta)$ (since $\mu(A_\beta) = 1$) and so $\exists\, \alpha \in P \cap \lambda$ such that $P \cap \lambda \notin B'_\alpha$. On the other hand $\forall\, \alpha \in P$, $P \in t' \circ \pi(\alpha)$, in particular for every $\alpha \in P \cap \lambda$, $P \in t' \circ \pi(\alpha) = t'(f(\alpha)) = F(A_{f(\alpha)})$ so $P \cap \lambda \in A_{f(\alpha)} = B'_\alpha$, a contradiction.

Then $\mu(\underset{\alpha<\lambda}{\triangle} B'_\alpha) = 1$, and μ is normal.

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I. V. I. C.
Caracas, Venezuela
UNION COLLEGE
Schenectady, New York

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On saturated sets of ideals and Ulam's problem

by

Alan D. Taylor * (Schenectady, N. Y.)

Abstract. A set $\mathscr{I}$ of countably complete ideals on ω_1 is called λ-saturated iff for every collection $\{X_\alpha\colon \alpha<\lambda\} \subseteq \mathcal{P}(\omega_1) - \bigcup \mathscr{I}$ there exists $\{\alpha, \beta\} \in [\lambda]^2$ such that $X_\alpha \cap X_\beta \notin \bigcap \mathscr{I}$. An old problem of Ulam asks if there can exist a 2-saturated set $\mathscr{I}$ of size ω_1. We show that a weak version of Kurepa's hypothesis implies that if $|\mathscr{I}| \leqslant \omega_1$ then $\mathscr{I}$ is not even ω_2-saturated. This answers a question of Prikry. Some related results are obtained and several questions are stated.

§ 0. Introduction. Over thirty years ago S. Ulam raised the following question (see [6]). Let $\varkappa$ be an uncountable cardinal less than the first weakly inaccessible cardinal. What is the smallest cardinal λ having the property that there exists a family of λ two valued countable additive measures defined for the subsets of $\varkappa$ (singletons having measure 0 and $\varkappa$ having measure 1 for each of them) such that every subset of $\varkappa$ is measurable with respect to at least one of these measures? The following version of this question was stated as Problem 81 of [7] and will be referred to here as *Ulam's problem*.

PROBLEM (S. Ulam). Can one define $\aleph_1$ σ-additive 0-1 measures on ω_1 so that each subset is measurable with respect to one of them?

In this paper we will consider several generalizations of Ulam's problem. Several new results are obtained and many older results from the literature are collected together. Some eighteen open problems are also stated.

We begin by establishing some notation. ν will denote an arbitrary cardinal, while λ and μ will be reserved for infinite cardinals and $\varkappa$ for an uncountable cardinal. We will use the phrase "ideal on $\varkappa$" to mean "proper uniform ideal on $\varkappa$". (An ideal I on $\varkappa$ is called *uniform* iff $[\varkappa]^{<\varkappa} \subseteq I$.) The (normal) ideal of non-stationary subsets of the regular cardinal $\varkappa$ is denoted by $NS_\varkappa$.

If I is an ideal on $\varkappa$ then I^+ denotes $\mathscr{P}(\varkappa) - I$ (the sets of "positive I-measure") and I^* denotes $\{X \subseteq \varkappa\colon \varkappa - X \in I\}$ (the sets of "I-measure one"). If $A \in I^+$ then the restriction of I to A is the ideal

$$I \upharpoonright A = \{X \subseteq \varkappa\colon X \cap A \in I\}.$$

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Notice that $I\restriction A$ is the ideal generated by $I\cup\{\varkappa-A\}$. If $f\colon \varkappa\to\varkappa$ is such that $f^{-1}(\alpha)\in I$ for every $\alpha<\varkappa$ then the ideal $f_*(I)$ is defined by

$$f_*(I)=\{X\subseteq\varkappa\colon f^{-1}(X)\in I\}.$$

If $\mathscr{I}$ is a set of ideals on $\varkappa$ then $\mathscr{I}^+=\bigcap\{I^+\colon I\in\mathscr{I}\}$ (so $\mathscr{I}^+=\mathscr{P}(\varkappa)-\bigcup\mathscr{I}$), and if $A\in\mathscr{I}^+$ then $\mathscr{I}\restriction A=\{I\restriction A\colon I\in\mathscr{I}\}$. If $f\colon\varkappa\to\varkappa$ is such that $f^{-1}(\alpha)\in\bigcap\mathscr{I}$ for all $\alpha<\varkappa$ then the collection $f_*(\mathscr{I})$ is defined by

$$f_*(\mathscr{I})=\{f_*(I)\colon I\in\mathscr{I}\}.$$

The following definition is central to the considerations of this paper.

DEFINITION 0.1. A set $\mathscr{I}$ of ideals on $\varkappa$ will be called ν-*saturated* iff for every collection $\{X_\alpha\colon\alpha<\nu\}\subseteq\mathscr{I}^+$ there exists $\{\alpha,\beta\}\in[\nu]^2$ such that $X_\alpha\cap X_\beta\notin\bigcap\mathscr{I}$.

Hence, to say that $\mathscr{I}$ is *not* ν-saturated means that there are ν sets in $\mathscr{I}^+$ such that pairwise intersections are of I-measure zero for every $I\in\mathscr{I}$. In terms of Definition 0.1, Ulam's problem asks if there is a 2-saturated collection $\mathscr{I}$ of size ω_1 consisting of $\aleph_1$-complete ideals on ω_1. In discussing some results and questions concerning generalizations of Ulam's problem, it is convenient to have available the following notation.

NOTATION 0.2. If $\mathscr{R}$ is a set of ideals on $\varkappa$ then the symbol

$$\text{“}\langle\varkappa\colon\lambda,\mu\rangle\xrightarrow{\mathscr{R}}\nu\text{”}$$

denotes the following assertion.

If $\mathscr{I}\subseteq\mathscr{R}$ and $|\mathscr{I}|=\lambda$ and every ideal in $\mathscr{I}$ is at least μ-complete then $\mathscr{I}$ is no ν-saturated.

Of course our primary interest is with the special case in which $\mathscr{R}$ is the set of all ideals on $\varkappa$. For this case, we suppress the “$\mathscr{R}$” in the notation and simply write

$$\langle\varkappa\colon\lambda,\mu\rangle\to\nu.$$

For a regular cardinal $\varkappa$ we will also be concerned with the relations obtainable when one of the following two sets of ideals is playing the role of “$\mathscr{R}$”.

1. $\mathscr{R}=\mathscr{N}$: the set of all normal ideals on $\varkappa$.
2. $\mathscr{R}=\mathscr{E}$: the set of all ideals on $\varkappa$ extending $\mathrm{NS}_\varkappa$.

Notice that for a fixed set $\mathscr{R}$ of ideals on $\varkappa$, the relation $\langle\varkappa\colon\lambda,\mu\rangle\xrightarrow{\mathscr{R}}\nu$ gets stronger as λ or ν increases or μ decreases. The negation of the assertion in 0.2 is denoted by striking out the arrow.

In the way of illustration, we restate Ulam's problem with the notation introduced above.

PROBLEM A (Ulam). Does $\langle\omega_1\colon\omega_1,\omega_1\rangle\to2$ hold?

In section 1 we prove some general results, the most important of which deal with reducing the consideration of arbitrary sets of ideals on $\varkappa$ to just those extending $\mathrm{NS}_\varkappa$. Section 2 contains the major results known to be provable in ZFC, and we generalize (via a different proof) the Erdös–Alaoglu theorem to arbitrary successor cardinals. In section 3 we introduce a sequence of successively weaker axioms (ranging from Kurepa's hypothesis through the assumption that there is no ω_2-saturated ideal on ω_1). These axioms are applied to several generalized versions of Ulam's problem in section 4. In particular, we show that $\mathrm{KH}\omega_1$ implies that $(\omega_1\colon\omega_1,\omega_1)\to\omega_2$ holds, answering a question of Prikry [15] and strengthening results of Prikry [15] and Grzegorek [8]. Section 5 considers ideals on $\varkappa$ that are not $\varkappa$-complete.

§ 1. General results. In this section we obtain several results that allow us to derive "nicer" saturated sets of ideals from a given saturated set of ideals. The key result is the following.

THEOREM 1.1. *Suppose $\varkappa$ is regular and $\mathscr{I}$ is a set of $\aleph_1$-complete ideals on $\varkappa$ such that $|\mathscr{I}|\leqslant\varkappa$. Then there exists a $<\varkappa$ to 1 function $g\colon\varkappa\to\varkappa$ such that $g_*(I)\cup\mathrm{NS}_\varkappa$ generates a proper ideal for every $I\in\mathscr{I}$.*

Proof. If I is an ideal on $\varkappa$ then let $\langle I\cup\mathrm{NS}_\varkappa\rangle$ be defined as follows

$$X\in\langle I\cup\mathrm{NS}_\varkappa\rangle\quad\text{iff}\quad\exists\,Y\in I\,\exists\,Z\in\mathrm{NS}_\varkappa\ (X=Y\cup Z).$$

Notice that $\langle I\cup\mathrm{NS}_\varkappa\rangle$ is an ideal on $\varkappa$ iff $\varkappa\notin\langle I\cup\mathrm{NS}_\varkappa\rangle$ iff $\mathrm{NS}_\varkappa\cap I^*=0$. Moreover, if I is λ-complete then so is $\langle I\cup\mathrm{NS}_\varkappa\rangle$. If $\varkappa\notin\langle I\cup\mathrm{NS}_\varkappa\rangle$ then we will say $\langle I\cup\mathrm{NS}_\varkappa\rangle$ is *proper*.

Let $S=\{f\in{}^\varkappa\varkappa\colon f\text{ is }<\varkappa\text{ to }1\}$ and for an ideal I on $\varkappa$ define $<_I$ on S by

$$f<_I g\quad\text{iff}\quad\{\alpha<\varkappa\colon f(\alpha)<g(\alpha)\}\in I^*.$$

Theorem 1.1 is an easy consequence of the following sequence of claims. Proofs of the easy ones have been omitted, and we fix an $\aleph_1$-complete uniform ideal I on the regular cardinal $\varkappa$.

CLAIM 1 (Well-known). *$\langle I\cup\mathrm{NS}_\varkappa\rangle$ fails to be proper iff there is a set $A\in I^*$ and a regressive function $f\colon A\to\varkappa$ such that f is $<\varkappa$ to 1.*

CLAIM 2. *If I is $\aleph_1$-complete then $<_I$ is well founded on S.*

CLAIM 3. *If $f\in S$ then f is $<_I$ minimal iff $\langle f_*(I)\cup\mathrm{NS}_\varkappa\rangle$ is proper.*

CLAIM 4. *Suppose $f,g\in S$ and $\langle f_*(I)\cup\mathrm{NS}_\varkappa\rangle$ is proper. If $\{\xi<\varkappa\colon f(\xi)<g(\xi)\}\in I$ then $\langle g_*(I)\cup\mathrm{NS}_\varkappa\rangle$ is proper.*

CLAIM 5. *If $\{f_\alpha\colon\alpha<\varkappa\}\subseteq S$ then there exists $g\in S$ such that for every $\alpha<\varkappa$ $|\{\xi\colon f_\alpha(\xi)<g(\xi)\}|<\varkappa$.*

Proof. For each $\delta<\varkappa$ define $X_\delta\subseteq\varkappa$ by

$$\xi\in X_\delta\quad\text{iff}\quad\exists\,\alpha\leqslant\delta\ \text{s.t.}\ f_\alpha(\xi)\leqslant\delta.$$

Now define $g\colon \varkappa \to \varkappa$ by $g(\xi) = \inf\{\delta\colon \xi \in X_\delta\}$. Since $|X_\delta| < \varkappa$ it is easy to see that $g \in S$. Now fix $\alpha < \varkappa$ and suppose $\xi > \sup \bigcup \{f_\alpha^{-1}(\gamma)\colon \gamma \leqslant \alpha\}$. Let $\delta = f_\alpha(\xi)$. Then $\delta > \alpha$ so $\xi \in X_\delta$. Hence $g(\xi) \leqslant \delta = f_\alpha(\xi)$.

To complete the proof of Theorem 1.1. Let $\mathscr{I} = \{I_\alpha\colon \alpha < \varkappa\}$ be the given set of uniform $\aleph_1$-complete ideals on $\varkappa$. For each $\alpha < \varkappa$ choose $f_\alpha \in S$ (by Claim 2) so that f_α is $<_{I_\alpha}$ minimal. Now choose $g \in S$ as guaranteed to exist by Claim 5. That g is the desired function follows immediately from Claims 1, 3, 4 and 5.

COROLLARY 1.2. *Suppose $\varkappa$ is a regular cardinal, $\lambda \leqslant \varkappa$ and $\omega_1 \cdot \nu \leqslant \mu$. If $\langle \varkappa\colon \lambda, \mu\rangle \overset{\mathscr{E}}{\to} \nu$ then $\langle \varkappa\colon \lambda, \mu\rangle \to \nu$.*

Proof. Suppose $\mathscr{I}$ shows that $\langle \varkappa\colon \lambda, \mu\rangle \nrightarrow \nu$. Choose g as guaranteed to exist by Theorem 1.1. For each $I \in \mathscr{I}$ let $J_I = \langle g_*(I) \cup \mathrm{NS}_\varkappa\rangle$. Then J_I is proper and $J_I \supseteq \mathrm{NS}_\varkappa$. We claim that $\mathscr{J} = \{J_I\colon I \in \mathscr{I}\}$ shows that $\langle \varkappa\colon \lambda, \mu\rangle \overset{\mathscr{E}}{\nrightarrow} \nu$. If not then there exists $\{X_\alpha\colon \alpha < \nu\} \subseteq \mathscr{J}^+$ such that $X_\alpha \cap X_\beta \in \bigcap \mathscr{J}$ for $\alpha < \beta < \nu$. Since $\mu \geqslant \nu$ we lose no generality in assuming that $X_\alpha \cap X_\beta = 0$ for $\alpha < \beta < \nu$. But then $\{g^{-1}(X_\alpha)\colon \alpha < \nu\} \subseteq \mathscr{I}^+$ and $g^{-1}(X_\alpha) \cap g^{-1}(X_\beta) = 0$ for $\alpha < \beta < \nu$. This contradicts our assumption that $\mathscr{I}$ shows $\langle \varkappa\colon \lambda, \mu\rangle \nrightarrow \nu$.

A well known result of Solovay [17] shows that if a regular cardinal $\varkappa$ carries a $\varkappa$-complete $\varkappa^+$-saturated ideal then it carries a normal $\varkappa^+$-saturated ideal. In analogy with this one would hope that a sufficiently saturated ("small") set of $\varkappa$-complete ideals on $\varkappa$ would give rise to an equally saturated ("small") set of normal ideals on $\varkappa$. In view of this (and the results in section 4), we would be very interested in a (positive) solution to the following:

PROBLEM B. For a regular cardinal $\varkappa$, does $\langle \varkappa\colon \varkappa, \varkappa\rangle \overset{\mathscr{N}}{\to} 2$ imply $\langle \varkappa\colon \varkappa, \varkappa\rangle \to 2$?

Of course a consequence of Theorem 1.1 is that to answer Problem B affirmatively it suffices to show that $\langle \varkappa\colon \varkappa, \varkappa\rangle \overset{\mathscr{N}}{\to} 2$ implies $\langle \varkappa\colon \varkappa, \varkappa\rangle \overset{\mathscr{E}}{\to} 2$.

In order to present one other partial result related to Problem B, we need to introduce two more sets of ideals to play the role of "$\mathscr{R}$" in the notation we are using.

1. $\mathscr{P}$ denotes the set of all P-point ideals on $\varkappa$. (A $\varkappa$-complete ideal I on $\varkappa$ is called a *P-point* iff for every $f\colon \varkappa \to \varkappa$ such that $f^{-1}(\alpha) \in I$ for every $\alpha < \varkappa$ there exists $X \in I^*$ such that $f \restriction X$ is $< \varkappa$ to 1.)

2. $\mathscr{Q}$ denotes the set of all Q-point ideals on $\varkappa$. (A $\varkappa$-complete ideal I on $\varkappa$ is called a *Q-point* iff for every $f\colon \varkappa \to \varkappa$ such that f is $< \varkappa$ to 1 there exists $X \in I^*$ such that $f \restriction X$ is 1 to 1.)

Suppose I is a $\varkappa$-complete ideal on $\varkappa$. Then Kanamori [9] has shown that if $I \supseteq \mathrm{NS}_\varkappa$ then I is a Q-point and Węglorz [20] has shown that if I is normal then I is a P-point. Hence, one immediately obtains the following:

PROPOSITION 1.3. (i) *If $\langle \varkappa\colon \lambda, \varkappa\rangle \overset{\mathscr{Q}}{\to} \nu$ then $\langle \varkappa\colon \lambda, \varkappa\rangle \overset{\mathscr{E}}{\to} \nu$.* (ii) *If $\langle \varkappa\colon \lambda, \varkappa\rangle \overset{\mathscr{P}}{\to} \nu$ then $\langle \varkappa\colon \lambda, \varkappa\rangle \overset{\mathscr{N}}{\to} \nu$.*

Corollary 1.2 shows that in many cases the converse of Proposition 1.3 (i) holds. The next corollary shows the same for Proposition 1.3 (ii).

COROLLARY 1.4. *Suppose $\varkappa$ is a regular cardinal, $\lambda \leqslant \varkappa$ and $\nu \leqslant \varkappa$. If $\langle \varkappa\colon \lambda, \varkappa\rangle \overset{\mathscr{N}}{\to} \nu$ then $\langle \varkappa\colon \lambda, \varkappa\rangle \overset{\mathscr{P}}{\to} \nu$.*

Proof. Suppose $\mathscr{I}$ shows that $\langle \varkappa\colon \lambda, \varkappa\rangle \overset{\mathscr{P}}{\nrightarrow} \nu$. Let $\mathscr{J} = \{J_I\colon I \in \mathscr{I}\}$ be as in the proof of Corollary 1.2 above. Now since $I \in \mathscr{I}$ is a P-point, it is an easy exercise to check that $g_*(I)$ is also a P-point and therefore so is $J_I = \langle g_*(I) \cup \mathrm{NS}_\varkappa\rangle$. But any P-point extending $\mathrm{NS}_\varkappa$ must, in fact, be a normal ideal on $\varkappa$ (see [3]). Again, as in the proof of Corollary 1.2, we see that $\mathscr{J}$ shows $\langle \varkappa\colon \lambda, \varkappa\rangle \overset{\mathscr{N}}{\nrightarrow} \nu$.

Corollary 1.2 also shows that if $\langle \varkappa\colon \varkappa, \varkappa\rangle \overset{\mathscr{Q}}{\to} 2$ holds then $\langle \varkappa\colon \varkappa, \varkappa\rangle \overset{\mathscr{P}}{\to} 2$ holds. An affirmative answer to Problem B would follow from the converse of this.

PROBLEM C. For a regular cardinal $\varkappa$, does $\langle \varkappa\colon \varkappa, \varkappa\rangle \overset{\mathscr{P}}{\to} 2$ imply $\langle \varkappa\colon \varkappa, \varkappa\rangle \overset{\mathscr{Q}}{\to} 2$?

§ 2. Results in ZFC. A classic result of Ulam shows that if $\varkappa$ is less than the first weakly inaccessible then $\varkappa$ does not carry a countably complete ultrafilter. In our notation, this says $\langle \varkappa\colon 1, \omega_1\rangle \to 2$. At the time Ulam originally stated his problem (see [6]) he was also aware that $\langle \varkappa\colon n, \omega_1\rangle \to 2$ for every $n \in \omega$. The first extension of this was provided in 1948 by Alaoglu and Erdös [6]. Their theorem (as stated in [6]) asserts that for $\varkappa$ as above one has $\langle \varkappa\colon \omega, \omega_1\rangle \to 2$, but it is easy to see that their proof actually yields the following:

THEOREM 2.1 (Alaoglu–Erdös [6]). $\langle \varkappa\colon \omega, \omega_1\rangle \to \omega_1$ *iff* $\langle \varkappa\colon 1, \omega_1\rangle \to \omega_1$.

Although our concern in this paper is with 2 valued measures (i.e. ideals), there are several natural questions analogous to the ones we are considering but with respect to real valued measures, or even arbitrary σ-algebras satisfying certain chain conditions. For example Prikry showed [15] that if $\langle \gamma\colon 1, \omega_1\rangle \to \omega_1$ for every $\gamma \leqslant \varkappa$ then for every countable family of countably additive real valued measures on $\varkappa$, one can find $\aleph_1$ pairwise disjoint subsets of $\varkappa$ each of which is of outer measure one with respect to every measure in the collection. Grzegorek [8] showed that Prikry's arguments even extend to the more general context of σ-algebras satisfying certain chain conditions.

The Alaoglu–Erdös argument does not generalize to $\lambda > \omega$. Nevertheless, a different approach (suggested bq an argument in [1]) yields the following:

THEOREM 2.2. *If $\lambda < \varkappa$ then $\langle \varkappa\colon \lambda, \lambda^+\rangle \to \lambda^+$ iff $\langle \varkappa\colon 1, \lambda^+\rangle \to \lambda^+$.*

Proof. The implication from right to left is trivial. Suppose then that no λ^+-complete ideal on $\varkappa$ is λ^+-saturated and let $\mathscr{I} = \{I_\alpha\colon \alpha < \lambda\}$ be a set of λ^+-complete ideals on $\varkappa$. We will show that $\mathscr{I}$ is not λ^+-saturated. For each $\alpha < \lambda$ let $\mathscr{X}_\alpha = \{X_\beta^\alpha\colon \beta < \lambda^+\}$ be a pairwise disjoint partition of $\varkappa$ into sets in I_α^+. This is possible since I_α is λ^+-complete (and not λ^+-saturated). Define a function $H\colon \lambda \to \lambda$ by

$$H(\xi) = \inf\{\alpha < \lambda\colon |\{\beta < \lambda^+\colon X_\beta^\alpha \in I_\xi^+\}| = \lambda^+\}.$$

Notice that $H(\xi)\leqslant\xi$ for every $\xi<\lambda$. Choose $\delta<\lambda^+$ such that for every $\xi<\lambda$ if $\alpha<H(\xi)$ and $X^\alpha_\beta\in I^+_\xi$ then $\beta<\delta$. For each $\alpha<\lambda$ let $\mathscr{X}'_\alpha=\{X^\alpha_\beta:\ \delta<\beta<\lambda^+\}$. It is now easy to inductively construct a sequence $\{Y_\xi:\ \xi<\lambda\}$ of distinct sets such that for every $\xi<\lambda$ $Y_\xi\in\mathscr{X}'_{H(\xi)}\cap I^+_\xi$. For every $\xi<\lambda$ define Z_ξ by

$$Z_\xi = Y_\xi-\bigcup\{Y_\alpha:\ H(\alpha)<H(\xi)\}\,.$$

Then $Z_\xi\in I^+_\xi$ since $Y_\xi\in I^+_\xi$ and we are subtracting off at most λ sets each of which is in I_ξ. Hence $\{Z_\xi:\ \xi<\lambda\}$ is a collection of pairwise disjoint sets such that $Z_\xi\in I^+_\xi$ for every $\xi<\lambda$. Now for each $\xi<\lambda$, let $\{Z^\xi_\alpha:\ \alpha<\lambda^+\}$ be a pairwise disjoint partition of Z_ξ into sets in I^+_ξ. Finally, we define A_α for each $\alpha<\lambda^+$ by

$$A_\alpha=\bigcup\{Z^\xi_\alpha:\ \xi<\lambda\}\,.$$

Then $\{A_\alpha:\ \alpha<\lambda^+\}$ shows that $\mathscr{I}$ is not λ^+-saturated.

COROLLARY 2.3. $\langle\mu^+:\mu,\mu^+\rangle\to\mu^+$.

Of course Corollary 2.3 suggests the following generalized version of Ulam's problem.

PROBLEM D. Does $\langle\mu^+:\mu^+,\mu^+\rangle\not\to\mu^+$ always hold?

Theorems 2.1 and 2.2 show (roughly) that the only way a cardinal $\varkappa$ can carry a sufficiently saturated "small" set of sufficiently complete ideals is if $\varkappa$ in fact carries a single ideal that is this saturated and complete. Of course "small" here means "of size less than the completeness", and so these results say nothing about sets of ideals of size $\varkappa$. Nevertheless, by using a "normal version" of the proof of Theorem 2.2, the following result was established in [18], Lemma 6.5.

THEOREM 2.4. *If $\varkappa$ is regular then* $\langle\varkappa:\varkappa,\varkappa\rangle\overset{\mathcal{N}}{\to}\varkappa^+$ *iff* $\langle\varkappa:1,\varkappa\rangle\overset{\mathcal{N}}{\to}\varkappa^+$.

§ 3. From $\mathrm{KH}_\varkappa$ to $\mathrm{FK}_\varkappa$.

In this section we collect together several additional axioms that are relevant to our considerations. We list these now, with the assumption in each that $\varkappa=\mu^+$.

1. $\mathrm{KH}_\varkappa$ (Kurepa's hypothesis for $\varkappa$): There exists $F\subseteq\mathscr{P}(\varkappa)$ such that $|F|=\varkappa^+$ and for each $\alpha<\varkappa$

$$|\{X\cap\alpha:\ X\in F\}|\leqslant|\alpha|\,.$$

2. $\mathrm{TH}_\varkappa$ (Transversals hypothesis for $\varkappa$): There exists $F\subseteq{}^\varkappa\mu$ such that $|F|=\varkappa^+$ and $|\{\alpha<\varkappa:\ f(\alpha)=g(\alpha)\}|<\varkappa$ whenever $\{f,g\}\in[F]^2$.

3. $\mathrm{Th}_\varkappa(\mathrm{NS}_\varkappa)$: There exists $F\subseteq{}^\varkappa\mu$ such that $|F|=\varkappa^+$ and $\{\alpha<\varkappa:\ f(\alpha)=g(\alpha)\}\in\mathrm{NS}_\varkappa$ whenever $\{f,g\}\in[F]^2$.

4. $\mathrm{SpH}_\varkappa$ (Splitting hypothesis for $\varkappa$): If I is a $\varkappa$-complete ideal on $\varkappa$ then there exists $\{X_\alpha:\ \alpha<\varkappa^+\}\subseteq I^+$ such that $|X_\alpha\cap X_\beta|<\varkappa$ for $\alpha<\beta<\varkappa^+$.

5. $\mathrm{SatH}_\varkappa$ (Saturation hypothesis for $\varkappa$): There is no $\varkappa^+$-saturated $\varkappa$-complete ideal on $\varkappa$.

6. $\mathrm{FH}_\varkappa$ (Fodor's hypothesis for $\varkappa$): If I is a $\varkappa$-complete ideal on $\varkappa$ and $\{A_\alpha:\ \alpha<\varkappa\}\subseteq I^+$ then there exists a pairwise disjoint collection $\{B_\alpha:\ \alpha<\varkappa\}\subseteq I^+$ such that for every $\alpha<\varkappa$ $B_\alpha\subseteq A_\alpha$.

THEOREM 3.1. $\mathrm{KH}_\varkappa\to\mathrm{TH}_\varkappa\to\mathrm{TH}_\varkappa(\mathrm{NS}_\varkappa)\to\mathrm{SpH}_\varkappa\to\mathrm{SatH}_\varkappa\to\mathrm{FH}_\varkappa$.

Proof. The first implication is well known (see [4] where "$\mathrm{TH}_\varkappa$" is referred to as "$\mathrm{wKH}_\varkappa$"), and the second implication is trivial. The fact that $\mathrm{TH}_\varkappa$ implies $\mathrm{SpH}_\varkappa$ is due to Prikry [14], and the following proof that $\mathrm{Th}_\varkappa(\mathrm{NS}_\varkappa)$ implies $\mathrm{SpH}_\varkappa$ is a modification of his proof.

Let $\{f_\alpha:\ \alpha<\varkappa^+\}$ be as guaranteed to exist by $\mathrm{TH}_\varkappa(\mathrm{NS}_\varkappa)$ and let I be a $\varkappa$-complete ideal on $\varkappa$. By claims 1-3 in the proof of Theorem 1.1 we can choose $g:\ \varkappa\to\varkappa$ such that g is $<\varkappa$ to 1 and such that $J=\langle g_*(I)\cup\mathrm{NS}_\varkappa\rangle$ is a proper $\varkappa$-complete ideal on $\varkappa$. For each $\alpha<\varkappa^+$ choose $X_\alpha\in J^+$ and $n_\alpha\in\mu$ such that $f_\alpha(X_\alpha)=\{n_\alpha\}$. (Recall that $\varkappa=\mu^+$). Choose $Y\in[\varkappa^+]^{\varkappa^+}$ and $n\in\mu$ such that $n_\alpha=n$ for every $\alpha\in Y$. If $\{\alpha,\beta\}\in[Y]^2$ let $A(\alpha,\beta)=\{\xi<\varkappa:\ f_\alpha(\xi)=f_\beta(\xi)\}$. Now for each $\alpha\in Y$ let $Z_\alpha=X_\alpha-\nabla\{A(\beta,\alpha):\ \beta<\alpha\}$ where "∇" denotes diagonal union. Since $A(\beta,\alpha)\in\mathrm{NS}_\varkappa$ we see that $\nabla\{A(\beta,\alpha):\ \beta<\alpha\}\in\mathrm{NS}_\varkappa\subseteq J$. Hence $Z_\alpha\in I^+$ for each $\alpha\in Y$ and it is easy to see that if $\{\alpha,\beta\}\in[Y]^2$ then $|Z_\alpha\cap Z_\beta|<\varkappa$. Since $g_*(I)\subseteq J$ the set $\{Z_\alpha:\ \alpha\in Y\}\subseteq(g_*(I))^+$, and so $\{g^{-1}(Z_\alpha):\ \alpha\in Y\}\subseteq I^+$. Moreover, since $|Z_\alpha\cap Z_\beta|<\varkappa$ for $\alpha\neq\beta$ and since g is $<\varkappa$ to 1, it follows that

$$|g^{-1}(Z_\alpha)\cap g^{-1}(Z_\beta)|<\varkappa$$

for $\alpha\neq\beta$. This shows that $\mathrm{SpH}_\varkappa$ holds.

To complete the proof of Theorem 3.1, we note that the next to last implication is trivial while the last implication is a theorem of Baumgartner–Hajnal–Máté [1].

An unpublished result of Baumgartner shows that the first implication is not reversible. All of the others are open, but we are most concerned with the following ones.

PROBLEM E. Does $\mathrm{SpH}_{\omega_1}\to\mathrm{TH}_{\omega_1}$?

PROBLEM F. Does $\mathrm{SatH}_{\omega_1}\to\mathrm{SpH}_{\omega_1}$?

PROBLEM G. Does $\mathrm{FH}_\varkappa\to\mathrm{SatH}_\varkappa$?

A more subtle comparison of the relative strengths of these hypotheses is given by their various large cardinal consequences. For example Silver has shown [1] that $\neg\mathrm{KH}_{\omega_1}$ implies there is a strongly inaccessible cardinal in an inner model (i.e. ω_2 in L) while Kunen has shown [10] that $\neg\mathrm{SatH}_{\omega_1}$ gives 2 measurable cardinals in an inner model, and hence $0^\#$ exists. This suggests the following.

PROBLEM H. Does $\neg\mathrm{SpH}_{\omega_1}$ imply $0^\#$ exists?

Of course an affirmative answer to Problem F would give an affirmative answer to Problem H. The following proposition, however, shows that the naive approach to Problem F fails.

PROPOSITION 3.2. *Suppose I is an ω_2-saturated ω_1-complete ideal on ω_1. Then there exists an ω_1-complete ideal J on ω_1 having the following properties*

(i) $NS_{\omega_1} \subseteq J$.

(ii) *J is not ω_2-saturated.*

(iii) *If $\{X_\alpha: \alpha<\omega_2\} \subseteq J^+$ then $|X_\alpha \cap X_\beta| = \omega_1$ for some $\{\alpha, \beta\} \in [\omega_2]^2$.*

Proof. By Solovay's theorem [17] we can assume that I is normal. Let $\{A_\alpha: \alpha<\omega_1\}$ be a pairwise disjoint partition of ω_1 into sets in I^+ such that $A_\alpha \cap (\alpha+1) = 0$. Let J be the ideal generated by $I \cup \{A_\alpha: \alpha<\omega_1\}$. Then J clearly satisfies (i) and (iii). But J is clearly not a P-point so by a result in [18], J is not ω_2-saturated.

Kunen has shown [11] that $\neg SatH_{\omega_1}$ is consistent relative to the existence of a huge cardinal. Nevertheless the following is open.

PROBLEM I. Is $\neg FH_{\omega_1}$ consistent?

§ 4. Sets of $\varkappa$-complete ideals on $\varkappa$. In this section we shall study the effect of the various axioms introduced in section 3 on the saturation of sets of $\varkappa$-complete ideals on $\varkappa$. In [13], Prikry showed (via a forcing argument) that $\langle\omega_1:\omega_1, \omega_1\rangle \to 2$ is consistent with ZFC+GCH, and Jensen later showed (see [5]) that $\langle\omega_1: \omega_1, \omega_1\rangle \to 2$ holds if $V = L$. The following strengthening of these results is important because it shows the large cardinal nature of Ulam's problem.

THEOREM 4.1 (K. Prikry [15]). *Assume TH_{ω_1}. Let $\mathcal{M}$ be a family of σ-additive real valued (proper non-trivial) measures on ω_1 such that $|\mathcal{M}| = \omega_1$. Then there exists ω_1 pairwise disjoint subsets of ω_1 each of which is non-measurable with respect to every measure in $\mathcal{M}$.*

COROLLARY 4.2 (Prikry). $TH_{\omega_1} \vdash \langle\omega_1: \omega_1, \omega_1\rangle \to \omega_1$.

If $\mathcal{A}$ is a σ-algebra of subsets of ω_1 then we let $I_{\mathcal{A}} = \{X \subseteq \omega_1: \mathcal{P}(X) \subseteq \mathcal{A}\}$ and we say that $\mathcal{A}$ is *proper* if $\mathcal{A} \neq \mathcal{P}(\omega_1)$. If $\mathcal{A}$ contains all singletons then $\mathcal{A}$ is called *uniform*. Hence, if $\mathcal{A}$ is proper and uniform then $I_{\mathcal{A}}$ is an ω_1-complete ideal on ω_1. To say that $\mathcal{A}$ has the $\varkappa$-chain condition ($\varkappa$.c.c.) means that if $\mathcal{F} \subseteq \mathcal{A} - I_{\mathcal{A}}$ and $F \cap G \in I_{\mathcal{A}}$ for $\{F, G\} \in [\mathcal{F}]^2$ then $|\mathcal{F}|<\varkappa$. Notice that if μ is a σ-additive real valued (proper nontrivial) measure on ω_1 and $\mathcal{A} = \{X \subseteq \omega_1: X \text{ is } \mu\text{-measurable}\}$ then $\mathcal{A}$ is a proper uniform σ-algebra on ω_1 satisfying ω_1.c.c. The relevance of these considerations lies in Grzegorek's observation [8] that Prikry's arguments can be generalized to yield the following.

THEOREM 4.3 (Grzegorek [8]). *Assume TH_{ω_1}. Let $\mathcal{M}$ be a family of proper uniform σ-algebras on ω_1 such that each $\mathcal{A} \in \mathcal{M}$ satisfies ω_1.c.c. Then if $|\mathcal{M}| = \omega_1$ there exists ω_1 pairwise disjoint sets in $\mathcal{P}(\omega_1) - \bigcup \mathcal{M}$.*

The following is the main result of this paper. It answers a question of Prikry [15] and allows an immediate strengthening of the results of Prikry and Grzegorek.

THEOREM 4.4. $SpH_{\omega_1} \vdash \langle\omega_1:\omega_1, \omega_1\rangle \to \omega_2$.

The proof of Theorem 4.4 requires the following sequence of lemmas and definitions.

LEMMA 4.5 (Ulam [19]). *If $\varkappa = \mu^+$ then there exists a "μ by $\varkappa$ matrix" $\mathcal{U} = \{\mathcal{U}(\alpha, \xi): \alpha<\mu, \xi<\varkappa\} \subseteq \mathcal{P}(\varkappa)$, called an Ulam matrix for $\varkappa$, and having the property that each row and each column is a pairwise disjoint partition of an end segment of $\varkappa$.*

LEMMA 4.6. *Suppose $\varkappa = \mu^+$ and $\mathcal{I}$ is a set of $\varkappa$-complete ideals on $\varkappa$ such that $|\mathcal{I}| = \varkappa$. Then $\mathcal{I}$ can be expressed as the union $\mathcal{I} = \bigcup\{\mathcal{I}_\alpha: \alpha<\mu\}$ of μ sets of ideals on $\varkappa$ such that each $\mathcal{I}_\alpha$ fails to be $\varkappa$-saturated.*

Proof. Suppose $\mathcal{I}$ is given and let $\mathcal{U} = \{\mathcal{U}(\alpha, \xi): \alpha<\mu, \xi<\varkappa\}$ be an Ulam matrix for $\varkappa = \mu^+$. For each $\alpha<\mu$ define $\mathcal{I}_\alpha \subseteq \mathcal{I}$ by

$$I \in \mathcal{I}_\alpha \text{ iff } I \in \mathcal{I} \text{ and } \alpha = \inf\{\beta<\mu: |\{\xi<\varkappa: \mathcal{U}(\beta, \xi) \in I^+\}| = \varkappa\}.$$

If I is a $\varkappa$-complete ideal on $\varkappa$ then every column of $\mathcal{U}$ contains a set of positive I-measure, and so some row of $\mathcal{U}$ must contain $\varkappa$ sets of positive I-measure. This shows that $\mathcal{I} = \bigcup\{\mathcal{I}_\alpha: \alpha<\mu\}$.

Now fix $\alpha<\mu$ and enumerate $\mathcal{I}_\alpha$ as $\{I_\beta: \beta<\varkappa\}$. It is easy to see that we can inductively construct an increasing sequence $\langle\xi_\beta: \beta<\varkappa\rangle$ of ordinals such that $\mathcal{U}(\alpha, \xi_\beta) \in I_\beta^+$. For each $\beta<\varkappa$ let $A_\beta = \mathcal{U}(\alpha, \xi_\beta)$ and let $\{A_\beta^\xi: \xi<\varkappa\}$ be a pairwise disjoint partition of A_β into sets of positive I_β-measure. Finally, for each $\xi<\varkappa$ let $B_\xi = \bigcup\{A_\beta^\xi: \beta<\varkappa\}$. Then $\{B_\xi: \xi<\varkappa\}$ shows that $\mathcal{I}_\alpha$ fails to be $\varkappa$-saturated.

DEFINITION 4.7. A $\varkappa$-complete ideal I on $\varkappa$ will be called *splitable* iff for every $X \in I^+$ there exists $\{X_\alpha: \alpha<\varkappa^+\} \subseteq \mathcal{P}(X) \cap I^+$ such that $|X_\alpha \cap X_\beta|<\varkappa$ whenever $\alpha<\beta<\varkappa^+$.

DEFINITION 4.8. Let $\mathcal{I}$ be a set of $\varkappa$-complete ideals on $\varkappa$ such that $\mathcal{I}$ is not $\varkappa$-saturated.

(i) If $X \in \mathcal{I}^+$ then X will be called *$\mathcal{I}$-large* iff $\mathcal{I} \restriction X$ is not $\varkappa$-saturated.

(ii) $\mathcal{I}$ will be called *splitable* iff for every $\mathcal{I}$-large set X there exists $\{Z_\alpha: \alpha<\varkappa^+\} \subseteq \mathcal{P}(X)$ such that Z_α is $\mathcal{I}$-large for all $\alpha<\varkappa^+$ and such that $|Z_\alpha \cap Z_\beta|<\varkappa$ for $\alpha<\beta<\varkappa^+$.

In terms of Definition 4.7, $SpH_\varkappa$ asserts that every $\varkappa$-complete ideal on $\varkappa$ is splitable. In Definition 4.8 notice that $\varkappa$ is $\mathcal{I}$-large and that every $\mathcal{I}$-large set X is in $\mathcal{I}^+$. In particular, if $\mathcal{I}$ is splitable then $\mathcal{I}$ fails (badly) to be $\varkappa^+$-saturated.

LEMMA 4.9. *Suppose $\mathcal{I}$ is a set of $\varkappa$-complete ideals on $\varkappa$ such that every $I \in \mathcal{I}$ is a splitable ideal extending $NS_\varkappa$. Suppose also that $|\mathcal{I}| \leqslant \varkappa$ and $\mathcal{I}$ fails to be $\varkappa$-saturated. Then $\mathcal{I}$ is splitable.*

Proof. Let $\mathcal{I} = \{I_\alpha: \alpha<\varkappa\}$ and suppose X is $\mathcal{I}$-large. Then $\mathcal{I} \restriction X$ is not $\varkappa$-saturated so it easily follows that there exists a pairwise disjoint partition $\{X_\alpha: \alpha<\varkappa\}$ of X such that $X_\alpha \in \mathcal{I}^+$ and $X_\alpha \cap (\alpha+1) = 0$ for every $\alpha<\varkappa$. Since each I_α is a splitable ideal, there exists a collection $\{X_\alpha^\beta: \beta<\varkappa^+\} \subseteq \mathcal{P}(X_\alpha) \cap I_\alpha^+$ such that $|X_\alpha^{\beta_1} \cap X_\alpha^{\beta_2}|<\varkappa$ for $\beta_1 \neq \beta_2$. For each $\beta<\varkappa^+$ let $Y_\beta = \bigcup\{X_\alpha^\beta: \alpha<\varkappa\}$. Then

$Y_\beta \subseteq X$ and $Y_\beta \in \mathscr{I}^+$. Notice that if $\alpha_1 \neq \alpha_2$ then $X_{\alpha_1}^{\beta_1} \cap X_{\alpha_2}^{\beta_2} = 0$ since $X_{\alpha_1}^{\beta_1} \subseteq X_{\alpha_1}$ and $X_{\alpha_2}^{\beta_2} \subseteq X_{\alpha_2}$ and $X_{\alpha_1} \cap X_{\alpha_2} = 0$. Thus, if $\beta_1 \neq \beta_2$ then

$$Y_{\beta_1} \cap Y_{\beta_2} \subseteq \bigcup \{X_{\alpha_1}^{\beta_1} \cap X_{\alpha_2}^{\beta_2}\colon \alpha_1, \alpha_2 < \varkappa\} = \bigcup \{X_\alpha^{\beta_1} \cap X_\alpha^{\beta_2}\colon \alpha < \varkappa\}.$$

But $|X_\alpha^{\beta_1} \cap X_\alpha^{\beta_2}| < \varkappa$ and $X_\alpha^{\beta_1} \cap X_\alpha^{\beta_2} \cap (\alpha+1) = 0$ (since $X_\alpha \cap (\alpha+1) = 0$). Thus $Y_{\beta_1} \cap Y_{\beta_2} \in \mathrm{NS}_\varkappa$. Now, for each $\alpha < \varkappa^+$ let $Z_\alpha = Y_\alpha - \nabla\{Y_\beta \cap Y_\alpha\colon \beta < \alpha\}$. Since $Y_\alpha \in \mathscr{I}^+$ and $\mathrm{NS}_\varkappa \subseteq \bigcap \mathscr{I}$ we have that $Z_\alpha \in \mathscr{I}^+$ and $|Z_\alpha \cap Z_\beta| < \varkappa$ for $\alpha \neq \beta$.

It remains only to show that Z_β is $\mathscr{I}$-large. But $Y_\beta = \bigcup \{X_\alpha^\beta\colon \alpha < \varkappa\}$ and $X_\alpha^\beta \in I_\alpha^+$. For each $\alpha < \varkappa$ let $A_\alpha = X_\alpha^\beta \cap Z_\beta$. Then $\{A_\alpha\colon \alpha < \varkappa\}$ is a disjoint partition of Z_β such that $A_\alpha \in I_\alpha^+$ for each $\alpha < \varkappa$. Let $\{A_\alpha^\xi\colon \xi < \varkappa\}$ be a disjoint partition of A_α into sets in I_α^+ and for each $\xi < \varkappa$ let $B_\xi = \bigcup \{A_\alpha^\xi\colon \alpha < \varkappa\}$. Then $\{B_\xi\colon \xi < \varkappa\}$ is a disjoint partition of Z_β such that $B_\xi \in \mathscr{I}^+$ for every $\xi < \varkappa$. Hence $\mathscr{I} \restriction Z_\beta$ is not $\varkappa$-saturated so Z_β is $\mathscr{I}$-large as desired.

LEMMA 4.10. *Suppose that for each $n \in \omega$ $\mathscr{I}_n$ is a set of $\varkappa$-complete ideals on $\varkappa$ such that $|\mathscr{I}_n| \leqslant \varkappa$. Suppose $\mathscr{I}_n$ is splitable and X is $\mathscr{I}_n$-large for every $n \in \omega$. Then there exists a disjoint partition $X = X_0 \cup X_1$ such that X_0 is $\mathscr{I}_0$-large and X_1 is $\mathscr{I}_n$-large for every $n \in \omega$.*

Proof. Since $\mathscr{I}_0$ is splitable and X is $\mathscr{I}_0$-large there exists (by Definition 4.8) $\{Z_\alpha\colon \alpha < \varkappa^+\} \subseteq \mathscr{P}(X)$ such that Z_α is $\mathscr{I}_0$-large for every $\alpha < \varkappa^+$ and such that $|Z_\alpha \cap Z_\beta| < \varkappa$ for $\alpha < \beta < \varkappa^+$. Since X is $\mathscr{I}_n$-large for each $n > 0$ we can choose a pairwise disjoint partition $\{X_\alpha^n\colon \alpha < \varkappa\}$ of X into sets in $\mathscr{I}_n^+$. Let $\mathscr{J}$ be the following set of ideals.

$$\mathscr{J} = \{I \restriction X_\alpha^n\colon I \in \bigcup_{j \in \omega} \mathscr{I}_j;\ n \in \omega;\ \alpha \in \varkappa \text{ and } X_\alpha^n \in I^+\}.$$

Notice that $|\mathscr{J}| \leqslant \varkappa$. For each $\alpha < \varkappa^+$ at most one of the sets Z_β can be of J-measure one for any single $J \in \mathscr{J}$. Hence, we can choose β such that Z_β is not of J-measure one for any $J \in \mathscr{J}$. Let $X_0 = Z_\beta$ and $X_1 = X - X_0$. Then X_0 is clearly $\mathscr{I}_0$-large. Now fix $n \in \omega$. We claim that $\{X_\alpha^n \cap X_1\colon \alpha < \varkappa\}$ shows that X_1 is $\mathscr{I}_n$-large. If not, then for some $\alpha < \varkappa$ we have $X_\alpha^n \cap X_1 \notin \mathscr{I}_n^+$. Hence $X_\alpha^n \cap X_1 \in I$ for some $I \in \mathscr{I}_n$ so $X_0 \in (I \restriction X_\alpha^n)^*$. But $X_0 = Z_\beta$ and this contradicts our choice of β.

LEMMA 4.11. *Assume that every ω_1-complete ideal I on ω_1 such that $\mathrm{NS}_{\omega_1} \subseteq I$ is splitable. Then every set $\mathscr{I}$ of ω_1-complete ideals on ω_1 such that $|\mathscr{I}| \leqslant \omega_1$ fails to be ω_1-saturated.*

Proof. Suppose the conclusion fails. Then $\langle \omega_1 : \omega_1, \omega_1 \rangle \not\to \omega_1$, so by Corollary 1.2 $\langle \omega_1 : \omega_1, \omega_1 \rangle \overset{\varepsilon}{\not\to} \omega_1$. Hence there is an ω_1-saturated set $\mathscr{I}$ of ω_1-complete ideals on ω_1 extending NS_{ω_1} such that $|\mathscr{I}| = \omega_1$. By Lemma 4.6 $\mathscr{I}$ can be expressed as a union $\mathscr{I} = \bigcup \{\mathscr{I}_n\colon n \in \omega\}$ such that for each $n \in \omega$ $\mathscr{I}_n$ is not ω_1-saturated.

We now inductively construct a decreasing sequence $\langle Z_j\colon j \in \omega\rangle$ of subsets of ω_1 so that for each $j \in \omega$ Z_j is $\mathscr{I}_n$-large for all n and $Z_j - Z_{j+1}$ is $\mathscr{I}_j$-large. Set $Z_0 = \omega_1$. Then Z_0 is $\mathscr{I}_n$-large for all n since $\mathscr{I}_n$ is not ω_1-saturated. Suppose Z_j has been constructed and Z_j is $\mathscr{I}_n$-large for all n. Consider $\mathscr{I}_j$. Since $|\mathscr{I}_j| \leqslant \omega_1$ and every $I \in \mathscr{I}_j$ is a splitable ideal extending NS_{ω_1}, and $\mathscr{I}_j$ is not ω_1-saturated we can appeal to Lemma 4.9 to conclude that $\mathscr{I}_j$ is splitable. Since Z_j is $\mathscr{I}_n$-large for all n, Lemma 4.10 yields a disjoint partition $Z_j = X_0 \cup X_1$ such that X_0 is $\mathscr{I}_j$ large and X_1 is $\mathscr{I}_n$-large for all n. Set $Z_{j+1} = X_1$. Then $Z_j - Z_{j+1} = X_0$ so $Z_j - Z_{j+1}$ is $\mathscr{I}_j$-large as desired. This completes the construction of $\langle Z_j\colon j \in \omega\rangle$.

For each $n \in \omega$ let $X_n = Z_n - Z_{n+1}$. Then $\{X_n\colon n \in \omega\}$ is a pairwise disjoint collection such that X_n is $\mathscr{I}_n$-large. Hence $\mathscr{I}_n \restriction X_n$ is not ω_1-saturated. For each $n \in \omega$ let $\{X_n^\alpha\colon \alpha < \omega_1\}$ be a pairwise disjoint partition of X_n into sets in $\mathscr{I}_n^+$. For $\alpha < \omega_1$ let $Y_\alpha = \bigcup \{X_n^\alpha\colon n \in \omega\}$. Then $\{Y_\alpha\colon \alpha < \omega_1\}$ shows that $\mathscr{I}$ is not ω_1-saturated and this contradiction completes the proof.

Theorem 4.4 is now an immediate consequence of the following.

THEOREM 4.12. *Assume that every ω_1-complete ideal I on ω_1 such that $\mathrm{NS}_{\omega_1} \subseteq I$ is splitable. Then every set $\mathscr{I}$ of ω_1 complete ideals on ω_1 such that $|\mathscr{I}| = \omega_1$ is splitable.*

Proof. It follows from Lemma 4.11 that under the assumptions of the theorem no set $\mathscr{I}$ such that $|\mathscr{I}| = \omega_1$ can be ω_1-saturated. Hence, X is $\mathscr{I}$-large iff $X \in \mathscr{I}^+$ so to prove 4.11 it suffices to show that if $\mathscr{I}$ is a set of ω_1 complete ideals on ω_1 and $|\mathscr{I}| = \omega_1$ then there exists $\{X_\alpha\colon \alpha < \omega_2\} \subseteq \mathscr{I}^+$ such that $|X_\alpha \cap X_\beta| < \omega_1$ for $\alpha < \beta < \omega_2$. (Then for a given set $X \in \mathscr{I}^+$ we can apply this to $\mathscr{I} \restriction X$).

Suppose then that $|\mathscr{I}| = \omega_1$. By Theorem 1.1 there is a $<\omega_1$ to 1 function $g\colon \omega_1 \to \omega_1$ such that for every $I \in \mathscr{I}$ the ideal J_I generated by $g_*(I) \cup \mathrm{NS}_{\omega_1}$ is proper. Let $\mathscr{J} = \{J_I\colon I \in \mathscr{I}\}$. By Lemma 4.11, $\mathscr{J}$ is not ω_1-saturated. Moreover, each $J_I \in \mathscr{J}$ is a splitable ideal extending NS_{ω_1} so Lemma 4.9 guarantees that $\mathscr{J}$ is splitable. Let $\{Z_\alpha\colon \alpha < \omega_2\} \subseteq \mathscr{J}^+$ be such that $|Z_\alpha \cap Z_\beta| < \omega_1$ for $\alpha < \beta < \omega_2$ and for each $\alpha < \omega_2$ let $X_\alpha = g^{-1}(Z_\alpha)$. If $I \in \mathscr{I}$ then $Z_\alpha \in (J_I)^+ \subseteq g_*(I)^+$ so $X_\alpha \in I^+$. Moreover, if $\alpha \neq \beta$ then $X_\alpha \cap X_\beta = g^{-1}(Z_\alpha \cap Z_\beta)$ and this set is countable since g is $<\omega_1$ to 1. Hence $\{X_\alpha\colon \alpha < \varkappa^+\} \subseteq \mathscr{I}^+$ is such that $|X_\alpha \cap X_\beta| < \omega_1$ for $\alpha \neq \beta$ as desired.

A consequence of Theorem 2.4 is that $\langle \omega_1 : \omega_1, \omega_1 \rangle \overset{\mathscr{N}}{\to} \omega_2$ iff $\langle \omega_1 : 1, \omega_1 \rangle \overset{\mathscr{N}}{\to} \omega_2$. Notice that Theorem 4.12 yields a result of the same spirit. That is, ω_1 carries a non-splitable set $\mathscr{I}$ of size ω_1 (consisting of ω_1-complete ideals) iff it carries such a set of size one.

The following corollary of Theorem 4.12 strengthens the results of Prikry and Grzegorek (stated previously as Theorems 4.1 and 4.3).

COROLLARY 4.13. *Assume SpH_{ω_1}. Let $\mathscr{M}$ be a family of proper uniform σ-algebras on ω_1 such that each $\mathscr{A} \in \mathscr{M}$ satisfies ω_2-c.c. Then if $|\mathscr{M}| = \omega_1$ there exists ω_2 sets in $\mathscr{P}(\omega_1) - \bigcup \mathscr{M}$ such that pairwise intersections are countable.*

Proof. We apply Theorem 4.12 to the collection $\{I_{\mathscr{A}}\colon \mathscr{A} \in \mathscr{M}\}$ to obtain $\{X_\alpha\colon \alpha < \omega_2\}$ such that $|X_\alpha \cap X_\beta| < \omega_1$ for $\alpha < \beta < \omega_2$ and such that for every $\alpha < \omega_2$ and every $\mathscr{A} \in \mathscr{M}$, $X_\alpha \notin I_{\mathscr{A}}$. Since each $\mathscr{A} \in \mathscr{M}$ satisfies ω_2-c.c., at most ω_1 of the sets $\{X_\alpha\colon \alpha<\omega_2\}$ can be in $\mathscr{A}$ for any single $\mathscr{A}\in\mathscr{M}$. Hence, if

$$Y = \{\alpha<\omega_2\colon\ \exists\mathscr{A}\in\mathscr{M}\ (X_\alpha\in\mathscr{A})\}$$

then $|Y|\leqslant\omega_1$, so $\{X_\alpha\colon \alpha\in\omega_2-Y\}$ is the desired collection.

Notice that our proof of Theorem 4.12 does not seem to generalize from ω_1 to $\varkappa^+$. Hence, the following remains open.

Problem J. Does $\mathrm{SpH}_{\varkappa^+}\vdash\langle\varkappa^+:\varkappa^+,\varkappa^+\rangle\to\varkappa^+$?

An affirmative answer to Problem F would also yield an affirmative answer to the following.

Problem K. Does $\mathrm{SatH}_{\omega_1}\vdash\langle\omega_1:\omega_1,\omega_1\rangle\to\omega_2$?

Problem L. Does $\langle\omega_1:\omega_1,\omega_1\rangle\nrightarrow 2$ imply $0^\#$ exists?

The weakest axiom from our list in section 3 is $\mathrm{FH}_\varkappa$. This hypothesis was our primary concern in [18], and we refer the reader there for more information on (and motivation for) $\mathrm{FH}_\varkappa$. To conclude this section however, we will extract a few results from [18] in order to motivate some additional problems that we are interested in. The first one we state emphasizes the relevance of $\mathrm{SatH}_\varkappa$ and $\mathrm{FH}_\varkappa$ to our present considerations. For uniformity, we chose to state all the axioms in section 3 for successor cardinals, but it is easy to see that $\mathrm{SatH}_\varkappa$ and $\mathrm{FH}_\varkappa$ are meaningful for any regular cardinal $\varkappa$.

Theorem 4.14 [18]. *If $\varkappa$ is a regular cardinal then:*

(i) $\mathrm{SatH}_\varkappa$ *holds iff* $\langle\varkappa:\varkappa,\varkappa\rangle\overset{\mathcal{N}}{\to}\varkappa^+$.

(ii) $\mathrm{FH}_\varkappa$ *holds iff* $\langle\varkappa:\varkappa,\varkappa\rangle\overset{\mathcal{N}}{\to}\varkappa$.

We say that an ideal I on ω_1 has a *dense set of size* ω_1 iff $\mathscr{P}(\omega_1)/I$ has a dense set of size ω_1 in the "forcing theoretic" sense. Two other results from [18] are the following.

Theorem 4.15 [18]. $\langle\omega_1:\omega_1,\omega_1\rangle\overset{\mathcal{N}}{\nrightarrow}\omega_1$ *iff some ω_1-complete ideal on ω_1 has a dense set of size ω_1.*

Corollary 4.16. $\langle\omega_1:\omega_1,\omega_1\rangle\overset{\mathcal{N}}{\to}\omega_1$ *iff* $\langle\omega_1:\omega_1,\omega_1\rangle\overset{\mathcal{N}}{\to}2$.

It is easy to see that if $\langle\omega_1:\omega_1,\omega_1\rangle\to 2$ then $\langle\omega_1:\omega_1,\omega_1\rangle\to\omega_0$. Nevertheless, we have been unable to settle the following

Problem M. Does $\langle\omega_1:\omega_1,\omega_1\rangle\to 2$ imply $\langle\omega_1:\omega_1,\omega_1\rangle\to\omega_1$?

It is also shown in [18] that $\mathrm{MA}_{\aleph_1}\vdash\langle\omega_1:\omega_1,\omega_1\rangle\overset{\mathcal{N}}{\to}\omega_1$. This suggests the following two problems

Problem N. Does $\mathrm{MA}_{\aleph_1}\vdash\langle\omega_1:\omega_1,\omega_1\rangle\to 2$?

Problem O. Does $\mathrm{MA}_{\aleph_1}\vdash\mathrm{SatH}_{\omega_1}$?

§ 5. Sets of countably complete ideals. In this section we turn our attention to sets of ideals on $\varkappa$ that are only countably complete. Results of the form $\langle\varkappa:\varkappa,\lambda\rangle\to 2$ become more difficult to obtain when λ is less than $\varkappa$, and this is perhaps best emphasized by the following.

Theorem 5.1 (Magidor [12]). *Assuming the consistency of a huge cardinal one obtains the consistency of* $\langle\omega_3:\omega_3,\omega_1\rangle\nrightarrow 2$.

There is a rather large gap between Magidor's negative result above and the positive results we have been able to obtain. In fact, the following question of Prikry is still open.

Problem P (Prikry [15]). Does $\mathrm{TH}_{\omega_2}\vdash\langle\omega_2:\omega_1,\omega_1\rangle\to\omega_2$?

The partial solution to Problem P that we have been able to obtain involves consideration of the following class of ideals to play the role of "$\mathscr{R}$" in our notation $\langle\varkappa:\lambda,\mu\rangle\overset{\mathscr{R}}{\to}\nu$.

$\mathscr{W}$ denotes the set of all weakly normal ideals on $\varkappa$. (I is *weakly normal* iff every regressive function defined on a set of positive I-measure is bounded on a set of positive I-measure.)

Theorem 5.2. *Suppose that for every countably complete weakly normal ideal I on $\varkappa^+$ there are $\varkappa^+$ pairwise disjoint sets in I^+. Then for every set $\mathscr{I}$ consisting of at most $\varkappa$ countably complete weakly normal ideals on $\varkappa^+$ there are $\varkappa^+$ pairwise disjoint sets in $\mathscr{I}^+$.*

Proof. Let $\mathscr{I} = \{I_\alpha\colon \alpha<\varkappa\}$ be a set of countably complete weakly normal ideals on $\varkappa^+$. We will inductively construct $\{\mathscr{X}_\alpha\colon \alpha<\varkappa\}$ satisfying the following.

(a) $\mathscr{X}_\alpha$ is a collection of $\varkappa^+$ pairwise disjoint sets in I_α^+.

(b) If $\alpha<\beta$ then either $\mathscr{X}_\alpha\supseteq\mathscr{X}_\beta$ or

$$|\{X\in\mathscr{X}_\alpha\colon\ \exists Y\in\mathscr{X}_\beta(X\cap Y\neq 0)\}|\leqslant\varkappa.$$

(c) If $\forall\alpha<\beta\ \mathscr{X}_\alpha\not\supseteq\mathscr{X}_\beta$ then $\mathscr{X}_\beta = \{X_\xi^\beta\colon \xi<\varkappa^+\}$ and $X_\xi^\beta\cap(\xi+1) = 0$.

Suppose $\beta<\varkappa$ and $\mathscr{X}_\alpha$ has been constructed for all $\alpha<\beta$. Let A be defined by

$$A = \{\beta'<\beta\colon\ \forall\alpha'<\beta'\ \mathscr{X}_{\alpha'}\not\supseteq\mathscr{X}_{\beta'}\}.$$

Intuitively, $\{\mathscr{X}_\alpha\colon \alpha\in A\}$ is a collection of partitions of "essentially disjoint" sets and every other $\mathscr{X}_\alpha$ for $\alpha<\beta$ is a subcollection of one of these.

If $|\mathscr{X}_\alpha\cap I_\beta^+| = \varkappa^+$ for some $\alpha<\beta$ then we simply set $\mathscr{X}_\beta = \mathscr{X}_\alpha\cap I_\beta^+$ and we are done with the βth step of the construction.

Suppose then that $|\mathscr{X}_\alpha\cap I_\beta^+|\leqslant\varkappa$ for every $\alpha<\beta$. Choose $\gamma<\varkappa^+$ such that if $\alpha<\beta$ and $\mathscr{X}_\alpha = \{X_\xi^\alpha\colon \xi<\varkappa^+\}$ then $X_\xi^\alpha\in I_\beta$ for every $\xi>\gamma$. Moreover, because of condition (b) we can assume that γ is large enough so that if $\{\beta,\delta\}\in[A]^2$ and $\xi,\eta>\gamma$ then $X_\xi^\beta\cap X_\eta^\delta = 0$.

Define $f\colon \bigcup\{X_\xi^\alpha\colon \alpha\in A;\ \xi>\gamma\}\to\varkappa^+$ so that

$$f(\bigcup\{X_\xi^\alpha\colon \alpha\in A\}) = \{\xi\}.$$

Then f is well defined since $\{X^\alpha_\xi\colon \alpha\in A,\ \xi>\gamma\}$ is a collection of pairwise disjoint sets. Moreover, because of (c) we have that $f^{-1}(\xi)\cap(\xi+1)=0$, so f is regressive.

If domain $(f)\in I_\beta$ then we simply choose $\mathscr{X}_\beta$ to be a collection of subsets of $\varkappa^+$-domain (f) satisfying (a) and (c).

If domain $(f)\in I_\beta^+$ then we appeal to the weak normality of I_β to obtain a set $B\subseteq$ domain (f) and $\delta<\varkappa^+$ such that $B\in I_\beta^+$ and $f(B)\subseteq\delta$. Hence if $\xi>\delta$ and $\alpha\in A$ then $X^\alpha_\xi\cap B=0$. Now let $\mathscr{X}_\beta$ be a pairwise disjoint partition of B into sets in I_β^+ such that (c) is satisfied.

This yields $\{\mathscr{X}_\alpha\colon \alpha<\varkappa\}$ satisfying (a), (b) and (c). By deleting initial segments of each $\mathscr{X}_\alpha$ we can clearly assume (because of (b)) that if $\alpha<\beta$ then either $\mathscr{X}_\alpha\supseteq\mathscr{X}_\beta$ or every set in $\mathscr{X}_\alpha$ is disjoint from every set in $\mathscr{X}_\beta$. An easy inductive construction now yields a pairwise disjoint collection $\{Y_\alpha\colon \alpha<\varkappa\}$ such that $Y_\alpha\in I_\alpha^+$ for every $\alpha<\varkappa$. As usual, we now let $\{Y^\alpha_\xi\colon \xi<\varkappa^+\}$ be a pairwise disjoint partition of Y_α into sets in I_α^+ and for each $\xi<\varkappa^+$ we set $Z_\xi=\bigcup\{Y^\alpha_\xi\colon \alpha<\varkappa\}$. Then $\{Z_\xi\colon \xi<\varkappa^+\}\subseteq\mathscr{I}^+$ is a pairwise disjoint collection.

Theorem 5.2 is another result that essentially says that the only way a "small" set of ideals can have a certain saturation property is if a single ideal has this same property. An affirmative answer to the following question would let us conclude from Theorem 5.2 that $\langle\varkappa^+\colon\varkappa,\omega_1\rangle\xrightarrow{\mathscr{W}}\varkappa^+$ iff $\langle\varkappa^+\colon 1,\omega_1\rangle\xrightarrow{\mathscr{W}}\varkappa^+$.

PROBLEM Q. Suppose I is a weakly normal countably complete ideal on $\varkappa^+$ that is not $\varkappa^+$-saturated. Can one then find $\varkappa^+$ pairwise disjoint sets in I^+?

Nevertheless, if we return to our list of axioms in section 3, then Theorem 5.2 will allow us to conclude the following.

THEOREM 5.3. $\mathrm{TH}_{\omega_2}\vdash\langle\omega_2\colon\omega_1,\omega_1\rangle\xrightarrow{\mathscr{W}}\omega_2$.

Proof. Let $\{f_\alpha\colon \alpha<\omega_3\}$ show that TH_{ω_2} holds and let I be a countably complete ideal on ω_2. By Theorem 5.2 it will suffice to show that there are ω_2 pairwise disjoint sets in I^+.

Define a relation R on $\{f_\alpha\colon \alpha<\omega_3\}$ by

$$f_\alpha R f_\beta \quad\text{iff}\quad \{\xi<\omega_2\colon f_\alpha(\xi)<f_\beta(\xi)\}\in I^+\,.$$

Then for $\alpha\neq\beta$ we have $f_\alpha Rf_\beta$ or $f_\beta Rf_\alpha$ so standard arguments produce a $\beta<\omega_2$ such that $|\{\alpha<\omega_3\colon f_\alpha Rf_\beta\}|=\omega_2$. For each $\delta<\omega_1$ let $h_\delta\colon\delta\to|\delta|$ be one to one. Let $A=\{\alpha\colon f_\alpha Rf_\beta\}$ and for each $\alpha\in A$ let $A_\alpha=\{\xi<\omega_2\colon f_\alpha(\xi)<f_\beta(\xi)\}$. Now since $f_\beta\colon\omega_2\to\omega_1$ we can define a function $g_\alpha\colon A_\alpha\to\omega$ for each $\alpha\in A$ as follows.

If $\xi\in A_\alpha$ and $f_\beta(\xi)=\delta$ then set $g_\alpha(\xi)=h_\delta(f_\alpha(\xi))$. For each $\alpha\in A$ choose $B_\alpha\in\mathscr{P}(A_\alpha)\cap I^+$ and $n_\alpha\in\omega$ such that $g_\alpha(B_\alpha)\equiv n_\alpha$. Choose $A'\subseteq A$ and $n\in\omega$ such that $|A'|=\omega_2$ and $n_\alpha=n$ for all $\alpha\in A'$. Now it is easy to see that $\{B_\alpha\colon \alpha\in A'\}$ is a set of ω_2 sets of positive I-measure such that $|B_\alpha\cap B_\beta|\leqslant\omega_1$ for $\alpha\neq\beta$. For each $\alpha\in A'$ let $C_\alpha=B_\alpha-\bigcup\{B_\beta\colon \beta\in A'\text{ and }\beta<\alpha\}$. Then $\{C_\alpha\colon \alpha\in A'\}$ is the desired set of ω_2 pairwise disjoint sets in I^+.

Theorem 5.3 is the partial solution to Prikry's Problem P that we referred to earlier.

Our final consideration of this section is motivated by Solovay's theorem [17] that it is consistent (relative to the consistency of a measurable cardinal) that $2^{\aleph_0}$ be real valued measurable. Of course $\langle 2^{\aleph_0},1,\omega_1\rangle\to 2$. Nevertheless, the following is open.

PROBLEM R. Does $\langle 2^{\aleph_0},\omega,\omega_1\rangle\to 2$ hold?

An ideal I on $\varkappa$ is said to have a *dense set of size* λ if there exists $\{X_\alpha\colon \alpha<\lambda\}\subseteq I^+$ such that for every $X\in I^+$ there exists $\alpha<\lambda$ such that $X_\alpha-X\in I$. It is easy to see that if I has a dense set of size λ then I is λ^+-saturated. A consequence of the results in section 4 is the following.

1. $\langle\omega_1\colon\omega_1,\omega_1\rangle\xrightarrow{\mathscr{N}}\omega_2$ iff there is no ω_1-complete ω_2-saturated ideal on ω_1.
2. $\langle\omega_1\colon\omega_1,\omega_1\rangle\xrightarrow{\mathscr{N}}\omega_1$ iff there is no ω_1-complete ideal on ω_1 having a dense set of size ω_1.

These should be compared with the following.

THEOREM 5.4. 1. $\langle\varkappa\colon\omega,\omega_1\rangle\to\omega_1$ *iff there is no ω_1-complete ω_1-saturated ideal on $\varkappa$.*

2. $\langle\varkappa\colon\omega,\omega_1\rangle\to\omega$ *iff there is no ω_1-complete ideal on $\varkappa$ having a dense set of size ω.*

Proof. Part 1 follows immediately from Theorem 2.2. For part 2, notice first that if $\{X_n\colon n\in\omega\}$ is a dense set for the ω_1-complete ideal I on $\varkappa$ then $\mathscr{I}=\{I\restriction X_n\colon n\in\omega\}$ easily shows that $\langle\varkappa\colon\omega,\omega_1\rangle\nrightarrow 2$.

For the converse of part 2, suppose that no ω_1-complete ideal on $\varkappa$ has a countable dense set. Let $\mathscr{I}=\{I_n\colon n\in\omega\}$ be the given set of ω_1-complete ideals on $\varkappa$. To show that $\mathscr{I}$ is not $\aleph_0$-saturated we need the following two lemmas.

LEMMA 5.5. *Suppose that $\mathscr{I}$ is a countable set of nowhere ω_1-saturated countably complete ideals on $\varkappa$. Then $\mathscr{I}$ is not ω_1-saturated.*

Proof. This is an immediate consequence of (the proof of) Theorem 2.2.

LEMMA 5.6. *Suppose that no ω_1-complete ideal on $\varkappa$ has a countable dense set. Then if $\mathscr{I}$ is a countable set of ω_1-saturated countably complete ideals on $\varkappa$, then $\mathscr{I}$ is not ω_0-saturated.*

Proof. Let $\mathscr{I}=\{I_n\colon n\in\omega\}$ and let $I=\bigcap\{I_n\colon n\in\omega\}$. Then it is easy to see that I must be ω_1-saturated since I is countably complete and each I_n is ω_1-saturated. Now, for each n let $\mathscr{A}_n$ be a maximal collection of sets in I_n-I that are almost disjoint (mod I). Then $|\mathscr{A}_n|<\omega_1$ so $A_n=\bigcup\mathscr{A}_n$ is in I_n. Hence $B_n=\varkappa-A_n\in I^+$ and it is easy to see that $I_n=I\restriction B_n$ (cf. Theorem 3.1 of [2]). We now construct a pairwise disjoint refinement $\{C_n\colon n\in\omega\}\subseteq I^+$ of $\{B_n\colon n\in\omega\}$ as follows. Since $\{B_n\colon n>0\}$ is not a dense set for $I\restriction B_n$ we can choose $C_0\subseteq B_0$ such that $C_0\in I^+$ and such that for each $n>0$ $B_n^0=B_n-C_0\in I^+$. If C_j has been defined and for each $n>j$ we have $B_n^j\in I^+$ then we can similarly choose $C_{j+1}\subseteq B_{j+1}^j$ such that $C_{j+1}\in I^+$ and such that for each $n>j+1$ $B_n^{j+1}=B_n^j-C_{j+1}\in I^+$. This yields $\{C_n\colon n\in\omega\}\subseteq I^+$ such that $C_n\in\mathscr{P}(B_n)\cap I^+$ for each n and $\{C_n\colon n\in\omega\}$ is pairwise disjoint. Since

$C_n \in I_n^+$ for each n, the desired collection of $\aleph_0$ pairwise disjoint sets in $\mathscr{I}^+$ follows easily since $I \restriction C_n$ is definitely not $\aleph_0$-saturated. (I.e. if $I \restriction C_n$ is $\aleph_0$-saturated then $\varkappa$ must carry a countably complete ultrafilter $\mathscr{U}$ in which case $\mathscr{U}^*$ has a dense set of size one.)

To return to the proof of Theorem 5.4, we have our given set $\mathscr{I} = \{I_n: n \in \omega\}$ of ω_1-complete ideals on $\varkappa$. We define sets $\mathscr{I}_0$, $\mathscr{I}_1$ and $\mathscr{I}_2$ as follows.

$$\mathscr{I}_0 = \{I \in \mathscr{I}: I \text{ is nowhere } \omega_1\text{-saturated}\}.$$

$$\mathscr{I}_1 = \{I \in \mathscr{I}: \exists A_I \in I^+ \text{ s.t. } I \restriction A_I \text{ is } \omega_1\text{-saturated}\}.$$

$$\mathscr{I}_2 = \{I \restriction A_I: I \in \mathscr{I}_1\}.$$

By Lemma 5.5 there exists a set $\{Y_\alpha: \alpha < \omega_1\}$ of pairwise disjoint sets in $\mathscr{I}_0^+$. (This uses the fact that each $I \in \mathscr{I}$ is countably complete in order to get the sets disjoint). At most one Y_α can be of J-measure one for any single $J \in \mathscr{I}_2$ so we can choose $\gamma < \omega_1$ such that $Y_\gamma \notin J^*$ for any $J \in \mathscr{I}_2$. Let $A = Y_\gamma$ and let $B = \varkappa - Y_\gamma$. Then $A \in \mathscr{I}_0^+$ and $B \in \mathscr{I}_2^+$ and $A \cap B = 0$.

Applying Lemma 5.6 to $\{J \restriction B: J \in \mathscr{I}_2\}$ yields a pairwise disjoint partition $\{B_n: n \in \omega\}$ of B such that for each $n \in \omega$ $B_n \in \mathscr{I}_2^+ \subseteq \mathscr{I}_1^+$. Similarly, if we apply Lemma 5.5 to $\{I \restriction A: I \in \mathscr{I}_0\}$ we obtain a pairwise disjoint partition $\{A_n: n \in \omega\}$ of A such that for each $n \in \omega$ $A_n \in \mathscr{I}_0^+$. But now $\{B_n \cup A_n: n \in \omega\}$ shows that $\mathscr{I} = \mathscr{I}_0 \cup \mathscr{I}_1$ is not $\aleph_0$-saturated.

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UNION COLLEGE
Schenectady, New York

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