Commutative quasi-trivial superassociative systems

by

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Abstract. In this note we give a classification of a certain class of algebras (A, f) with one commutative $(n+1)$-ary operation $(n \geqslant 1)$ such that any subset of A is a subalgebra of (A, f) and such that f satisfies the "superassociative law"

$$f(f(x_0, \ldots, x_n), x_{n+1}, \ldots, x_{2n}) = f(x_0, f(x_1, x_{n+1}, \ldots, x_{2n}), \ldots, f(x_n, \ldots, x_{2n}))$$

for any $x_0, \ldots, x_{2n} \in A$.

1. Introduction. Quite a natural generalization of the concept of a semigroup is that of a superassociative system, the latter being an algebra with one $(n+1)$-ary operation $(n \geqslant 1)$ satisfying some law which in case $n = 1$ reduces to the well-known associative law.

Superassociativity turns out to be the essential property of composition of functions since for superassociative systems there holds some sort of Cayley-representation theorem generalizing that one valid for semigroups. Superassociative systems have already been considered e.g. by R. M. Dicker ([1]) and K. Menger ([3], [4]). K. Menger was the first to fully realize the significance of the concept of superassociativity. Some material concerning superassociative systems can also be found in a book by H. Lausch and W. Nöbauer ([2], chapter 3). In [5] H. Skala investigated quasi-trivial superassociative systems, i.e. superassociative systems, any subset of which being a subalgebra. The present paper is devoted to the study of such algebras, too. Our motivation is the following: In lattice theory, operations $m_{i,\leqslant}$, $1 \leqslant i \leqslant n+1$, n some fixed positive integer, of the following kind are considered:

$$m_{i,\leqslant}(x_0, \ldots, x_n) := \bigwedge \{\bigvee \{x_j \mid j \in I\} \mid I \subseteq \{0, \ldots, n\},\ |I| = i\}$$

$(x_0, \ldots, x_n \in L$, $(L, \leqslant) = (L, \vee, \wedge)$ being some distributive lattice). The operations $m_{i,\leqslant}$, $1 \leqslant i \leqslant n+1$, on L turn out to be commutative and superassociative and in case $(L, \leqslant)$ is a chain they are quasi-trivial, too (the latter means $m_{i,\leqslant}(x_0, \ldots, x_n) \in \{x_0, \ldots, x_n\}$ for any $x_0, \ldots, x_n \in L$). Hence, the problem of classifying all $(n+1)$-ary commutative quasi-trivial superassociative operations arises. In our paper we give a complete solution of this problem in case n is odd and a partial solution in case n is even. Moreover, we give a characterization of the operations $m_{i,\leqslant}$, $1 \leqslant i \leqslant n+1$, $i \neq \frac{1}{2}n+1$, on chains $(L, \leqslant)$.

2. Definitions and basic results. In the following let n be some fixed positive integer.

DEFINITION 1. Let (A,f) be some algebra with one $(n+1)$-ary operation. f is called *commutative* if

$$f(x_{\pi 0}, \ldots, x_{\pi n}) = f(x_0, \ldots, x_n)$$
$$\text{for any } x_0, \ldots, x_n \in A \text{ and for any } \pi \in \operatorname{Sym}\{0, \ldots, n\}.$$

f is called *quasi-trivial* if

$$f(x_0, \ldots, x_n) \in \{x_0, \ldots, x_n\} \quad \text{for any } x_0, \ldots, x_n \in A.$$

f is called *superassociative* if

$$f(f(x_0, \ldots, x_n), x_{n+1}, \ldots, x_{2n}) = f(x_0, f(x_1, x_{n+1}, \ldots, x_{2n}), \ldots, f(x_n, \ldots, x_{2n}))$$
$$\text{for any } x_0, \ldots, x_{2n} \in A.$$

(A,f) is called *commutative*, *quasi-trivial* or an *n-dimensional superassociative system*, respectively, if f has the corresponding property.

In the following let (A,f) be some fixed n-dimensional commutative quasi-trivial superassociative system. If i is some non-negative integer and if a is an element of some algebra then $a(i)$ will stand for the sequence $a, \ldots, a$ of length i.

LEMMA 2. *$f(f(x_0, \ldots, x_n), x_1, \ldots, x_n) = f(x_0, \ldots, x_n)$ for any $x_0, \ldots, x_n \in A$.*

Proof. Suppose, Lemma 2 does not hold. Then there exist $a_0, \ldots, a_n \in A$ such that $f(f(a_0, \ldots, a_n), a_1, \ldots, a_n) \neq f(a_0, \ldots, a_n)$. Let g denote the mapping $x \mapsto f(x, a_1, \ldots, a_n)$ from A to A. Then $g^2 a_0 \neq g a_0$ whence

$$(1) \qquad a_0 \neq g a_0 \neq g^2 a_0 .$$

Using commutativity and superassociativity of f we obtain

$$\begin{aligned}(2_i) \qquad g f(x_0, \ldots, x_n) &= g f(x_i, x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &= f(x_i, g x_0, \ldots, g x_{i-1}, g x_{i+1}, \ldots, g x_n) \\ &= f(g x_0, \ldots, g x_{i-1}, x_i, g x_{i+1}, \ldots, g x_n) \\ &\text{for any } x_0, \ldots, x_n \in A \text{ and } i = 0, \ldots, n.\end{aligned}$$

Using quasi-triviality of f together with (2_0)-(2_n) we conclude

$$g^{n+1} a_0 = g^{n+1} f(a_0, \ldots, a_0) = f(g^n a_0, \ldots, g^n a_0) = g^n a_0 .$$

Now put $j := \min\{i \mid i \geqslant 0,\ g^{i+1} a_0 = g^i a_0\}$. Because of (1) we have $j > 1$. Put $a := g^{j-2} a_0$, $b := g^{j-1} a_0$ and $c := g^j a_0$. Then from the definition of j it follows

$$(3) \qquad a \neq b \neq c .$$

Now $f(b, a, \ldots, a) \neq a$ would imply $f(b, a, \ldots, a) = b$ (by quasi-triviality of f) whence $b = f(b, \ldots, b) = c \neq b$ (by quasi-triviality of f, (2_0) and (3)), a contradiction. Hence,

$$(4) \qquad f(b, a, \ldots, a) = a .$$

Now let us consider the case $n > 1$. Let $0 < k < n$ and assume $f(c(k-1), b, a, \ldots, a) = a$ already proved. Then

$$(5) \qquad f(c(k), a, b, \ldots, b) = b$$

(by (2_k)). Now $f(c(k), a, \ldots, a) \neq a$ would imply $f(c(k), a, \ldots, a) = c$ (by quasi-triviality of f) whence $b = f(c(k), a, b, \ldots, b) = c \neq b$ (by (5), (2_k) and (3)), a contradiction. Hence $f(c(k), a, \ldots, a) = a$ and therefore

$$(6) \qquad f(c(k), b, \ldots, b) = b$$

(by (2_{k-1})). Now $f(c(k), b, a, \ldots, a) \neq a$ would imply $f(c(k), b, a, \ldots, a) \in \{b, c\}$ (by quasi-triviality of f) whence $b = f(c(k), b, \ldots, b) = c \neq b$ (by (6), (2_k) and (3)), a contradiction. Hence $f(c(k), b, a, \ldots, a) = a$. By induction argument,

$$f(c, \ldots, c, b, a) = a$$

which also holds in case $n = 1$ because of (4). Therefore, in any case $(n \geqslant 1)$ we obtain $\{a, c\} \ni f(c, \ldots, c, a) = b \notin \{a, c\}$ (by quasi-triviality of f, (2_n) and (3)), a contradiction. This completes the proof of Lemma 2.

Remark. Using commutativity of f together with Lemma 2 we obtain

$$\begin{aligned} f(x_0, \ldots, x_{i-1}, f(x_0, \ldots, x_n), x_{i+1}, \ldots, x_n) &= f(f(x_0, \ldots, x_n), x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &= f(f(x_i, x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n), x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \\ &= f(x_i, x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = f(x_0, \ldots, x_n)\end{aligned}$$

for any $x_0, \ldots, x_n \in A$ and $i = 0, \ldots, n$. Applying this result finitely many times one obtains

$$(7) \qquad f(\{x_0, f(x_0, \ldots, x_n)\} \times \ldots \times \{x_n, f(x_0, \ldots, x_n)\}) = f(x_0, \ldots, x_n)$$
$$\text{for any } x_0, \ldots, x_n \in A .$$

This important property of f will often be used in the sequel.

In the following put $p := [\frac{1}{2}(n+1)]$.

DEFINITION 3. For any $i = 1, \ldots, p$ we define a binary relation $\leqslant_i$ on A as follows:

$$x \leqslant_i y \text{ iff } f(x(i), y, \ldots, y) = x \quad (x, y \in A).$$

Remark. From (7) immediately follows $\leqslant_1 \subseteq \ldots \subseteq \leqslant_p$.

LEMMA 4. *$(A, \leqslant_i)$ is a poset for any $i = 1, \ldots, p$.*

Proof. Let $1 \leqslant j \leqslant p$. Reflexivity and antisymmetry of $\leqslant_j$ follow from quasi-triviality of f and from (7) and commutativity of f, respectively. Now assume $\leqslant_j$ not to be transitive. Then there exist $a, b, c \in A$ such that

$$(8) \qquad a \leqslant_j b \leqslant_j c \quad \text{and} \quad a \not\leqslant_j c .$$

Of course,

$$(9) \qquad a \neq b \neq c \neq a .$$

Now put $a_i := f(a(i), b(j-i), c, \ldots, c)$ for any $i = 0, \ldots, j$. Because of (8) and (9) we have $a_0 = b \neq a, c$. Now let $0 < k \leqslant j$ and assume $a_{k-1} \neq a, c$ already proved. Then

(10) $$a_{k-1} = b$$

(by quasi-triviality of f). Now $a_k = a$ would imply $a \leqslant_j c \not\geqslant_j a$ (by (7) and (8)), a contradiction. Hence,

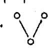
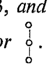
(11) $$a_k \neq a.$$

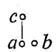
On the other hand, $a_k = c$ would imply

$$c = f\big(f(a(j), b, \ldots, b), a(k-1), b(j-k), c, \ldots, c\big) = f(a, c(j-1), b, \ldots, b)$$

(by (8), superassociativity and commutativity of f and (10)) whence $c \leqslant_j b$ (by (7)) which together with (8) and antisymmetry of $\leqslant_j$ yields $b = c$ contradicting (9). Hence $a_k \neq c$ which together with (11) yields $a_k \neq a, c$. By induction argument, $a_j \neq a, c$ contradicting quasi-triviality of f. Therefore $\leqslant_j$ is transitive and thus $(A, \leqslant_j)$ is a poset. This completes the proof of Lemma 4.

3. Main results

LEMMA 5. *Assume $|A| > 2$, let $B \subseteq A$, $|B| = 3$, and let $1 \leqslant k \leqslant p$. Then the Hasse-diagram of $(B, \leqslant_k)$ is of the type* $\circ\circ\circ$ *or*  *or* .

Proof. Let $B = \{a, b, c\}$. First assume  to be the Hasse-diagram of $(B, \leqslant_k)$. Then $f(a(k), b, \ldots, b) \neq b$ would imply $f(a(k), b, \ldots, b) = a$ (by quasi-triviality of f), i.e. $a \leqslant_k b$ contradicting our assumption. Hence,

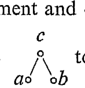
(12) $$f(a(k), b, \ldots, b) = b.$$

Put $a_i := f(a(i), b(k-i), c, \ldots, c)$ for any $i = 0, \ldots, k$. Now $a_0 \neq c$ would imply $a_0 = b$ (by quasi-triviality of f), i.e. $b \leqslant_k c$ contradicting our assumption. Hence, $a_0 = c$. Now let $0 < l \leqslant k$ and assume $a_{l-1} = c$ already proved. Then $a_l = b$ would imply $b \leqslant_k c$ (by (7)) contradicting our assumption. Hence,

(13) $$a_l \neq b.$$

On the other hand, $a_l = a$ would imply

$$c = f\big(f(a(k), b, \ldots, b), a(l-1), b(k-l), c, \ldots, c\big) = f(a(k), c, \ldots, c) = a \neq c$$

(by (12), commutativity of f, induction hypothesis, superassociativity of f and our assumption), a contradiction. Hence $a_l \neq a$ which together with (13) and quasi-triviality of f implies $a_l = c$. By induction argument and our assumption we obtain $c = a_k = a \neq c$, a contradiction. Now assume  to be the Hasse-diagram of $(B, \leqslant_k)$. Put $a_{ij} := f(a(i), b(j), c, \ldots, c)$ for any $i = 0, \ldots, k$ and $j = k, \ldots, 2k-i$.

Then $a_{0j} = b$ for any $j = k, \ldots, 2k$ (by our assumption and (7)). Now let $0 < m \leqslant k$ and assume $a_{m-1,j} = b$ for any $j = k, \ldots, 2k-m+1$ already proved. Further assume $k \leqslant q \leqslant 2k-m$. Then $a_{mq} = c$ would imply $b = f(b(k), c, \ldots, c) = c \neq b$ (by our assumption, (7) and commutativity of f), a contradiction. Hence

(14) $$a_{mq} \neq c.$$

On the other hand, $a_{mq} = a$ would imply

$$a = f\big(f(a(m), b(q), c, \ldots, c), a(m-1), b(q), c, \ldots, c\big) = f(a(m), b, \ldots, b)$$

(by superassociativity and commutativity of f and induction hypothesis) whence $a \leqslant_k b$ (by (7)) contradicting our assumption. Hence $a_{mq} \neq a$ which together with (14) (and quasi-triviality of f) implies $a_{mq} = b$. Since q was an arbitrarily chosen element of $\{k, \ldots, 2k-m\}$ we have proved $a_{mj} = b$ for any $j = k, \ldots, 2k-m$. By induction argument, $a_{kk} = b$ and hence by symmetry argument, $a_{kk} = a$, a contradiction. This completes the proof of Lemma 5.

DEFINITION 6. For any $i = 1, \ldots, p$ put

$$M_i := \{x \in A \mid x \text{ is maximal with respect to } \leqslant_i\} \text{ and } K_i := A \setminus M_i.$$

Remark. Let $1 \leqslant j \leqslant p$. Then

$$M_j = \{x \in A \mid f(x(j), y, \ldots, y) = y \text{ for any } y \in A\} \text{ and } M_1 \supseteq \ldots \supseteq M_p$$

(cf. remark after Definition 3). From Lemma 5 it follows that $(K_j, \leqslant_j)$ is a chain as well as that $x <_j y$ for any $x \in K_j$ and for any $y \in M_j$. Hence $(A, \leqslant_j)$ is a chain iff $|M_j| \leqslant 1$. Finally, let $a_0, \ldots, a_n \in A$. Then

$$\begin{aligned} f(a_i, f(a_0, \ldots, a_n), \ldots, f(a_0, \ldots, a_n)) &= f\big((f(a_0, \ldots, a_n))(i), a_i, f(a_0, \ldots, a_n), \ldots, f(a_0, \ldots, a_n)\big) \\ &= f(a_0, \ldots, a_n) \end{aligned}$$

for any $i = 0, \ldots, n$ (by commutativity of f and (7)) whence for any $i = 0, \ldots, n$ either $f(a_0, \ldots, a_n) = a_i$ or $a_i \not\leqslant_1 f(a_0, \ldots, a_n)$. From this and from the fact that all elements of K_1 are comparable with all elements of A with respect to $\leqslant_1$ we conclude $f(a_0, \ldots, a_n) \leqslant_1 x$ for any $x \in \{a_0, \ldots, a_n\} \cap K_1$. Hence (using quasi-triviality of f) we see that

$$f(a_0, \ldots, a_n) = \min_{\leqslant_1}(\{a_0, \ldots, a_n\} \cap K_1) \quad \text{if} \quad (a_0, \ldots, a_n) \in A^{n+1} \setminus M_1^{n+1}.$$

Thus we obtain

THEOREM 7. *If $|M_1| \leqslant 1$ then $(A, \leqslant_1)$ is a chain and $f(x_0, \ldots, x_n) = m_{1,\leqslant_1}(x_0, \ldots, x_n)$ for any $x_0, \ldots, x_n \in A$.*

COROLLARY. *Let n be some positive integer and let (B, g) be some algebra with one $(n+1)$-ary operation. Then t.f.a.e.:*

(i) *(B, g) is an n-dimensional commutative quasi-trivial superassociative system and there exists at most one $x \in B$ such that $g(x, y, \ldots, y) = y$ for any $y \in B$.*

(ii) *There exists some total ordering* $\leqslant$ *on* B *such that* $g(x_0, \dots, x_n) = m_{1,\leqslant}(x_0, \dots, x_n)$ *for any* $x_0, \dots, x_n \in B$.

Remark. This corollary characterizes the operations $m_{1,\leqslant}, m_{n+1,\leqslant}$ on chains $(L, \leqslant)$.

THEOREM 8. *Assume* $|M_1|>1$ *and* $M_p \neq M_1$ *and put* $k := \min\{i \mid 1<i\leqslant p, M_i \neq M_1\}$. *Then* $(M_1, \leqslant_k)$ *is a chain and* $f(x_0, \dots, x_n) = m_{k,\leqslant_k}(x_0, \dots, x_n)$ *for any* $x_0, \dots, x_n \in M_1$.

Proof. Applying the remark after Definition 6 we obtain

$$f(x(k-1), y, \dots, y) = y \quad \text{for any} \quad x, y \in M_1 \tag{15}$$

and there exist $x'_0 \in M_1$ and $y'_0 \in A$ such that

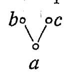
$$f(x'_0(k), y'_0, \dots, y'_0) = x'_0 \neq y'_0 \tag{16}$$

(here also quasi-triviality of f was used). Now $y'_0 \notin M_1$ would imply $y'_0 \in K_1$ whence $y'_0 <_1 x'_0$ (cf. remark after Definition 6) which implies $y'_0 <_k x'_0$ (cf. remark after Definition 3) whence $y'_0 <_k x'_0 <_k y'_0$ (by (16)), a contradiction. Hence,

$$y'_0 \in M_1 . \tag{17}$$

First consider the case $|M_1| = 2$. Then (16) together with (17) implies $M_1 = \{x'_0, y'_0\}$ and $x'_0 <_k y'_0$. Hence $(M_1, \leqslant_k)$ is a chain. Moreover,

$$f(x_0, \dots, x_n) = m_{k,\leqslant_k}(x_0, \dots, x_n) \quad \text{for any } x_0, \dots, x_n \in M_1$$

because of (15), (16), (7) and commutativity of f. Therefore Theorem 8 is proved in this case. Thus, for the rest of the proof suppose $|M_1|>2$. Now let $B \subseteq M_1$, $|B| = 3$, say $B = \{a, b, c\}$. Assume  to be the Hasse-diagram of $(B, \leqslant_k)$. Put

$$a_i := f(a(i), b(k-i), c, \dots, c) \quad \text{for any } i = 0, \dots, k-1 .$$

Now $a_0 \neq c$ would imply $a_0 = b$ (by quasi-triviality of f), i.e. $b \leqslant_k c$ contradicting our assumption. Hence $a_0 = c$. Now let $0<l<k$ and assume $a_{l-1} = c$ already proved. Then $a_l = b$ would imply $b \leqslant_k c$ (by (7)) contradicting our assumption. Hence,

$$a_l \neq b . \tag{18}$$

On the other hand, $a_l = a$ would imply

$$\begin{aligned} a &= f\big(f(a(l), b(k-l), c, \dots, c), a(l-1), b(k-l), c, \dots, c\big) \\ &= f(a(l), c, \dots, c) = c \neq a \end{aligned}$$

(by superassociativity and commutativity of f, induction hypothesis, (7) and (15)), a contradiction. Hence $a_l \neq a$ which together with (18) and quasi-triviality of f implies $a_l = c$. By induction argument,

$$a_i = c \quad \text{for any} \quad i = 0, \dots, k-1 . \tag{19}$$

Now $f(b(k), c, \dots, c) \neq c$ would imply $f(b(k), c, \dots, c) = b$ (by quasi-triviality of f), i.e. $b \leqslant_k c$ contradicting our assumption. Hence $f(b(k), c, \dots, c) = c$. Put $m := \max\{i \mid 0 \leqslant i \leqslant n+1, f(b(i), c, \dots, c) = c\}$. Now $m>n-k$ would imply $f(c(k), b, \dots, b) = f(b(n-k+1), c, \dots, c) = c$ (by commutativity of f, definition of m and (7)), i.e. $c \leqslant_k b$ contradicting our assumption. Hence,

$$m \leqslant n-k . \tag{20}$$

Therefore $f(b(m+1), c, \dots, c)$ is well-defined and $f(b(m+1), c, \dots, c) \neq c$ (by definition of m) whence

$$f(b(m+1), c, \dots, c) = b \tag{21}$$

(by quasi-triviality of f). Now put $d := f(a, b(m), c, \dots, c)$. Then d is well-defined because of (20). Now $d = a$ would imply

$$a = f\big(f(a, b(m), c, \dots, c), a(k-1), c, \dots, c\big) = f(a, c, \dots, c) = c \neq a$$

(by our assumption, superassociativity and commutativity of f, (19), (15) and (7)), a contradiction. Hence,

$$d \neq a . \tag{22}$$

Now $d = b$ would imply

$$b = f\big(f(a(k), c, \dots, c), b(m), c, \dots, c\big) = f(a, b(k-1), c, \dots, c) = c \neq b$$

(by our assumption, (21), superassociativity and commutativity of f, definition of m and (19)), a contradiction. Hence $d \neq b$ which together with (22) and quasi-triviality of f yields $d = c$. But now

$$c = f\big(f(a(k), b, \dots, b), b(m), c, \dots, c\big) = f(a, c(k-1), b, \dots, b)$$

(by our assumption, superassociativity of f and (21)) whence $c \leqslant_k b$ (by (7)) contradicting our assumption. Thus, using (16), (17) and Lemma 5 we conclude that

$$(M_1, \leqslant_k) \text{ is a chain.} \tag{23}$$

Now let $b_0, \dots, b_n \in M_1$ such that

$$b_0 \leqslant_k \dots \leqslant_k b_n . \tag{24}$$

Since f is quasi-trivial there exists some q, $0 \leqslant q \leqslant n$, such that

$$f(b_0, \dots, b_n) = b_q . \tag{25}$$

First suppose

$$b_q <_k b_{k-1} . \tag{26}$$

Let r be some fixed integer, $k-1 \leqslant r \leqslant n$, and put

$$a_{ij} := f(b_q(i), b_r(j), b_{k-1}, \dots, b_{k-1}) \quad \text{for any } i = 0, \dots, k-1 \text{ and } j = 1, \dots, k-i .$$

Using (24) we conclude

$$b_{k-1} \leqslant_k b_r . \tag{27}$$

Hence $a_{0j} = b_{k-1}$ for any $j = 1, \ldots, k$ (by (7) and commutativity of f). Let $0<s<k$ and assume $a_{s-1,j} = b_{k-1}$ for any $j = 1, \ldots, k-s+1$ already proved. Further assume $1 \leqslant t \leqslant k-s$. Then $a_{st} \notin \{b_q, b_{k-1}\}$ would imply $a_{st} = b_r \neq b_{k-1}$ (by quasi-triviality of f) whence $b_r <_k b_{k-1} \leqslant_k b_r$ (by (7) and (27)), a contradiction. Hence,

$$a_{st} \in \{b_q, b_{k-1}\}. \tag{28}$$

On the other hand, $a_{st} = b_q$ would imply

$$\begin{aligned} b_q &= f\big(f(b_q(s), b_r(t), b_{k-1}, \ldots, b_{k-1}), b_q(s-1), b_r(t), b_{k-1}, \ldots, b_{k-1}\big) \\ &= f(b_q(s), b_{k-1}, \ldots, b_{k-1}) = b_{k-1} >_k b_q \end{aligned}$$

(by superassociativity and commutativity of f, induction hypothesis, (15), (7) and (26)), a contradiction. Hence $a_{st} \neq b_q$ which together with (28) yields $a_{st} = b_{k-1}$. Since t was an arbitrarily chosen element of $\{1, \ldots, k-s\}$ we have proved $a_{sj} = b_{k-1}$ for any $j = 1, \ldots, k-s$. By induction argument, $a_{k-1,1} = b_{k-1}$, i.e.

$$f(b_q(k-1), b_r, b_{k-1}, \ldots, b_{k-1}) = b_{k-1}.$$

Since r was an arbitrarily chosen element of $\{k-1, \ldots, n\}$ we have proved

$$f(b_q(k-1), b_i, b_{k-1}, \ldots, b_{k-1}) = b_{k-1} \quad \text{for any } i = k-1, \ldots, n.$$

Therefore

$$\begin{aligned} b_q &= f(b_q(k), b_{k-1}, \ldots, b_{k-1}) \\ &= f\big(f(b_q(k-1), b_{k-1}, \ldots, b_n), b_q(k-1), b_{k-1}, \ldots, b_{k-1}\big) \\ &= f(b_q(k-1), b_{k-1}, \ldots, b_{k-1}) = b_{k-1} >_k b_q \end{aligned}$$

(by (26), (25), (7) and superassociativity and commutativity of f), a contradiction. Hence,

$$b_q \geqslant_k b_{k-1} \tag{29}$$

(by (23)). Finally suppose

$$b_q >_k b_{k-1}. \tag{30}$$

Put

$$c_i := f(b_{k-i}, \ldots, b_{k-2}, b_{k-1}(k-i+1), b_q(n-k+1)) \quad \text{for any } i = 1, \ldots, k.$$

Then $c_1 = f(b_{k-1}(k), b_q, \ldots, b_q) = b_{k-1} \neq b_q$ (by (30)). Now let $1<u \leqslant k$ and assume $c_{u-1} \neq b_q$ already proved. Then $c_{u-1} \in \{b_{k-u+1}, \ldots, b_{k-1}\}$ (by quasi-triviality of f) and hence

$$c_{u-1} \leqslant_k b_{k-1} \tag{31}$$

(by (24)). Now $c_u = b_q$ would imply

$$\begin{aligned} b_q &= f\big(f(b_{k-u}(k), b_{k-1}, \ldots, b_{k-1}), b_{k-u+1}, \ldots, b_{k-2}, b_{k-1}(k-u+1), b_q(n-k+1)\big) \\ &= f(b_{k-u}, b_q(k-1), c_{u-1}, \ldots, c_{u-1}) \end{aligned}$$

(by (24) and superassociativity and commutativity of f) whence $b_q \leqslant_k c_{u-1} \leqslant_k b_{k-1} <_k b_q$ (by (7), (31) and (30)), a contradiction. Hence, $c_u \neq b_q$. By induction argument, (25) and (7) we obtain $b_q \neq c_k = b_q$, a contradiction. Hence, $b_q \leqslant_k b_{k-1}$ (by (23)) which together with (29) yields $b_q = b_{k-1}$. Therefore, by (23) and commutativity of f we have

$$f(x_0, \ldots, x_n) = m_{k,\leqslant_k}(x_0, \ldots, x_n) \quad \text{for any } x_0, \ldots, x_n \in M_1.$$

This completes the proof of Theorem 8.

Corollary. *Let n, i be positive integers such that $1<i \leqslant \frac{1}{2}(n+1)$ and let (B, g), $|B|>1$, be some algebra with one $(n+1)$-ary operation. Then t. f. a. e.:*

(i) *(B, g) is an n-dimensional commutative quasi-trivial superassociative system, $g(x(i-1), y, \ldots, y) = y$ for any $x, y \in B$ and there exist $a, b \in B$ such that $g(a(i), b, \ldots, b) \neq b$.*

(ii) *There exists some total ordering $\leqslant$ on B such that*

$$g(x_0, \ldots, x_n) = m_{i,\leqslant}(x_0, \ldots, x_n) \quad \textit{for any } x_0, \ldots, x_n \in B.$$

Remark. This corollary characterizes the operations $m_{i,\leqslant}$, $1<i \leqslant n$, $i \neq \frac{1}{2}n+1$, on chains $(L, \leqslant)$.

From the remark following Definition 6 and from Theorem 8 we conclude

Proposition 9. *Assume $|M_1|>1$ and $M_p \neq M_1$ and put $j := \min\{i \mid 1<i \leqslant p, M_i \neq M_1\}$. Further, define a binary relation $\leqslant$ on A as follows: For any $x, y \in A$ let $x \leqslant y$ iff one of the following conditions* (i)-(iii) *is satisfied:*

(i) *$x, y \in K_1$ and $x \leqslant_1 y$.*

(ii) *$x \in K_1$ and $y \in M_1$.*

(iii) *$x, y \in M_1$ and $x \leqslant_j y$.*

Then $(A, \leqslant)$ is a chain and

$$f(x_0, \ldots, x_n) = \begin{cases} m_{j,\leqslant}(x_0, \ldots, x_n) & \textit{if } (x_0, \ldots, x_n) \in M_1^{n+1}, \\ m_{1,\leqslant}(x_0, \ldots, x_n) & \textit{otherwise} \end{cases}$$

$(x_0, \ldots, x_n \in A)$.

Remark. Until now we have considered the cases $|M_1| \leqslant 1$ (Theorem 7) and $|M_1|>1$, $M_p \neq M_1$ (Proposition 9). Further note that in case n is odd quasi-triviality of f implies connexity of $\leqslant_{\frac{1}{2}(n+1)}$.

Now we are able to formulate our main results:

Theorem 10 (Classification Theorem for commutative quasi-trivial superassociative operations of even arity). *Let n be some odd positive integer and let (B, g) be some algebra with one $(n+1)$-ary operation. Then t. f. a. e.:*

(i) *(B, g) is an n-dimensional commutative quasi-trivial superassociative system.*

(ii) *There exists some total ordering $\leqslant$ on B, there exists some final segment C of $(B, \leqslant)$ and there exists some integer i, $1 \leqslant i \leqslant \frac{1}{2}(n+1)$, such that*

$$g(x_0, \ldots, x_n) = \begin{cases} m_{i,\leqslant}(x_0, \ldots, x_n) & \textit{if } (x_0, \ldots, x_n) \in C^{n+1}, \\ m_{1,\leqslant}(x_0, \ldots, x_n) & \textit{otherwise} \end{cases}$$

$(x_0, \ldots, x_n \in B)$.

Hence, for any positive integer m, up to isomorphism there exist exactly $1+\frac{1}{2}(m-1)(n-1)$ m-element n-dimensional commutative quasi-trivial superassociative systems.

THEOREM 11 (classification of a certain class of commutative quasi-trivial superassociative operations of odd arity). *Let n be some even positive integer and let (B, g), $|B|>1$, be some algebra with one $(n+1)$-ary operation. Then t.f.a.e.:*

(i) *(B, g) is an n-dimensional commutative quasi-trivial superassociative system and there hold* (a) *or* (b):

(a) *There exists at most one $x \in B$ such that $g(x, y, \ldots, y) = y$ for any $y \in B$.*

(b) *There exists some $a \in B$ such that $g(a, y, \ldots, y) = y$ for any $y \in B$ and such that there exists some $b \in B$ with $g(a(\frac{1}{2}n), b, \ldots, b) \neq b$.*

(ii) *There exists some total ordering $\leqslant$ on B, there exists some final segment C of $(B, \leqslant)$ and there exists some integer i, $1 \leqslant i \leqslant \frac{1}{2}n$, such that*

$$g(x_0, \ldots, x_n) = \begin{cases} m_{i,\leqslant}(x_0, \ldots, x_n) & \text{if } (x_0, \ldots, x_n) \in C^{n+1}, \\ m_{1,\leqslant}(x_0, \ldots, x_n) & \text{otherwise} \end{cases}$$

$(x_0, \ldots, x_n \in B)$.

Remark. The following example shows that there exist n-dimensional commutative quasi-trivial superassociative systems, n even, neither satisfying (i) (a) nor (i) (b): Put $B := \{0, 1, 2, 3\}$, $n := 2$, $g(x, x, y) = g(x, y, x) = g(y, x, x) := x$ for any $x, y \in B$ and $g(x, y, z) \equiv -(x+y+z) \bmod 4$ for any three mutually distinct elements $x, y, z \in B$.

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On vertices and edges in maximum path-factors of a tree

by

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Abstract. The paper presents proofs for part of the results announced in [11]. It develops a method of classifying the edges and the vertices of a tree T with respect to their appearance in maximum path-factors of T.

1. Introduction. Since Ore's pioneering work [7] in 1961, different publications concerning Hamiltonian graphs have dealt with the covering of vertices by (or partition of vertices into) disjoint (possibly trivial) paths in an ordinary graph, say G. Most of these papers deal with the invariant of G introduced by Barnette [1]. Following Skupień [8] we will denote this invariant by $\pi_0(G)$, and call it the vertex-path partition number of G, where $\pi_0(G)$ is the minimum number of paths among the path partitions of vertices of G.

Recently, new related invariants, namely Hamiltonian completion number $\mathrm{hc}(G)$ and Hamiltonian shortage $s_{\mathrm{H}}(G)$, have been independently introduced by Goodman and Hedetniemi [3], and Skupień [8], [9]. In general, these new invariants coincide. Namely, both equal 0 when G is Hamiltonian, and both equal $\pi_0(G)$ when G is a non-trivial non-Hamiltonian graph. Only for $G = K_1$ we have $\pi_0(K_1) = \mathrm{hc}(K_1) = s_{\mathrm{H}}(K_1)-1 = 1$.

In a series of papers sufficient conditions have been found for either $\pi_0(G) \leqslant s$ or $s_{\mathrm{H}}(G) \leqslant s$, where s is an integer.

The problem of determining $\pi_0(G)$ or $s_{\mathrm{H}}(G)$ is considered independently in [2], [3], and [8]. In each of these papers algorithms for determining $\pi_0(G)$ in the case where G is a tree or forest are developed. Algorithms presented in [2] and [3] are very similar to each other. Two other algorithms, based on labelling the vertices of a tree, are presented in [8].

Evaluating π_0 for trees is of special importance. Namely, in [2] and [3] it is noted that, for a connected graph G,

$$\pi_0(G) = \min\{\pi_0(T)\colon T \text{ is spanning tree of } G\}.$$

In general (cf. [10]),

$$\pi_0(G) = \min\{\pi_0(F)\colon F \text{ is a spanning forest of } G, \text{ with components which are spanning trees of components of } G\}.$$