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Approximation by continuous vector valued functions

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Abstract. Let X be a topological space, let E be a uniformly convex space and let $C(X, E)$ denote the space of bounded continuous functions from X into E. We show that for any bounded set-valued (or single valued) map Φ from X into 2^E and for any closed $C(X)$-submodule M in $C(X, E)$, there exists a best approximation from M to Φ. We use this result to study various approximation problems in $C(X, E)$.

§ 1. Introduction. Let $C(X, E)$ ($B(X, E)$) be the set of bounded continuous (respectively bounded) functions f from a topological space X into a Banach space E; these are Banach spaces under the supremum norm defined by $\|f\| = \sup_{t\in X}\|f(t)\|$. If Φ is a map from X into the family of subsets of E, we define the distance of Φ to an $f \in B(X, E)$ by

$$d(f, \Phi) = \sup_{t\in X}\sup_{y\in\Phi(t)}\|f(t)-y\|.$$

The main result of this paper is concerned with the existence of the best approximation to a set-valued map Φ by continuous point-valued functions in $C(X, E)$:

Let X be a topological space and let E be a uniformly convex space. Then for any bounded set-valued map Φ: $X\to 2^E$ and for any closed $C(X)$-submodule M in $C(X, E)$, there exists an $f \in M$ such that $d(f, \Phi) =$ $\inf\{d(g, \Phi)\colon g \in M\}$.

This generalizes a result of Olech [7] where, in order to apply Michael's selection theorem, it was assumed in addition, that X is paracompact, Φ is upper semicontinuous, and $M = C(X, E)$. Our proof differs significantly from his and is, in fact, inspired by a construction of approximation by Ward in [10].

We prove the above theorem in § 2 and § 3. In § 4, we apply the theorem to study some approximation problems of bounded functions by continuous functions (cf. [4], [6], [7], [10]) and bounded linear operators by compact operators (cf. [5]).

§ 2. Some lemmas. Let E be a (real) Banach space and let E^* be the dual of E. For any $r > 0$, $x \in E$, we let $B_r(x) = \{x\colon \|x\| \leqslant r\}, U_r(x) = \{x\colon \|x\| < r\}$ and $S_r(x) = \{x\colon \|x\| = r\}$. For any $r > \delta > 0$, we define

$$\varepsilon_r(\delta) = \sup_{\|x^*\|=1} (\operatorname{diam}\{x\colon x^*(x) = r-\delta,\ \|x\| = r\})$$

where $\operatorname{diam} A = \sup\{\|x-y\|\colon x, y \in A\}$. If $r = 1$, we simply use $\varepsilon(\delta)$ to denote $\varepsilon_1(\delta)$. It is clear that $\varepsilon_r(\delta) = r\varepsilon(\delta/r)$.

LEMMA 2.1. *Let g be a concave function defined on $[0, 1]$ with $g(0) > 0$ and $g(1) = 0$. Let $0 < a < 1$ and let h be a function defined on $[0, a]$ by $h(x) = ag(x/a)$, $x \in [0, a]$. Then $g(a) - h(a) \geqslant g(x) - h(x)$ for $x \in [0, a]$.*

Proof. Note that the derivative $g'(x)$ exists and decreases almost everywhere. Hence

$$g'(x) - h'(x) = g'(x) - g'(x/a) \geqslant 0 \text{ a.e. on } [0, a]$$

and $g - h$ is an increasing function on $[0, a]$. This completes the proof.

LEMMA 2.2. *Let E be a Banach space. Let $r > \delta > 0$ be given. Then for any line segment $[x\colon y]$ in between $S_r(0)$ and $S_{r-\delta}(0)$ (i.e. $z \in [x\colon y]$ implies $r - \delta \leqslant \|z\| \leqslant r$), $\|x-y\| \leqslant \varepsilon_r(\delta)$. In particular, we have $\delta \leqslant \varepsilon_r(\delta)$.*

Proof. We need only consider the two dimensional space generated by x and y. We may also assume that x and y are on the spheres $S_r(0)$ or $S_{r-\delta}(0)$. Let L_1 be the line parallel to $[x\colon y]$ and pass through 0. Let L be the maximal line segment contained in $B_r(0)$, which is parallel to L_1, on the same side of $[x\colon y]$ determined by L_1 and is a tangent to the ball $B_{r-\delta}(0)$. Let $|L|$ denote the length of L. If x is in $S_r(0)$ and y is in $S_{r-\delta}(0)$, then simple application of Lemma 2.1 will imply that $\|x-y\| \leqslant |L|$. If both x and y are in $S_r(0)$, then we consider the trapezoid determined by x, y and the two points of $L_1 \cap S_r(0)$, say x' and y'. Note that L is in between the line segments $[x\colon y]$ and $[x'\colon y']$ and $\|x-y\| \leqslant 2r = \|x'-y'\|$. By the convexity of the ball, we conclude that $\|x-y\| \leqslant |L|$. Hence in both cases we have $\|x-y\| \leqslant |L| \leqslant \varepsilon_r(\delta)$.

Our main lemma is

LEMMA 2.3. *Let E be a Banach space. For any $r > \delta > 0$ and for any $x, y \in E$ with $\|y-x\| > \varepsilon_r(\delta)$, let*

$$z = x + \frac{\varepsilon_r(\delta)}{\|y-x\|}(y-x).$$

Then

$$B_r(x) \cap B_{r-\delta}(y) \subseteq B_{r-\delta}(z).$$

(We remark that the condition $\|y-x\| > \varepsilon_r(\delta)$ implies that z is a convex combination of x and y.)

Proof. Without loss of generality, we assume that $x = 0$. For any $w \in B_r(0) \cap B_{r-\delta}(y)$, let a, b $(a \neq b)$ be the two end points of the line segment $\{w + \alpha y\colon \alpha \in \mathbf{R}\} \cap B_{r-\delta}(0)$; write $w = \lambda a + (1-\lambda) b$. Consider the following cases:

(i) $0 \leqslant \lambda \leqslant 1$. It follows that $\|w\| \leqslant r - \delta$. By assumption, $\|w-y\| \leqslant r-\delta$. Since z is a convex combination of 0 and y, we have $\|w-z\| \leqslant r-\delta$.

(ii) $\lambda > 1$ *or* $\lambda < 0$. We only consider $\lambda > 1$, the other case is proved by interchanging the role of a and b. Note that

$$\begin{aligned} w - z &= \lambda a + (1-\lambda) b - \varepsilon_r(\delta)\frac{y}{\|y\|} = \lambda a + (1-\lambda) b - \varepsilon_r(\delta)\frac{a-b}{\|a-b\|} \\ &= \Big(\lambda - \frac{\varepsilon_r(\delta)}{\|a-b\|}\Big) a + \Big(1 - \Big(\lambda - \frac{\varepsilon_r(\delta)}{\|a-b\|}\Big)\Big) b. \end{aligned}$$

We will show that $0 \leqslant \lambda - \dfrac{\varepsilon_r(\delta)}{\|a-b\|} \leqslant 1$. This will imply $\|w-z\| \leqslant r-\delta$. To this end, observe that $\|w-y\| \leqslant r-\delta$ and $w-y$ is on the line $\{w+\alpha y\colon \alpha \in \mathbf{R}\}$, so

$$w - y = \alpha a + (1-\alpha) b, \qquad 0 \leqslant \alpha \leqslant 1$$

and

$$\begin{aligned} w - z &= (w-y) + (y-z) \\ &= \alpha a + (1-\alpha) b + \beta(a-b) \qquad (\beta > 0) \\ &= (\alpha+\beta) a + \big(1 - (\alpha+\beta)\big) b. \end{aligned}$$

(That $\beta > 0$ follows from $\lambda > 1$.) It follows that

$$0 < \alpha + \beta = \lambda - \frac{\varepsilon_r(\delta)}{\|a-b\|}.$$

On the other hand, since $\lambda > 1$, the line segment $[a\colon w]$ is in between $S_r(0)$ and $S_{r-\delta}(0)$. By Lemma 2.2, $\|w-a\| \leqslant \varepsilon_r(\delta)$. This implies that

$$(\lambda - 1)\|a-b\| = \|w-a\| \leqslant \varepsilon_r(\delta) \quad \text{and} \quad \lambda - \frac{\varepsilon_r(\delta)}{\|a-b\|} \leqslant 1.$$

A Banach space is called *uniformly convex* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any x, y in $S_1(0)$ with $\|x-y\| > \varepsilon$, $\|(x+y)/2\| < 1 - \delta$.

LEMMA 2.4. *Let E be a uniformly convex space. Then* $\lim_{\delta\to 0^+} \varepsilon(\delta) = 0$.

Proof. It follows from Lemma 2.2 that $\varepsilon(\delta)$ is increasing on δ. Suppose $\lim_{\delta\to 0^+} \varepsilon(\delta) = \delta_0 > 0$. Let δ_0 be the corresponding number for ε_0 in the definition of uniform convexity. There exists an $x^* \in X^*$, $\|x^*\| = 1$

and $x, y \in S_1(0)$ such that

$$x^*(x) = x^*(y) = 1-\delta_0/2 \quad \text{and} \quad \|x-y\| \geqslant \varepsilon_0.$$

This implies that $\|(x+y)/2\| \geqslant 1-\delta_0/2 > 1-\delta_0$, which is a contradiction.

We remark that the converse of the above lemma is also true. Nevertheless, we do not need that fact here.

§3. The main theorem. Let X be a topological space and let E be a Banach space. A $C(X)$-*submodule* M in $C(X, E)$ is a linear subspace in $C(X, E)$ which is closed under multiplication by scalar valued functions in $C(X)$. The reader may refer to [1] for some properties of $C(X)$-submodules. Similarly, we can define $B(X)$-submodules in $B(X, E)$. We use 2^A to denote the family of subsets of A.

THEOREM 3.1. *Let X be a topological space and let E be a uniformly convex space. Then for any $\Phi\colon X\to 2^E$ and for any closed $C(X)$-submodule M in $C(X, E)$ with $\inf\{d(g,\varphi)\colon g \in M\} = r < \infty$, there exists an $f \in M$ such that $d(f, \varphi) = r$.*

Proof. Without loss of generality, we may assume that $r \geqslant 1$. By Lemma 2.4, we can choose a strictly decreasing sequence of positive numbers $\{\delta_n\}$ converges to 0 such that $\sum_{n=1}^{\infty}\varepsilon(\delta_n) < \infty$. Let $r_n = r+\delta_n$, then

$$\sum_{n=1}^{\infty}\varepsilon_{r_n}(\delta_n) = \sum_{n=1}^{\infty} r_n\varepsilon(\delta_n/r_n) < \sum_{n=1}^{\infty} r_n\varepsilon(\delta_n) < \infty.$$

We will use induction to define a sequence of functions $\{f_n\}$ in M: Let $f_1 \in M$ satisfy $d(f_1, \Phi) \leqslant r+\delta_1$. Suppose we have chosen $f_n \in M$ such that $d(f_n, \Phi) \leqslant r+\delta_n$, choose $g \in M$ with $d(g, \Phi) \leqslant r+\delta_{n+1}$. Let

$$d(t) = \|g(t)-f_n(t)\|, \quad t \in X$$

and define

$$f_{n+1}(t) = f_n(t)+\beta(t)\big(g(t)-f_n(t)\big), \quad t \in X$$

where

$$\beta(t) = \begin{cases} 1 & \text{if} \quad \varepsilon_{r_n}(\delta_n-\delta_{n+1}) \geqslant d(t), \\ \dfrac{\varepsilon_{r_n}(\delta_n-\delta_{n+1})}{d(t)} & \text{if} \quad \varepsilon_{r_n}(\delta_n-\delta_{n+1}) < d(t). \end{cases}$$

It is clear that $\beta(t)$ is a continuous function with $0 \leqslant \beta(t) \leqslant 1$. We claim that (i) f_{n+1} is in M, (ii) $\|f_{n+1}-f_n\| \leqslant \varepsilon_{r_n}(\delta_n-\delta_{n+1})$, (iii) $d(f_{n+1}, \Phi) \leqslant r+\delta_{n+1}$. Indeed, (i), (ii) follows from the construction of f_{n+1} and the definition of M. For (iii), we note that $d(f_n, \Phi) \leqslant r+\delta_n$, $d(g, \Phi) \leqslant r+\delta_{n+1}$. If $\beta(t) = 1$, then $f_{n+1}(t) = g(t)$. Hence $\Phi(t) \subseteq B_{r+\delta_{n+1}}(f_{n+1}(t))$. If $\beta(t) < 1$, by Lemma 2.3, we have

$$\Phi(t) \subseteq B_{r+\delta_n}(f_n(t)) \cap B_{r+\delta_{n+1}}(g(t)) \subseteq B_{r+\delta_{n+1}}(f_{n+1}(t)).$$

Hence $d(f_{n+1}, \Phi) \leqslant r+\delta_{n+1}$. Now, for $m > k$,

$$\|f_m-f_k\| \leqslant \sum_{n=k}^{m}\varepsilon_{r_n}(\delta_n-\delta_{n+1}) \leqslant \sum_{n=k}^{m}\varepsilon_{r_n}(\delta_n).$$

Since $\sum_{n=k}^{m}\varepsilon_{r_n}(\delta_n)\to 0$ as $m, k\to\infty$, $\{f_n\}$ is a Cauchy sequence. Let $f \in M$ be the uniform limit of $\{f_n\}$. For any $\varepsilon > 0$, there exists $t \in X$, $y \in \Phi(t)$ such that

$$d(f, \Phi) \leqslant \|f(t)-y\|+\varepsilon/3$$

and there exists n_0 such that $\|f-f_{n_0}\| < \varepsilon/3$ and $\delta_{n_0} < \varepsilon/3$. Hence

$$r \leqslant d(f, \Phi) \leqslant \|f(t)-y\|+\varepsilon/3 \leqslant \|f(t)-f_{n_0}(t)\|+\|f_{n_0}(t)-y\|+\varepsilon/3 \leqslant r+\varepsilon.$$

This implies $d(f, \Phi) = r$ and the proof is completed.

COROLLARY 3.2. *The above theorem also holds if we replace X by a set and M by a closed $B(X)$-submodule.*

Proof. Give X the discrete topology, then we can apply Theorem 3.1.

In the following, we will consider a similar type of theorem concerning the essential supremum norm on functions over a measure space. Let (X, μ) be a measure space, let E be a Banach space. For any function $f\colon X\to E$, define

$$\|f\| = \operatorname*{ess\,sup}_{t\in X}\|f(t)\| = \inf_{N\in\mathcal{N}}\ \sup_{t\in X\setminus \mathcal{N}}\|f(t)\|$$

where $\mathcal{N}$ denotes the family of null sets in X. We use $B_*(X, E)$, $(B_*(X))$ to denote the Banach space of essentially bounded (scalar, respectively) functions $f\colon X\to E$ and use $L^\infty(X, E)$ $(L^\infty(X))$ to denote the closed subspace of Bochner (scalar) measurable functions [2]. For any $\Phi\colon X\to 2^E$ and for any $f\colon X\to E$, we let

$$d_*(f, \Phi) = \operatorname*{ess\,sup}_{t\in X}\sup_{y\in\Phi(t)}\|f(t)-y\|.$$

THEOREM 3.3. *Let (X, μ) be a measure space and let E be uniformly convex. Suppose M is either* (i) *a closed $B_*(X)$-submodule of $B_*(X, E)$ or* (ii) *a closed $L^\infty(X)$-submodule of $L^\infty(X, E)$. Then for any $\Phi\colon X\to 2^E$ such that $\inf\{d_*(g, \Phi)\colon g \in M\} = r < \infty$, there exists an $f \in M$ with $d_*(f, \Phi) = r$.*

Proof. We can follow the same proof as Theorem 3.1; the sequence $\{f_n\}$ will be defined on $X\setminus N$ for some null set N and the distance d will be replaced by d_*.

§ 4. Applications. Let E be a Banach space and let K be a subset in E. A point $x \in E$ is said to *have a best approximation from K* if there exists a $y \in K$ such that $\|x-y\| = \inf\{\|x-z\|: z \in K\}$; K is called *proximinal* if every point in E admits a best approximation from K. In [4], Holmes and Kripke proved that every bounded function on a paracompact space has a best approximation from the set of continuous functions. Olech [7] showed that the result is also true if the range is an Euclidean space. It follows directly from Theorem 3.1 that

THEOREM 4.1. *Let X be a topological space, let E be uniformly convex and let M be a closed $C(X)$-submodule in $C(X, E)$. Then every bounded function $g: X \to E$ admits a best approximation from M.*

COROLLARY 4.2. *With X, E given as above, every closed $C(X)$-submodule is proximinal in $C(X, E)$.*

Let X, Y be two sets, let $\varphi: Y \to X$ be a surjection and let E be a Banach space. For any bounded function $f: X \to E$, we define $\varphi^\circ f = f\circ\varphi$; φ° is then an isometry of $B(X, E)$ into $B(Y, E)$. If X, Y are topological spaces and φ is continuous, then $C(X, E)$ can be identified with $\varphi^\circ C(X, E)$ in $C(Y, E)$. In [8], Pełczyński asked, for $g \in C(Y, E)$, does there exist a best approximation from $\varphi^\circ C(X, E)$? Olech [7] showed that the conjecture is true if X, Y are both compact Hausdorff and E is uniformly convex. By using Theorem 3.1, we obtain a more general result with a simplier proof.

THEOREM 4.3. *Let X be a topological space, let E be a uniformly convex space and let M be a closed $C(X)$-submodule of $C(X, E)$. Suppose φ is a surjection from a set Y onto X. Then every bounded function $g: Y \to E$ has a best approximation from $\varphi^\circ M$.*

Proof. Define a topology on Y as follows: $A\ (\subseteq Y)$ is open if and only if $A = \varphi^{-1}(B)$ with B open in X. It is clear that $C(X, E)$ is isometrically isomorphic to $C(Y, E)$ under φ and M is a closed $C(Y)$-submodule in $C(Y, E)$. Hence, by Theorem 4.1, every bounded function $g: Y \to E$ has a best approximation from $\varphi^\circ M$.

Let E be a Banach space, let K be a bounded set and let F be a set in E. A point $x \in F$ is called a *restricted center* of K with respect to F if

$$d(x, K) = \inf\{d(y, K): y \in F\}.$$

If $F = E$, then x is called a *Chebyshev center* of K. Kadet and Zamyatin [6] proved that every bounded set F in $C[0,1]$ admits a Chebyshev center. This fact was improved by Ward to $C(X, E)$ where E is a Hilbert space [10].

THEOREM 4.4. *Let X be a topological space and let E be uniformly convex. Then every bounded set in $C(X, E)$ admits a restricted center with respect to closed $C(X)$-submodules.*

In particular, every bounded set in $C(X, E)$ admits a Chebyshev center.

Proof. For any bounded set K in $C(X, E)$ we need only define the set-valued map $\Phi: X \to 2^E$ by

$$\Phi(t) = \{f(t): f \in K\}, \quad t \in X$$

and apply Theorem 3.1.

Let E, F be Banach spaces and let $L(E, F)(K(E, F))$ be the space of bounded (compact) linear operators from E into F. In [5], Holmes and Kripke considered the question of proximity of $K(E, F)$ in $L(E, F)$. They showed that if E, F are both Hilbert spaces, then $K(E, F)$ is a proximinal subspace of $L(E, F)$. Little is known in general. Here, we add in two more special cases.

THEOREM 4.5. *Let E, F be Banach spaces such that either*

(i) *$E = L^1(X, \mu)$ where (X, μ) is a σ-finite measure space and F is uniformly convex or*

(ii) *E^* is uniformly convex and $F = C(Y)$ for some topological space Y.*

Then every $T \in L(E, F)$ has a best approximation from $K(E, F)$.

Proof. In (i), note that F has the Radon–Nikodym property, it follows that $L(E, F)$ is isometrically isomorphic to $L^\infty(X, F)$, the set of bounded Bochner measurable functions from X into F (cf. [2]). The set of compact operators $K(E, F)$ can be identified as the set of $f \in L^\infty(X, F)$ with $f(X \setminus N)$, N a null set, contained in a compact subset in F. We use $L_c^\infty(X, F)$ to denote this set. It is easy to show that $L_c^\infty(X, F)$ is an $L^\infty(X)$-submodule in $L^\infty(X, F)$. By Theorem 3.3, we conclude that every $f \in L^\infty(X, F)$ has a best approximation from $L_c^\infty(X, F)$.

For (ii), we observe that $C(Y)$ is an AM-space and has an order unit, hence it is isometrically isomorphic to $C(Z)$ for some compact Hausdorff space Z (cf. [9], p. 101). It is also well known that $K(E, F)$ can be identified as $C(Z, E^*)$ and $L(E, F)$ as $C(Z, (E^*, w^*))$ where (E^*, w^*) is the dual space E^* with the w^* topology (cf. [3], p. 490). Assertion (ii) now follows immediately from these remarks and Theorem 4.1.

COROLLARY 4.6. *Every bounded linear operator $T: L^1(\mu) \to L^p$ or $T: L^p \to C(X)$, where μ is a σ-finite measure, X is a topological space and $1 < p < \infty$, has a best approximation from the set of compact operators.*

§ 5. Some remarks. Let X and Y be topological spaces with Y compact Hausdorff; we can identify an $f \in C(X, C(Y))$ as a function $\tilde{f}$ in $C(X \times Y)$ where $\tilde{f}(x, y) = f(x)(y)$, $x \in X$, $y \in Y$. Moreover, this identification defines an isometric isomorphism of $C(X, C(Y))$ onto $C(X \times Y)$ [1]. For any set-valued map Φ from X into $C(Y)$, we define $\tilde{\Phi}$: $X \times Y \to 2^R$ by

$$\tilde{\Phi}(x, y) = \{f(y): f \in \Phi(x), \ x \in X, y \in Y\}.$$

Analogous to Theorem 3.1, we have

THEOREM 5.1. *Let X and Y be topological spaces. Let Φ: $X \to 2^{C(Y)}$ be* *a set-valued map such that $\Phi(x)$, $x \in X$ is contained in a bounded set of $C(Y)$. Then Φ admits a best approximation from $C(X, C(Y))$.*

Proof. Note that $C(Y)$ is isometric isomorphic to $C(Z)$ for some compact Hausdorff space Z ([9], p. 101). By Theorem 3.1, we can find a best approximation $\tilde{f}$ from the $C(X \times Z)$-submodule $C(X \times Z)$ to the set-valued function $\tilde{\Phi}$ on $X \times Z$. Hence the corresponding f in $C(X, C(Y))$ $(= C(X, C(Z)))$ is a best approximation to Φ.

We do not know whether the uniformly convex space E in Theorem 3.1 can be replaced by $L^1(\mu)$ or $M(K)$ where $M(K)$ denotes the set of regular Borel measures on a compact Hausdorff space K. In particular it would be interesting to know whether the space of compact operators $K(C(X), C(Y))$ is proximinal in the space of bounded linear operators $L(C(X), C(Y))$. We also do not know whether the condition of uniform convexity on E can be weakened.

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