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On functions satisfying a local Lipschitz condition of order α

by

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Abstract. It is proved a decomposition theorem for local Lipschitz functions with respect to a finite measure. This result is used to obtain a characterization of local Lipschitz functions of order α, when α is not an integer.

§ 1. Introduction. There is a classical theorem of Marcinkiewicz and Zygmund which states that if a function is k Riemann-bounded on a set, then at almost every point of that set it has a kth Peano derivative [3]. In [1], Ash introduced the concept of generalized-bounded function of order k, and he recaptured again the kth Peano derivative. Although this was essentially a one-dimensional result, the difficulties in extending it to higher dimensions can be easily overcome using the methods of Stein and Zygmund [10]. We refer the reader to [11], where a further interesting characterization of the n-dimensional Peano derivative is given.

It was pointed out by Stein and Zygmund [10], that the conditional continuity of a function, i.e. $\sum_{i=1}^{m} A_i f(x-a_i t) \to 0$ as $t \to 0$, implies almost everywhere the continuity of f at x, see also [4], [5].

The purpose of this note is to consider the analogous problem in the intermediate case between differentiability and continuity, the Lipschitz functions of order α. Thus for the expression $\sum_{i=1}^{m} A_i f(x-a_i t)$ it is generalized the following classical result for the second symmetric differences: if the function f, for every x in a measurable set E, satisfies $f(x+t)+f(x-t)-2f(x) = O(|t|^\alpha)$, then $f(x+t)-f(x) = O(|t|^\alpha)$ for $0<\alpha<1$, and $f(x+t)-f(x) = O\left(|t|\log\frac{1}{|t|}\right)$ for $\alpha = 1$, almost everywhere $x \in E$, [12].

The proof of our result relies in the conjunction of a localized version of a Tauberian theorem in [8] with some of the ideas used by Stein in his splitting theorem for harmonic derivatives [9].

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§ 2. Notations and main results. We consider only Lebesgue measurable function defined on $\mathbf{R}^n$, throughout this note E will denote a measurable subset of $\mathbf{R}^n$ and $|E|$ its Lebesgue measure. Recall that a bounded function f is said to be *Peano bounded of order α at x_0*, where α is a positive number, if there exists a polynomial P of degree strictly less than α with the property that $f(x_0+t)-P(t) = O(|t|^\alpha)$, for every $t \in \mathbf{R}^n$. The class of these functions is denoted by $T_\alpha(x_0)$. We say that $f \in \overline{T}_\alpha(x_0)$ if the function $|t|^\alpha$ is replaced by $|t|^\alpha \log|t|^{-1}$, for small values of $|t|$.

Let $A = \{A_1, A_2, \ldots, A_m, a_1, a_2, \ldots, a_m\}$ be a set of real numbers, such that $a_i \neq a_j$ if $i \neq j$ and each A_i is non zero. Then we write $f \in A_\alpha(x_0)$, $\alpha > 0$, $x_0 \in \mathbf{R}^n$, if f is a bounded function and

$$\sum_{i=1}^{m} A_i f(x_0 - ta_i) = O(|t|^\alpha) \quad \text{for all } t \in \mathbf{R}^n.$$

Now we state our main result.

THEOREM 1. *Suppose $f \in A_\alpha(x)$ for every $x \in E$; then $f \in T_\alpha(x)$* a.e. *$x \in E$ if α is not an integer, and $f \in \overline{T}_\alpha(x)$* a.e. *$x \in E$ for α integer.*

The natural concept of uniformly Lipschitz function of order α, $0 < \alpha \leqslant 1$, is generalized for higher values of α as follows. Let k be a non-negative integer and assume $k < \alpha \leqslant k+1$. A function f defined on a closed set F of $\mathbf{R}^n$ belongs to $\mathrm{Lip}(\alpha, F)$ if there exist bounded functions f_i, $i = (i_1, i_2, \ldots, i_n)$, $|i| = i_1+i_2+\ldots+i_n \leqslant k$, defined on F, with $f_0 = f$ such that

$$\Big| f_i(x) - \sum_{|i+j|\leqslant k} f_{i+j}(y)(x-y)^j \Big| \leqslant c|x-y|^{\alpha-|i|}; \qquad x, y \in F.$$

If the function $|x-y|^{\alpha-|i|}$ is replaced by $|x-y|^{\alpha-|i|}\log\dfrac{1}{|x-y|}$ for $|x-y|$ small, we write $f \in \overline{\mathrm{Lip}}(\alpha, F)$. We will use the following version of the Whitney extension theorem, [9]. Given a function in $\mathrm{Lip}(\alpha, F)$ (or $\overline{\mathrm{Lip}}(\alpha, F)$) it has an extension to $\mathrm{Lip}(\alpha, \mathbf{R}^n)$ (or $\overline{\mathrm{Lip}}(\alpha, \mathbf{R}^n)$).

Following the notation in [8], we say that a finite measure σ on $\mathbf{R}^n$ is *Tauberian* if its Fourier transform is a Tauberian function, i.e. for any $x \neq 0$ there exists $\lambda > 0$ such that $\hat{\sigma}(\lambda x) \neq 0$. For example, any real non-zero measure σ on the real line is Tauberian.

We introduce the following notation. A function f is *Lipschitz of order α, $\alpha > 0$, with respect to a finite measure σ at a point x_0*, and we write $f \in A_\alpha(\sigma, x_0)$, if the following conditions hold

(1) $f \in L^\infty(\mathbf{R}^n)$;

(2) $\int_{\mathbf{R}^n} (1+|x|^r)\, d|\sigma|(x) < \infty$;

(Here r is the smallest integer greater or equal than α, σ is a Tauberian measure and $|\sigma|$ stands for its total variation.)

(3) $\sigma_t * f(x_0+h) = \int_{\mathbf{R}^n} f(x_0+h-ty)\, d\sigma(y) = O(t^\alpha+|h|^\alpha)$, for all $t > 0$ and $h \in \mathbf{R}^n$.

The following decomposition theorem for Lipschitz functions will be an essential tool in the proof of Theorem 1.

THEOREM 2. *Suppose $f \in A_\alpha(\sigma, x)$ for every x in a set of finite measure E. Then given $\varepsilon > 0$ there exist a compact set F and a function g such that*

(1) *$F \subset E$, $|E-F| < \varepsilon$;*

(2) *$f-g = 0$ on F;*

(3) *$g \in \mathrm{Lip}(\alpha, \mathbf{R}^n)$ for α not integer;*

(4) *$g \in \overline{\mathrm{Lip}}(\alpha, \mathbf{R}^n)$ for α integer.*

§ 3. Auxiliary lemmas. The first lemma has its origin in [3] and it is a slight extension of Lemma 2 in [1]. The unpleasant presence of non necessarily Lebesgue measurable sets in the lemma inclined us to include its proof.

Set $\mu_t * f(x) = \sum_{i=1}^{m} A_i f(x-a_i t)$, where f is a bounded function defined in a neighborhood of a set $E \subset \mathbf{R}^n$. We assume that the condition on the set A is in force. The boundedness of the function f is assumed for simplicity, since our conditions on f will imply that f is bounded in the neighborhood of almost every point of the set E.

LEMMA 1. "Desymmetrization Lemma". *Suppose $\mu_t * f(x) = O(|t|^\alpha)$, $\alpha > 0$, for all $x \in E$. Then $\mu_t * f(x+h) = O(|t|^\alpha+|h|^\alpha)$ for almost every $x \in E$, and every small h, t.*

Proof. We may assume E to be contained in the unit cube I of $\mathbf{R}^n$, $a_i \neq a_j$ if $i \neq j$ and $A_1 \neq 0$, $a_1 \neq 0$. Let $F_i \subset F_{i+1} \ldots$, be a sequence of closed sets such that f restricted to F_i is continuous and $|I| = |H|$, $H = \bigcup_{i=1}^{\infty} F_i$.

Define for every positive integer k the following sets,

$$E_k = \{x \in E \cap H:\ |\mu_t * f(x)| < k|t|^\alpha,\ |t| \leqslant 1/k\},$$

$$\overline{E}_k = \{x \in E \cap H:\ |\mu_t * f(x)| < k|t|^\alpha \text{ if } |t| \leqslant 1/k \text{ and } x - a_i t \in H,\ i = 1, \ldots, m\}.$$

It is not difficult to see that the set $\overline{E}_k$ is measurable, for example the argument given in [4], Lemma 1 applies. Since

$$\sum_{i=1}^{m} A_i \mu_{ta_i} * f(x+a_i h) = \sum_{j=1}^{m} A_j \sum_{i=1}^{m} A_i f\big(x-a_i(a_j t-h)\big) = \sum_{j=1}^{m} A_j \mu_{ta_j-h} * f(x),$$

we have for every $x \in E_k$ and h, t small

$$\Big|\sum_{i=1}^{m} A_i \mu_{ta_i} * f(x+a_i h)\Big| \leqslant c(|t|^{\alpha}+|h|^{\alpha}),$$

c is a constant depending on k and the numbers $A_1, \ldots, A_m$.

Set $u_i = x+a_i h$, $i = 1, \ldots, m$. Then the above inequality can be restated as

$$\Big|\sum_{i=1}^{m} A_i \mu_{ta_i} * f(u_i)\Big| \leqslant c(|t|^{\alpha}+|u_1-u_2|^{\alpha}),$$

whenever $(u_1a_2-a_1u_2)/(a_2-a_1) \in E_k$, $|t|$ and $|u_1-u_2|$ small.

Fix k and t, t small, and let $u_0 \in \mathring{I}$ be a point of external density of E_k which is also a point of density of $\bar{E}_k$, as k tends to infinity these points u_0 cover E almost everywhere. Thus the lemma will follow if we prove: for every u_0 there exists $\varepsilon > 0$ such that for $|u_1-u_0| < \varepsilon$ and t small there exists u_2 with the following properties

$$\frac{u_1a_2-a_1u_2}{a_2-a_1} \in E_k,$$

$$u_i \in \bar{E}_k, \quad u_i - a_i a_j t \in H, \quad i = 2, \ldots, m,\ j = 1, \ldots, m,$$

$$|u_1-u_2| \leqslant |u_1-u_0|.$$

In fact, for $h = u_1-u_0$ we would have

$$|A_1\mu_{ta_1}*f(u_0+h)| \leqslant \Big|\sum_{i=1}^{m} A_i\mu_{ta_i}*f(u_i)\Big| + \sum_{i=2}^{m}|A_i||\mu_{ta_i}*f(u_i)|$$

$$\leqslant c(|t|^{\alpha}+|u_1-u_2|^{\alpha}) + \sum_{i=2}^{m}|A_i|k|a_i|^{\alpha}|t|^{\alpha} \leqslant c(|t|^{\alpha}+|h|^{\alpha}).$$

The constant c is not necessarily the same on each occurrence.

The existence of u_2 is provided by the following argument. Let F be a measurable set such that the measure of F is equal to the exterior measure of E_k, and u_0 a point of density of F. Then for small t and $u_1 \to u_0$ we have

$$\int\limits_{|u_1-u_2|\leqslant|u_1-u_0|} \chi_F\Big(\frac{u_1a_2-u_2a_1}{a_2-a_1}\Big)\prod_{i=2}^{m}\chi_{\bar{E}_k}(u_i)\prod_{j=1}^{m}\prod_{i=2}^{m}\chi_H(u_i-a_ia_jt)\,du_2$$

$$= \int\limits_{|u_1-u_2|\leqslant|u_1-u_0|} \chi_F\Big(\frac{u_1a_2-u_2a_1}{a_2-a_1}\Big)\prod_{i=1}^{m}\chi_{\bar{E}_k}(u_i)\,du_2 = (c_n+o(1))|u_1-u_0|^{n},$$

where c_n is a constant depending only on the dimension, and $o(1)$ tends to zero as $u_1 \to u_0$.

Remarks. (1) Suppose the function f is zero on E and $\mu_t*f(x) = O(|t|^{\alpha})$, $x \in E$. Then $f(x+h) = O(|h|^{\alpha})$ a.e. $x \in E$. Actually $O(|h|^{\alpha})$ can be replaced by $o(|h|^{\alpha})$, $h \to 0$, see pp. 251–252 [9].

(2) If the function $|t|^{\alpha}$ is replaced by $|t|^{\alpha}|\log|t||$ in the hypothesis of Lemma 1, then the same conclusion holds with $|t|^{\alpha}\log\frac{1}{|t|}+|h|^{\alpha}\log\frac{1}{|h|}$ instead of $|t|^{\alpha}+|h|^{\alpha}$.

LEMMA 2. *Let g be a Tauberian continuous complex valued function of one real variable, and K be a compact set in $\boldsymbol{R}^n$, $0 \notin K$. Then there exist $\lambda_{i,j} > 0$, complex numbers $a_{i,j}$ and smooth functions φ_j, $i = 1, \ldots, n$, $j = 1, \ldots, s$, such that $\sum_{j=1}^{s}\sum_{i=1}^{n} a_{i,j}\varphi_j(y)g(\lambda_{i,j}y_i) \neq 0$ for every $y = (y_1, \ldots, y_n) \in K$. The number s depends on the compact set K and the function g.*

Proof. For a given $x = (x_1, \ldots, x_n) \in K$ with $x_i \neq 0$, choose $\lambda_i = \lambda(x, i)$, $c_i = c(x, i)$ such that $\lambda_i > 0$, $|c_i| = 1$ and $\mathrm{Re}[c_ig(\lambda_ix_i)] > 0$. If $x_i = 0$, take $\lambda_i = 1$ and $c_i = 0$. Now for each $x \in K$ there exists $r(x) > 0$ such that $\mathrm{Re}\big[\sum_{i=1}^{n} c_ig(\lambda_iy_i)\big] > 0$ whenever $|y-x| < r(x)$. We cover K with a finite number of balls $B(x_k, r(x_k)/2)$ centered at $x_k \in K$ and radius $r(x_k)/2$, $k = 1, \ldots, s$.

To complete the proof of the lemma choose a nonnegative function φ in $C_0^{\infty}(\boldsymbol{R})$, infinitely differentiable with bounded support, which is one on $|x| \leqslant 1/2$ and zero on $|x| \geqslant 1$, and set $\varphi_k(y) = \varphi(|x_k-y|/r(x_k))$, $a_{i,k} = c(x_k, i)$ and $\lambda_{i,k} = \lambda(x_k, i)$.

For a function f defined on $\boldsymbol{R}^n$ let $f_t(x) = t^{-n}f(x/t)$ and $f_{(t)}(x) = f(tx)$. We will use the fact that $\hat{f}_t(x) = \hat{f}(tx)$, where $\hat{f}$ stands for the Fourier transform of f, whenever is defined. We denote by ψ a function such that $\hat{\psi} \in C_0^{\infty}(\boldsymbol{R}^n)$, the support of $\hat{\psi}$ is contained in $\{x\colon 1 < 2|x| \leqslant 4\}$ and $\sum_{j=-\infty}^{\infty}\hat{\psi}_{(2^j)}(x) = 1$ for $x \neq 0$. It is well known that such a function exists.

Recall that if f is a Lipschitz function of order α with respect to ψ at a point $x_0 \in \boldsymbol{R}^n$, we have the following condition

$$\text{(L)} \qquad \psi_t*f(x_0+h) = O(t^{\alpha}+|h|^{\alpha}), \quad t > 0,\ h \in \boldsymbol{R}^n.$$

Condition (L) allows us to prove the following pointwise version of Theorem (2.1) in [6], see also [8].

LEMMA 3. *Let $\varphi \in C_0^{\infty}(\boldsymbol{R}^n)$ and let P be an homogeneous polynomial of degree r, set $\hat{K} = P\varphi$, and assume* (L) *is in force. Then*

(i) $K_t*f(x_0+h) = O(t^{\alpha}+|h|^{\alpha})$, $t > 0$, $h \in \boldsymbol{R}^n$, $0 < \alpha < r$;

(ii) $K_t*f(x_0+h) = O\Big(t^{\alpha}\log\frac{1}{t}+|h|^{\alpha}\Big)$, *$t$ small, $h \in \boldsymbol{R}^n$ and $\alpha = r$.*

Proof. Let $\Phi \in C_0^\infty(\boldsymbol{R}^n)$ be one on $|x| \leqslant 1/2$ and zero on $|x| \geqslant 1$, and set $h = P\Phi$. If we assume that the function φ in the lemma is zero on $|x| \geqslant 1/2$, then $\hat{K} = P\varphi = h\varphi$. Now the $C_0^\infty(\boldsymbol{R}^n)$ function $\hat{\varrho}$ defined by $\hat{\varrho}(x) = h(x) - 2^{-r}h(2x)$ vanishes for either $|x| \geqslant 1$ or $|x| \leqslant 1/4$. Therefore $\hat{\varrho} = (\hat{\psi} + \hat{\psi}_{(2)} + \hat{\psi}_{(4)})\hat{\varrho}$, recall that $\sum_{j=-\infty}^{\infty} \psi_{(2^j)}(x) = 1$ for $x \neq 0$.

Since $h = \sum_{k=0}^{\infty} 2^{-kr}\hat{\varrho}_{(2^k)}$, we have the following representation of $\hat{K}$

$$\hat{K} = \varphi \sum_{k=0}^{\infty} 2^{-kr}(\hat{\psi}_{(2^k)} + \hat{\psi}_{(2^{k+1})} + \hat{\psi}_{(2^{k+2})})\hat{\varrho}_{(2^k)}.$$

For $f \in L^\infty(\boldsymbol{R}^n)$ the next equality holds true for every $x \in \boldsymbol{R}^n$.

$$K_t * f(x) = \sum_{k=0}^{\infty} 2^{-kr}(\psi_{2^k t} + \psi_{2^{k+1}t} + \psi_{2^{k+4}}) * \varrho_{2^k t} * \varphi_t * f(x).$$

If $0 < \alpha < r$ a straightforward calculation shows that $K_t * f(x_0 + h) = O(t^\alpha + |h|^\alpha)$. In order to obtain the logarithmic estimate use $|\psi_t * f(x_0 + h)| \leqslant c \min(1, t^r + |h|^r)$, and the lemma follows.

Set $u(x, y) = \varphi_y * f(x)$ where φ stands for any function in $\mathscr{S}$, the smooth rapidly decreasing functions, and assume $\hat{\varphi}(0) \neq 0$, $f \in L^\infty(\boldsymbol{R}^n)$. For $\alpha > 0$, $k < \alpha \leqslant k+1$ we define

$$L_\alpha(x, y) = \max_{|i|=k+1} |y^{k+1-\alpha} u_i(x, y)|,$$

$$\bar{L}_\alpha(x, y) = \max_{|i|=k+1} |u_i(x, y) \log^{-1} y|,$$

u_i, $i = (i_0, i_1, \ldots, i_n)$ denotes the partial derivatives of u with respect to the $n+1$ variables $y, x_1, x_2, \ldots, x_n$. We assume familiarity with the concept of a function being nontangentially bounded on a set, see p. 201 [9].

LEMMA 4. *Suppose L_α is nontangentially bounded on a set E of finite measure. Then given $\varepsilon > 0$ there exist a compact set F and a function $g \in \mathrm{Lip}(\alpha, \boldsymbol{R}^n)$ such that $|E - F| < \varepsilon$ and $f - g$ vanishes on F. If L_α is replaced by $\bar{L}_\alpha$ in the hypothesis, the conclusion also holds with $\overline{\mathrm{Lip}}(\alpha, \boldsymbol{R}^n)$ in place of $\mathrm{Lip}(\alpha, \boldsymbol{R}^n)$.*

Proof. We consider the case L_α nontangentially bounded on E. A parallel argument gives the desired result if $\bar{L}_\alpha$ is nontangentially bounded.

By an uniformization procedure, given $\varepsilon > 0$ there exist a set $F \subset E$, $|E - F| < \varepsilon$ and a constant c such that $|u_i(x, y)| \leqslant c y^{\alpha-k-1}$ if $|i| = k+1$ and $(x, y) \in \Gamma(x_0)$, $x_0 \in F$, where $\Gamma(x_0)$ is the cone $\{(x, y) \in \boldsymbol{R}_+^{n+1}: |x - x_0| < y < 1\}$. This uniformization can be founded in p. 201 [9]. Now the integral form of the mean value theorem yields $u_i \in \mathrm{Lip}(\alpha - k, \Gamma(x_0))$ for $|i| = k$, with a Lipschitz constant independent of $x_0 \in F$.

We denote by $\bar{x}, \bar{y}, \bar{z}$ variables in $\boldsymbol{R}^{n+1}$ and set

$$R_i(\bar{y}, \bar{x}) = u_i(\bar{y}) - \sum_{|i+j| \leqslant k} u_{i+j}(\bar{x}) \frac{(\bar{y} - \bar{x})^j}{j!},$$

then we have the following formulae for Taylor's expansion

$$(1) \qquad R_i(\bar{x}, \bar{y}) = \sum_{|i+j|=k} \int_0^1 R_{i+j}(\bar{y} + s(\bar{x} - \bar{y}), \bar{y}) \frac{(\bar{x} - \bar{y})^j}{j!} (1-s)^{k-|i|-1} ds, \quad |i| \leqslant k-1;$$

$$(2) \qquad R_i(\bar{y}, \bar{z}) - R_i(\bar{y}, \bar{x}) = \sum_{|i+j| \leqslant k} R_{i+j}(\bar{x}, \bar{z}) \frac{(\bar{y} - \bar{x})^j}{j!}, \qquad |i| \leqslant k.$$

Formula (1) is a reformulation of the elementary identity

$$f(x+t) - \sum_{k=0}^{r} f^{(k)}(x) \frac{t^k}{k!} = \frac{1}{(r-1)!} \int_x^{x+t} [f^{(r)}(y) - f^{(r)}(x)](x+t-y)^{r-1} dy,$$

and a proof of (2) can be founded in p. 177 [9].

An iterated use of (1) shows that $u \in \mathrm{Lip}(\alpha, \Gamma(x_0))$ with a Lipschitz bound independent of $x_0 \in F$. In order to see that $u \in \mathrm{Lip}(\alpha, \bigcup_{x_0 \in F} \Gamma(x_0))$, apply (2) to the triple $\bar{x}, \bar{y}, \bar{z}$ where $\bar{z} \in \Gamma(x_0) \cap \Gamma(x_1)$ if $\bar{x} \in \Gamma(x_0)$ and $\bar{y} \in \Gamma(x_1)$, $x_0, x_1 \in F$.

If we call U an extension of u to $\mathrm{Lip}(\alpha, \boldsymbol{R}^{n+1})$, the lemma follows with $g(x) = U(x, 0)$ and a possible smaller set F. In fact, since $\varphi \in \mathscr{S}$, $\hat{\varphi}(0) \neq 0$, the function $f(x)$ can be recaptured as the limit of $u(x, y) = \varphi_y * f(x)$, $y \to 0$ for almost every x.

We use the following result from Harmonic Analysis. Let $\hat{M}(\boldsymbol{R}^n)$ be the image under the Fourier transform of $M(\boldsymbol{R}^n)$, i.e. the finite measures on $\boldsymbol{R}^n$.

WIENER'S LEMMA. *Let $\hat{\mu}$ anad $\hat{\nu}$ belong to $\hat{M}(\boldsymbol{R}^n)$ and assume*

(i) $\hat{\mu}$ has support in a compact set K;

(ii) $|\hat{\nu}(x)| > 0$ in K.

Then $\hat{\mu}\hat{\nu}^{-1} \in \hat{M}(\boldsymbol{R}^n)$ (in fact, $\hat{\mu}\hat{\nu}^{-1} \in \hat{L}^1(\boldsymbol{R}^n)$).

We refer the reader to [7], or for a less abstract setting to [12]. Although a formulation for the periodic case is given in [12], the arguments can be easily adapted to $M(\boldsymbol{R}^n)$, as is done in [6].

§ 4. Proof of Theorem 1 and Theorem 2. First we show how Theorem 2 can be used to prove Theorem 1.

Let $f \in A_\alpha(x)$, $x \in E$, and $A = \{A_1, \ldots, A_m, a_1, \ldots, a_m\}$ be the set of numbers used in the definition of the class $A_\alpha(x)$.

Set $\mu^j(x) = \sum_{i=1}^m A_i \delta_{a_i e_j}(x)$ and $g(x) = \sum_{j=1}^m A_j e^{-2\pi i a_j s}$, where $\{e_1, \ldots, e_n\}$ is the standard orthonormal basis for $\boldsymbol{R}^n$. Then $\hat{\mu}^j(x) = g(x_j)$. Since the measure μ^j is not identically zero, the function g is Tauberian on the real line.

Next apply Lemma 3 to g and the compact set, say, $K = \{x: 1 \leqslant |x| \leqslant 2\}$. Thus there exists a Tauberian measure $\sigma \in M(\boldsymbol{R}^n)$ defined by $\hat{\sigma}(y) = \sum_{j=1}^r \sum_{i=1}^n a_{i,j}\varphi_j(y) g(\lambda_{i,j} y_i)$. If we write Φ^j for $\widehat{\Phi^j} = \varphi_j$, we have

$$\sigma_t * f(x) = \sum_{j=1}^r \sum_{i=1}^n a_{i,j} \Phi_t^j * \mu^j_{t\lambda_{i,j}} * f(x).$$

Now, Lemma 1 implies

$$f * \mu_t^j(x+h) = O(|h|^\alpha + t^\alpha), \quad h \in \boldsymbol{R}^n,\ t > 0,$$

a.e. $x \in E$, and $j = 1, \ldots, n$. Thus $\sigma_t * f(x+h) = O(t^\alpha + |h|^\alpha)$, a.e. $x \in E$, and we have proved $f \in A_\alpha(\sigma, x)$ a.e. $x \in E$. Note that the smoothness condition of σ required in the definition of $A_\alpha(\sigma, x)$, it is clearly satisfied for any value of α.

Finally for Theorem 2, we can write $f = g + b$, where $g \in \operatorname{Lip}(\alpha, \boldsymbol{R}^n)$ or $g \in \overline{\operatorname{Lip}}(\alpha, \boldsymbol{R}^n)$ for α not integer or α integer, respectively, and $b = 0$ on a set F of measure close to E. Theorem 1 follows applying Remark of Lemma 1 to the function b.

We now proceed to prove Theorem 2.

Assume α not integer, $k < \alpha < k+1$. The case α integer is treated in a similar way. Since σ is a Tauberian finite measure, there exist $\varphi_j \in C_0^\infty(\boldsymbol{R}^n)$ and $\lambda_j > 0$ such that $\sum_{j=1}^l \varphi_j(x)\hat{\sigma}(\lambda_j x)$ does not vanish on a given compact set $K \subset \boldsymbol{R}^n$, $0 \notin K$, see Lemma 2. If we take as K the support of the function $\hat{\psi}$ in Lemma 3, then, by Wiener's Lemma, there exists $h \in L^1(\boldsymbol{R}^n)$ such that $\hat{\psi}(x) = \hat{h}(x) \sum_{j=1}^l \varphi_j \hat{\sigma}(\lambda_j x)$. Furthermore it is not difficult to see that $\int (1+|x|)^{k+1} d|\sigma|(x) < \infty$ implies $\int (1+|x|)^{k+1} |h(x)|\, dx < \infty$. Indeed, the function $\hat{\sigma}$ has derivatives up to the $k+1$ order and so does $\hat{h}$. Thus, Wiener's Lemma assures that each of these derivatives of $\hat{h}$ belongs to $\widehat{L^1}(\boldsymbol{R}^n)$, i.e. h has finite momenta up to the order $k+1$.

For a fixed $x_0 \in E$, we have by hypothesis $\sigma_t * f(x_0 + h) = O(t^\alpha + |h|^\alpha)$, then taking in account that $\varphi_j \in C_0^\infty(\boldsymbol{R}^n)$ and the smoothness condition on $\hat{h}$, a basic calculation shows $\psi_t * f(x_0 + h) = O(t^\alpha + |h|^\alpha)$.

Now take any $\varphi \in \mathscr{S}$, $\hat{\varphi} \in C_0^\infty(\boldsymbol{R}^n)$, $\hat{\varphi}(0) \neq 0$, and set as before $u(x, y) = \varphi_y * f(x)$. Then for $i = (i_0, i_1, \ldots, i_n)$, $|i| = i_0 + i_1 + \ldots + i_n = k+1$, we have $y^{k+1} u_i(x, y) = K_y^i * f(x)$, where $\widehat{K^i} = P_i \varphi_i$, P_i an homogeneous polynomial of degree $k+1$ and $\varphi_i \in C_0^\infty(\boldsymbol{R}^n)$. Thus a use of Lemma 3 for these K^i shows, $K_t^i * f(x_0 + h) = O(t^\alpha + |h|^\alpha)$, in particular

$$|t^{k+1-\alpha} u_i(x_0 + h, t)| \leqslant M, \qquad |h| \leqslant t,$$

where the constant M depends on $x_0 \in E$. Therefore the function L_α in Lemma 4 is nontangentially bounded, and a use of this lemma completes the proof of Theorem 2 if α is not integer.

§ 5. Remarks. Theorem 1 gives the non-trivial implication in a characterization of the spaces $T_\alpha(x)$ when α is not integer.

(5.1) Suppose $k < \alpha < k+1$, and assume for the set of numbers defining $A_\alpha(x)$ the conditions $\sum_{j=1}^m a_j^i A_j = 0$, $i = 0, \ldots, k$. Then the following two conditions are equivalent:

(1) $\quad f \in T_\alpha(x)$, a.e. $x \in F$;

(2) $\quad f \in A_\alpha(x)$, a.e. $x \in E$.

We also observe that in the case $f \in A_k(x)$, k integer, Theorem 1 can be notably improved if a single condition is required on the set of numbers defining $A_k(x)$. We recall the definition of the kth Peano derivative at a point x_0, and we write as it is customary $f \in t_k(x_0)$, if there exists a polynomial P of degree at most k such that $f(x_0 + t) - P(t) = o(|t|^k)$ as $t \to 0$.

(5.2) Suppose $f \in A_k(x)$, $x \in E$, and $\sum_{i=1}^m A_i a_i^k \neq 0$, then $f \in t_k(x)$ a.e. $x \in E$. Compare with [1].

We sketch the proof of this remark.

For $\varphi(x) = e^{-|x|^2}$, say, set $u(x, y) = \varphi_y * f(x)$, then it can be seen that

(j) $$\sum_{j=1}^m A_j a_j^k u_i(x + a_j h, a_j y) = O_x(1)$$

for every $x \in E$, $|h| \leqslant y$ and $|i| = i_0 + i_1 + \ldots + i_n = k$.

Note that in proving (j), we can assume a_j, $0 < a_j < 1$ and still with the required condition in (5.2), apply Lemma 1.

Since $u_i(x, y)$ is analytic in $y > 0$, condition (j) and $\sum_{i=1}^m A_i a_i^k \neq 0$ imply

(jj) $$u_i(x, y) = O_x(1), \quad x \in E,\ y > 0 \text{ and } |i| = k.$$

But (j) and (jj) are the required conditions of the desymmetrization lemma in [9], thus u_i, $|i| = k$ are nontangentially bounded a.e. on E. Now Lemma 4 applies, and we can write $f = g+b$, $g \in \mathrm{Lip}(k, \boldsymbol{R}^n)$ and b vanishes on a set F of measure close to E. That $g \in t_k(x)$ a.e. $x \in \boldsymbol{R}^n$ can be seen in [2]. For the bad part b, we have $b(x+t) = O(|t|^k)$ a.e. $x \in F$, which again can be improved to $b(x+t) = o(|t|^k)$ a.e. $x \in F$, this time a basic argument on density points applies [9].

(5.3) The L^p counterpart in (5.2) is also true. The proof uses Theorem 2 [9], p. 248 in all its depth, i.e. the equivalences for non tangentially boundedness for conjugate harmonic functions. Could be of some interest to prove this result outside the framework of harmonic analysis as in (5.2).

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Basic sequences in a stable finite type power series space

by

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Abstract. A characterization is given of when a nuclear Fréchet space with basis is isomorphic to the subspace generated by a basic sequence in a stable finite type power series space. The characterization is in terms of an inequality very similar to the one obtained for basic sequences in (s) and a nuclearity condition. Several structural facts are obtained as applications of the main result.

In [7] and [8] we characterized, respectively, subspaces and quotient spaces (with basis) of the infinite type power series space (s). An interesting feature of this characterization is that it is done in terms of inequalities (type (d_3), (d_4) below) and the only difference between subspaces and quotient spaces is the sense of the inequality. Recently, Alpseymen [1] considered the case of a stable infinite type power series space $\Lambda_\infty(\alpha)$ and determined, for the characterization of subspaces, that the same inequality works with the additional requirement that the space be $\Lambda_{\boldsymbol{N}}(\alpha)$-nuclear in the sense of Ramanujan and Terzioglu [13].

In this paper we turn to finite type power series spaces. It turns out that subspaces with bases can again be characterized in terms of two kinds of conditions: an inequality and a stronger type of nuclearity. The inequality (type (d_5) below) is only slightly different from the one obtained for basic sequences in (s) [7] and the nuclearity condition is $\Lambda_1(\alpha)$-nuclearity as studied by Robinson [15].

Our results on subspaces and quotient spaces have been extended by Vogt and Wagner [16], [17] to eliminate the requirement of a basis. So far, this has only been done for subspaces and quotient spaces of (s).

We apply our main theorem to obtain several results about the structure of nuclear Köthe spaces. We are able to completely describe all power series subspaces of any stable power series space. We describe all $L_f(b, r)$ subspaces of a stable finite type power series space and obtain some new information in the absence of the stability assumption. Finally we obtain the interesting fact that the only type (d_2) subspace of a finite type power series space is a (finite type) power series space.

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