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Pointlike immersions of 2-manifolds

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Abstract. In [2] Vacarro gives an example of an immersion of S^2 in S^3 whose complement is an open 3-cell. The following will generalize this result to all compact 2-manifolds.

We will denote a compact 2-manifold (orientable or not) by M^2. By an immersion we will mean a local homeomorphism from M^2 into S^3. An immersion f will be called *pointlike* if $S^3 - f(M^2)$ is an open 3-cell.

THEOREM. *If M^2 is a 2-manifold, then there exists a pointlike immersion of M^2 into S^3.*

Proof. We will first consider the case when M is a 2-sphere. The immersion will essentially be Bing's "house with two rooms" [1]. Let $X = X_1 \cup X_2 \cup X_3 \cup X_4$ be the 2-dimensional complex (Fig. 1) consisting of

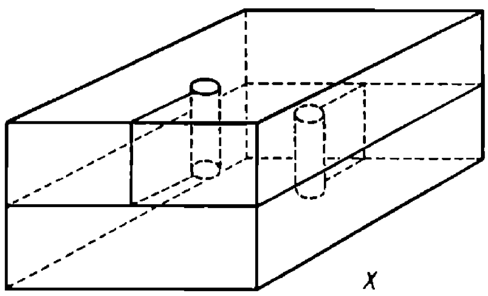

Fig. 1

$$X_1 = \{\text{boundary of the 3-cube } \{(x, y, z)|\ |x| \leqslant 1,\ |y| \leqslant 1,\ |z| \leqslant 1\}\} - \{(x, y, z)|\ z = 1,\ (x+\tfrac{1}{2})^2+y^2 < \tfrac{1}{16}\} \cup \{(x, y, z)|\ z = -1,\ (x-\tfrac{1}{2})^2+y^2 < \tfrac{1}{16}\},$$

$$X_2 = \{(x, y, z)|\ (x+\tfrac{1}{2})^2+y^2 = \tfrac{1}{16},\ 0 \leqslant z \leqslant 1\} \cup \{(x, y, z)|\ (x-\tfrac{1}{2})^2+y^2 = \tfrac{1}{16},\ -1 \leqslant z \leqslant 0\},$$

$$X_3 = \{(x, y, z)|\ |x| \leqslant 1,\ |y| \leqslant 1,\ z = 0\} - \{(x, y, z)|\ (x+\tfrac{1}{2})^2+y^2 < \tfrac{1}{16},\ z = 0\} \cup \{(x, y, z)|\ (x-\tfrac{1}{2})^2+y^2 < \tfrac{1}{16},\ z = 0\},$$

$$X_4 = \{(x, y, z)|\ x = 0,\ -\tfrac{3}{4} \geqslant y \geqslant -1, 0 \leqslant z \leqslant 1\} \cup \{(x, y, z)|\ x = 0,\ \tfrac{3}{4} \leqslant y \leqslant 1, 0 \geqslant z \geqslant -1\}.$$

Let S^2 be the unit 2-sphere broken into 5 regions Y_i, where $Y_i = \{(\varrho, \theta, \varphi)|\ \varrho = 1,\ 0 \leqslant \theta \leqslant 2\pi,\ (i-1)\tfrac{1}{5}\pi \leqslant \varphi \leqslant i\tfrac{1}{5}\pi\}$, Fig. 2. Define $f\colon S^2 \to X$ by the following four local homeomorphisms:

$f_1\colon Y_3 \to X_1$,

$f_2\colon Y_2 \cup Y_4 \to X_2$,

$f_3\colon Y_1 \to$ "lower room" of X,

$f_4\colon Y_5 \to$ "upper room" of X.

It should be clear that the complement of X is just on open 3-cel (a point).

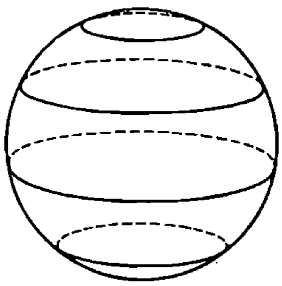

Fig. 2

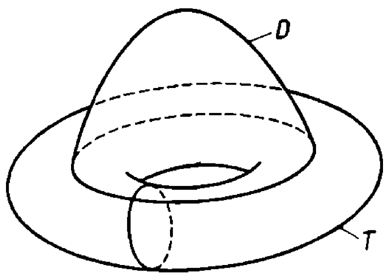

Fig. 3

To show the result for a torus, let T be an unknotted torus in S^3 with an attached 3-cell, $D^2 \times I$, such that $T \cap (D^2 \times I) =$ (boundary $D^2) \times I$ and the intersection is homotopic to a longitude (Fig. 3). Next, build a copy of X in $D^2 \times I$ so that (boundary $D^2) \times I$ corresponds to $\{(x, y, z)|\ |x| \leqslant 1, |y| \leqslant 1, z = \pm 1$, or $x = \pm 1, |y| \leqslant 1, |z| \leqslant 1\}$ in X. Now define a map f: (boundary of $D^2 \times I \cap T -$ copy $X)|$ onto the copy of X in the manner of f_2, f_3, and f_4. By repeating this construction on a latitude curve, we then have a local homeomorphism of T onto a complex whose complement is two open 3-cells. Then, as in the construction of the sphere case, we can obtain the desired map.

The case where M^2 is a Torus can be shown by constructing $2n+1$ copies of X in the manner of the Torus.

Finally, the non-orientable case follows in that all non-orientable surfaces can be immersed in S^3, and our Torus construction will reduce the complement to a set of open 3-cells.

References

[1] R. M. Bing, *Some aspects of the topology of 3-manifolds related to the Poincaré conjecture,* Lectures on Modern Mathematics, T. L. Saaty, Editor (Published by John Wiley and Sons, Inc., 1964).
[2] M. Vaccaro, *Proprieta topologiche delle cappre sentazioni localmente biumivoche,* Math. Ann. 133 (1957), p. 173–184.

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