ANNALES
POLONICI MATHEMATICI
XXXVII (1980)

Antisymmetric operator algebras, II

by WACŁAW SZYMAŃSKI (Kraków)

Abstract. By an analogy to antisymmetric sets for function algebras we introduce antisymmetric projections for subspaces of operators in a Hilbert space. We prove an antisymmetric decomposition theorem which may be regarded as a non-commutative generalization of the known Bishop decomposition theorem for function algebras. Some sufficient conditions for the decomposition of an operator algebra on antisymmetric parts are given. Finally, we consider some examples and applications of previous results to algebras of normal operators and to dilatable representations of function algebras.

1. PRELIMINARIES

1.1. Introduction. Problems of the antisymmetry in the function algebras theory were studied by many authors (see [4], [5]). In particular, Bishop proved a theorem on the decomposition of a function algebra on antisymmetric parts (Theorem 13.1, Chapter II, [4]). The investigation of antisymmetric subspaces and algebras by using Banach algebra and operator-theoretic methods has been initiated in [10]. Independently, Conway and Olin in [1], [2] have studied ultraweakly closed antisymmetric algebras generated by one subnormal operator. They used mostly the function algebra technique in the proof of an antisymmetric decomposition theorem, which is one of the main theorems of their works. In the present paper we introduce, by an analogy to antisymmetric sets for function algebras, antisymmetric projections associated with a given linear subspace of operators in a complex Hilbert space.

In Section 2 we prove basic properties of antisymmetric and maximal antisymmetric projections. The antisymmetric decomposition theorem is also proved.

We begin Section 3 with a simple corollary, that every reflexive operator algebra, whose invariant projections commute, admits the antisymmetric decomposition. Next we prove a sufficient condition for the existence of the antisymmetric decomposition of a reflexive operator algebra.

We conclude this section showing, how an antisymmetric decomposition theorem of Conway and Olin ([1], Theorem 3.1) follows from some theorems of the present paper.

In Section 4 some examples illustrating previous theorems are given. Moreover, we show the correspondence between maximal antisymmetric sets for the uniform closure of the algebra of polynomials on a compact subset X of the complex plane and maximal antisymmetric projections for the norm closure of the algebra of polynomials in one normal operator, whose spectrum is X.

Finally, we prove a theorem on antisymmetric projections for the range of a dilatable representation of a function algebra. The geometry of projections in von Neumann algebras and some methods of the unstarred operator algebras theory are main tools used in this paper. We refer to [11] for the theory of von Neumann algebras and to [8] for unstarred operator algebras.

1.2. Notations and definitions. Let H be a complex Hilbert space. $L(H)$ denotes the algebra of all linear bounded operators in H, I_H (or simply I) stands for the identity in H. By a subspace of H we mean always a closed subspace and by a projection we mean an operator $E = E^2 = E^* \in L(H)$. If E is a projection, then every projection $F \in L(H)$ such that $EF = FE = F$ (equivalently: $F \leqslant E$) is called a *subprojection* of E. For two projections E, $F \in L(H)$ we denote by $E \wedge F$, $E \vee F$ the projections onto $EH \cap FH$ and onto the closure of $\{Ex + Fy \colon x, y \in H\}$, respectively. A subspace $M \subset H$ is *invariant* under a set $\mathscr{S} \subset L(H)$ if every operator $T \in \mathscr{S}$ leaves M invariant. Projections onto invariant subspaces for $\mathscr{S}$ are called *invariant projections*. $\mathrm{Lat}\,\mathscr{S}$ denotes the lattice of all invariant projections for $\mathscr{S}$ with the lattice operations $\wedge$, $\vee$. $\mathscr{S}'$ stands for the commutant of $\mathscr{S}$ and $W^*(\mathscr{S})$ is the von Neumann algebra generated by $\mathscr{S}$ and I. If $E \in L(H)$ is a projection, then $\mathscr{S}_E = \{ES|_{EH} \colon S \in \mathscr{S}\}$ is a subset of $L(EH)$. If $\mathscr{A} \subset L(H)$ is an algebra with I, then its weak and strong operator closures coincide ([8], Corollary 7.2). If $T \in L(H)$, then $\mathrm{Lat}\,T$ denotes $\mathrm{Lat}\{T\}$, $\mathscr{A}(T)$ stands for the commutative algebra of all polynomials in T, $\overline{\mathscr{A}(T)}$ and $\overline{\mathscr{A}(T)}^s$ denote its norm and strong closures, resp. Let $\mathscr{F} \subset L(H)$ be a family of projections. The set $\mathrm{Alg}\,\mathscr{F} = \{T \in L(H) \colon \mathscr{F} \subset \mathrm{Lat}\,T\}$ is a weakly closed algebra with I. The inclusion $\mathscr{A} \subset \mathrm{Alg}\,\mathrm{Lat}\,\mathscr{A}$ is clear for any algebra $\mathscr{A} \subset L(H)$. A weakly closed algebra $\mathscr{A} \subset L(H)$ is called *reflexive* if $\mathscr{A} = \mathrm{Alg}\,\mathrm{Lat}\,\mathscr{A}$. A family of projections $\mathscr{F} \subset L(H)$ will be called a *net* if $E \vee F \in \mathscr{F}$, whenever $E, F \in \mathscr{F}$. Let now $\mathscr{R}$ be a von Neumann algebra with I. For two projections $E, F \in \mathscr{R}$ we write $E \sim F \pmod{\mathscr{R}}$ if there is a partial isometry $U \in \mathscr{R}$ such that $U^*U = E$, $UU^* = F$. We use the symbol $E \precsim F \pmod{\mathscr{R}}$ if there is a projection $E_1 \in \mathscr{R}$ such that $E_1 \sim E \pmod{\mathscr{R}}$ and $E_1 \leqslant F$.

2. ANTISYMMETRIC PROJECTIONS

2.1. Definitions and basic properties. Let us fix an arbitrary linear subspace $\mathscr{S}$ of $L(H)$. We assume that $I \in \mathscr{S}$ and we do not assume that $\mathscr{S}$ is closed in any topology. Recall from [10] that $\mathscr{S}$ is called *antisymmetric* if the only self-adjoint elements of $\mathscr{S}$ are real multiples of I. Define $\mathscr{R} = W^*(\mathscr{S})$.

DEFINITION. A projection $E \in \mathscr{S}'$ is called *antisymmetric* for $\mathscr{S}$ if, for each $T \in \mathscr{S}$, $TE = T^*E$ implies $TE = rE$ with some real r.

Comparing this definition with the definition of an antisymmetric subspace we see, that $\mathscr{S}$ is an antisymmetric subspace if and only if I is an antisymmetric projection for $\mathscr{S}$. Moreover, a projection $E \in \mathscr{S}'$ (equivalently, $E \in \mathscr{R}'$) is antisymmetric for $\mathscr{S}$ if and only if $\mathscr{S}_E$ is an antisymmetric subspace of $L(EH)$. Examples of antisymmetric projections will be given in Section 4. Observe that if $F \in \mathscr{S}$ is a subprojection of an antisymmetric projection E for $\mathscr{S}$, then either $F = 0$ or $F = E$. Hence if an antisymmetric projection for $\mathscr{S}$ belongs to $\mathscr{S}$, then it is a minimal projection in $\mathscr{S}$. It has been observed in [10] (Proposition 1) that a weakly closed algebra $\mathscr{A} \subset L(H)$ with I is antisymmetric if and only if $\mathscr{A}$ contains no projections except 0 and I.

PROPOSITION 1. *Suppose that $E \in \mathscr{R}'$ is an antisymmetric projection for $\mathscr{S}$. If $F \in \mathscr{R}'$ is a projection such that $E \sim F(\mathrm{mod}\,\mathscr{R}')$, then F is antisymmetric for $\mathscr{S}$. Moreover, for $T \in \mathscr{R}$ and for each complex z: $TE = zE$ if and only if $TF = zF$.*

Proof. Let $U \in \mathscr{R}'$ be a partial isometry such that $U^*U = E$, $UU^* = F$. We have then $UE = U$, $U^*F = U^*$ ([6], Problem 98). Suppose that $T \in \mathscr{S}$ satisfies $(T-T^*)F = 0$. This implies $U^*(T-T^*)U = U^*(T-T^*)FU = 0$, but $U \in \mathscr{R}'$, hence $(T-T^*)E = 0$. By the antisymmetry of E, there is a real r such that $(T-rI)E = 0$. We have now: $U(T-rI)U^* = U(T-rI)EU^* = 0$ and using again the commutativity of U with $\mathscr{R}$ we infer that $TF = rF$. To prove the remaining assertions we argue similarily and the proof is complete.

PROPOSITION 2. *Let $E \in \mathscr{R}'$ be a non-zero projection. If $F, G \in \mathscr{R}'$ are two antisymmetric projections for $\mathscr{S}$ such that $E \lesssim F$, $E \lesssim G(\mathrm{mod}\,\mathscr{R}')$, then $F \vee G$ is antisymmetric for $\mathscr{S}$.*

Proof. Take $T \in \mathscr{S}$ such that $T(F\vee G) = T^*(F \vee G)$. Hence $TF = T^*F$, $TG = T^*G$ and by the antisymmetry of F, G we find two reals r, s, such that $TF = rF$, $TG = sG$. By the assumptions, there are projections $E_1, E_2 \in \mathscr{R}'$ satisfying $E_1 \sim E \sim E_2(\mathrm{mod}\,\mathscr{R}')$, $E_1 \lesssim F$, $E_2 \lesssim G$. Hence we have $TE_1 = rE_1$, $TE_2 = sE_2$ and, by Proposition 1, $TE = rE = sE$; but $E \neq 0$, hence $r = s$, thus $FT = rF$, $TG = rG$. Consequently, $T(F \vee G) = r(F \vee G)$ and the proof is finished.

2.2. Maximal antisymmetric projections and the antisymmetric decomposition. We will say that an antisymmetric projection for $\mathscr{S}$ is *maximal* if for every antisymmetric projection G for $\mathscr{S}$, $F \leqslant G$ implies $F = G$. For every antisymmetric projection E for $\mathscr{S}$ there is a maximal antisymmetric projection F for $\mathscr{S}$ such that $E \leqslant F$. To see this, put $F = \mathrm{LUB}\mathscr{F}$, where $\mathscr{F} = \{G \in \mathscr{R}' : G$ is an antisymmetric projection for $\mathscr{S}$, $E \leqslant G\}$. Since $\mathscr{F}$ is a net (Proposition 2), F is the strong limit of elements of $\mathscr{F}$ ([11], p. 7) and one easily proves that $F \in \mathscr{F}$. Obviously, F is a maximal antisymmetric projection for $\mathscr{S}$ and $E \leqslant F$. Denote by $\mathfrak{M}$ the set of all maximal antisymmetric projections for $\mathscr{S}$.

PROPOSITION 3. *$\mathfrak{M} \subset \mathscr{R} \cap \mathscr{R}'$ and elements of $\mathfrak{M}$ are mutually orthogonal.*

Proof. To prove that $\mathfrak{M} \subset \mathscr{R} \cap \mathscr{R}'$, take $E \in \mathfrak{M}$, $E \neq 0$ and a unitary operator $U \in \mathscr{R}'$. By Proposition 1, the projection $F = UEU^*$ is antisymmetric for $\mathscr{S}$ and one easily cheks that $F \in \mathfrak{M}$. By the double commutant theorem, it is sufficient to prove $F = E$. Since $E \sim F (\mathrm{mod}\, \mathscr{R}')$, $E \vee F$ is an antisymmetric projection for $\mathscr{S}$, by Proposition 2. Thus $E = E \vee F = F$, because $E, F \in \mathfrak{M}$. To prove the second assertion, assume that $E, F \in \mathfrak{M}$ and $EF \neq 0$. Since E, F are central in $\mathscr{R}$, they commute and EF is a non-zero projection in $\mathscr{R}'$ such that $EF \leqslant E$, $EF \leqslant F$. Now our proposition follows from Proposition 2.

Define now two sets of projections:

$$\mathscr{E} = \{E \in \mathscr{R}' : E \textit{ is a projection and there is an antisymmetric projection } G \textit{ for } \mathscr{S} \textit{ such that } E \leqslant G\},$$

$$\mathfrak{N} = \{P \in \mathscr{R}' : P \textit{ is a projection, } P \textit{ has no non-zero subprojection which belongs to } \mathscr{E}\}.$$

The family $\mathfrak{N}$ is hereditary in the sense of [9], i.e. if $P \in \mathfrak{N}$ and $Q \in \mathscr{R}'$ is a subprojection of P, then $Q \in \mathfrak{N}$. Define $E_0 = \bigoplus_{F \in \mathfrak{M}} F = \mathrm{LUB}\,\mathscr{E}$ (the last equality verifies easily) and $E_1 = I - E_0$. It is immediate, that $E_1 \in \mathfrak{N}$, moreover, $E_1 = \mathrm{LUB}\,\mathfrak{N}$. Indeed, take $P \in \mathfrak{N}$ and $F \in \mathfrak{M}$. Since PF is a projection, PF belongs to $\mathscr{E}$. On the other hand, PF is a subprojection of P, hence, by the definition of $\mathfrak{N}$, $PF = 0$. Since F was arbitrary in $\mathfrak{M}$, $PE_0 = 0$, thus $P \leqslant E_1$. Summing up we get the following theorem:

THEOREM 1. *Let $\mathscr{S} \subset L(H)$ be an arbitrary subspace, which contains I and let $\mathfrak{M} \subset W^*(\mathscr{S}) \cap W^*(\mathscr{S})'$ be the family of all maximal antisymmetric projections for $\mathscr{S}$. Then:*

1° *$E_0 = \bigoplus_{F \in \mathfrak{M}} F$ is the smallest projection in $W^*(\mathscr{S})'$ containing all antisymmetric projections for $\mathscr{S}$ as subprojections,*

2° *$E_1 = I - E_0$ is the largest projection in the family $\mathfrak{N}$,*

3° *the decomposition $H = H_0 \oplus H_1$ ($H_i = E_i H$, $i = 0, 1$) is unique in the following sense: Suppose that $H = H_0' \oplus H_1'$ (E_i' are the projections onto*

H'_i, $i = 0,1$), $E'_1 \in \mathfrak{N}$ and E'_0 has no non-zero subprojection which belongs to $\mathfrak{N}$. Then $H'_0 = H_0$, $H'_1 = H_1$.

In this theorem 3° follows from (A) in [9] and 1° is a simple consequence of previous considerations. Observe that, if $W^*(\mathscr{S})$ is a factor (i.e. $W^*(\mathscr{S})$ has trivial center), then either $\mathscr{S}$ is antisymmetric or $\mathscr{S} = \mathscr{S}_{E_1}$, by Theorem 1. Theorem 1 gives, however, only the decomposition $H = (\bigoplus_{F\in\mathfrak{M}} FH)\oplus E_1 H$ of the Hilbert space H, associated with the subspace $\mathscr{S}$. Clearly, every subspace $\mathscr{S}_F$, $F \in \mathfrak{M}$ is antisymmetric, but in such generality one can not expect that

$$(*)\qquad \mathscr{S} = (\bigoplus_{F\in\mathfrak{M}} \mathscr{S}_F)\oplus\mathscr{S}_{\mathfrak{N}},$$

where $\mathscr{S}_{\mathfrak{N}} = \mathscr{S}_{E_1}$, i.e. that $\mathscr{S}$ itself decomposes on antisymmetric parts and $\mathscr{S}_{\mathfrak{N}}$. If the decomposition $(*)$ holds, then we say that $\mathscr{S}$ *admits the antisymmetric decomposition.* The following proposition gives a necessary and sufficient condition for the existence of the antisymmetric decomposition, when $\mathscr{S}$ is assumed to be an algebra:

PROPOSITION 4. *Let $\mathscr{A} \subset L(H)$ be an algebra with I and let $\mathfrak{M}$ be the family of all maximal antisymmetric projections for $\mathscr{A}$. $\mathscr{A}$ admists the antisymmetric decomposition if and only if $\mathfrak{M} \subset \mathscr{A}$.*

We omit the proof, because it follows from simple known facts.

3. SUFFICIENT CONDITIONS FOR THE EXISTENCE OF THE ANTISYMMETRIC DECOMPOSITION

In this section $\mathfrak{M}$ will denote always the set of all maximal antisymmetric projections for a given operator algebra $\mathscr{A}$ with I.

3.1. Algebras, whose invariant projections commute. Let $\mathscr{A} \subset L(H)$ be a weakly closed algebra with I. Assume that elements of $\operatorname{Lat}\mathscr{A}$ commute each other. If $F \in \mathfrak{M}$ and $P \in \operatorname{Lat}\mathscr{A}$, then $PF = FP$, hence $\operatorname{Lat}\mathscr{A} \subset \operatorname{Lat} F$, thus $F \in \operatorname{Alg}\operatorname{Lat}\mathscr{A}$. By Proposition 4 we get:

COROLLARY 1. *If $\mathscr{A} \subset L(H)$ is a reflexive operator algebra, whose invariant projections commute each other, then $\mathscr{A}$ admits the antisymmetric decomposition.*

There is a class of operator algebras, satisfying assumptions of Corollary 1, studied by many authors, so-called nest algebras. $\mathscr{A} \subset L(H)$ is called a ***nest algebra*** if $\mathscr{A}$ is reflexive and $\operatorname{Lat}\mathscr{A}$ is totally ordered. Nest algebras are examples of those operator algebras, which have no non-zero antisymmetric projections and in the antisymmetric decomposition $\mathscr{A} = \mathscr{A}_{\mathfrak{N}}$. Indeed, if $\mathscr{A}$ is a nest algebra, then $\mathscr{A}$ is irreducible, consequently $W^*(\mathscr{A})$ is a factor. Moreover, $\mathscr{A}$ contains always (in non-trivial cases) a self-adjoint operator, which is not any scalar multiple of I; namely the generator of the commutative von Neumann algebra $W^*(\mathrm{Lat}\,\mathscr{A})$ (here we assume that H is separable). By remarks after Theorem 1 we have $\mathscr{A} = \mathscr{A}_{\mathfrak{N}}$.

3.2. Remarks on operator algebras. Here we collect some simple remarks needed for the proof of the main theorem of this section.

Let $\mathscr{A} \subset L(H)$ be an algebra.

Remark 1. If $E \in \mathscr{A}$ is a projection, then $E \in (\mathrm{Lat}\,\mathscr{A})'$.

Remark 2. If $E \in \mathscr{A}'$ is a projection, then

$$\mathrm{Lat}(\mathscr{A}_E) = \{F_E = \text{projection } EH \text{ onto } \overline{EFH}\colon F \in \mathrm{Lat}\,\mathscr{A}\}.$$

The proof is standard.

Remark 3. If $E \in \mathscr{A}$ is a projection, then $\{F|_{EH}\colon F \in \mathrm{Lat}\,\mathscr{A}\} \subset \mathrm{Lat}(\mathscr{A}_E)$.

Remark 4. If $E \in \mathscr{A} \cap \mathscr{A}'$ is a projection, then $(\mathrm{Alg}\,\mathrm{Lat}\,\mathscr{A})_E = \mathrm{Alg}\,\mathrm{Lat}(\mathscr{A}_E)$.

The proof follows from Remarks 1 and 2.

Remark 5. If $\mathscr{A} \subset L(H)$ is a reflexive operator algebra and $E \in \mathscr{A} \cap \mathscr{A}'$ is a projection, then $\mathscr{A}_E$ is reflexive.

This is a consequence of Remark 4.

Remark 6. If $\mathscr{A} \subset L(H)$ is a reflexive algebra and $E \in \mathscr{A}$ is a projection, then for every projection $P \in \mathscr{A}_E$ there is a projection $P_0 \in \mathscr{A}$ such that $P = P_0|_{EH}$.

The proof follows from Remarks 1 and 3.

3.3. The theorem and corollaries. Let us fix now a weakly closed algebra $\mathscr{A} \subset L(H)$ containing I. Take a projection $F \in L(H)$ and define

$$\mathscr{K}_F = \{E \in \mathscr{A}\colon E \text{ is a projection, } EF = 0\},$$

$$\mathscr{D}_F = \{E \in \mathscr{A}\colon E \text{ is a projection, } EF = FE = F\}.$$

Put $K_F = \mathrm{LUB}\,\mathscr{K}_F$, $D_F = \mathrm{GLB}\,\mathscr{D}_F$. Since $\mathscr{A}$ is weakly closed, $K_F \in \mathscr{K}_F$, $D_F \in \mathscr{D}_F$, by [11], Lemma 1. Both K_F, D_F are uniquely determined by F and $\mathscr{A}$. It is plain that $E \in \mathscr{K}_F$ if and only if $I - E \in \mathscr{D}_F$. Consequently $I - K_F \in \mathscr{D}_F$ and $I - D_F \in \mathscr{K}_F$, hence $I - K_F = D_F$.

Now we are able to prove the following theorem:

THEOREM 2. *Suppose that $\mathscr{A} \subset L(H)$ is a reflexive algebra such that D_F belongs to $\mathscr{A}'$ for every $F \in \mathfrak{M}$. Then $\mathscr{A}$ admits the antisymmetric decomposition.*

Proof. By Proposition 4, it is sufficient to prove $\mathfrak{M} \subset \mathscr{A}$. Take $F \in \mathfrak{M}$ We claim, that $D = D_F$ is an antisymmetric projection for $\mathscr{A}$. Since $D \in \mathscr{A} \cap \mathscr{A}'$ and $\mathscr{A}$ is reflexive, $\mathscr{A}_D$ is also reflexive, by Remark 5, hence $\mathscr{A}_D$ is weakly closed in $L(DH)$. By the remarks after the definition of antisymmetric projections in 2.1, to prove the claim it is enough to prove that the only projections in $\mathscr{A}_D$ are 0 and I_{DH}. Let $P \in \mathscr{A}_D$ be a projection. By Remark 6, there is a projection $P_0 \in \mathscr{A}$ such that $P_0|_{DH} = P$. Since F is antisymmetric for $\mathscr{A}$, $\mathscr{A}_F$ can not contain any projection except 0 and I_{FH}, consequently, every projection, which belongs to $\mathscr{A}$, must belong either to $\mathscr{K}_F$ or to $\mathscr{D}_F$. If $P_0 \in \mathscr{K}_F$, then $P_0 \leqslant K_F$, thus $P_0 D = 0$, because $D = I - K_F$. Similarily, if $P_0 \in \mathscr{D}_F$, then $D \leqslant P_0$, thus $P_0 D = D$. It implies that either $P = 0$ or $P = I_{DH}$, which proves the claim. Since $F \in \mathfrak{M}$ and $F \leqslant D$, we have $F = D$ (F is, obviously, assumed to be different from zero) and theorem is now proved.

In particular, the assumptions of Theorem 2 are satisfied, if $\mathscr{A}$ is reflexive and commutative ($\mathscr{A} \subset \mathscr{A}'$). We have then

COROLLARY 2. *Every reflexive, commutative algebra admits the antisymmetric decomposition.*

COROLLARY 3. *If $A \in L(H)$ is an operator such that $\overline{\mathscr{A}(A)}^s$ is reflexive, then $\overline{\mathscr{A}(A)}^s$ admits the antisymmetric decomposition.*

3.4. Remarks on a theorem of Conway and Olin. Let (X, μ) be a finite measure space; put $H = L^2(\mu)$. By M we will denote the von Neumann algebra in $L(H)$ consisting of all operators $L_u f = uf$, $u \in L^\infty(\mu)$, $f \in H$. The map Φ: $L^\infty(\mu) \to M$ defined by $\Phi(u) = L_u$ is a $*$-isomorphism of these two algebras. If we endow $L^\infty(\mu)$ and M with the weak-star and weak operator topologies, respectively, then Φ is a homeomorphism. Conway and Olin have proven the following theorem ([1], Theorem 3.1):

THEOREM C–O. *Let $\mathscr{A}$ be a weak-star closed subalgebra of $L^\infty(\mu)$ such that $1 \in \mathscr{A}$; then there is a measurable partition $\{\Delta_0, \Delta_1, \ldots\}$ of X such that:*

(a) $\chi_{\Delta_n} \in \mathscr{A}$ *for every* $n \geqslant 0$,

(b) *for* $n \geqslant 1$, χ_{Δ_n} *is a minimal projection in* $\mathscr{A}$,

(c) $\mathscr{A}|_{\Delta_0}$ *is a pseudosymmetric subalgebra of* $L^\infty(\Delta_0, \mu)$ *(see* [1] *for the definition),*

(d) *for* $n \geqslant 1$, $\mathscr{A}|_{\Delta_n}$ *is an antisymmetric subalgebra of* $L^\infty(\Delta_n, \mu)$,

(e) $\mathscr{A} = \mathscr{A}|_{\Delta_0} \oplus \mathscr{A}|_{\Delta_1} \oplus \cdots$

We want to show, how this theorem can be derived from our previous results. We preserve the notation introduced before Theorem C–O. Suppose that $\mathscr{A} \subset L^\infty(\mu)$ is a weak-star closed algebra such that $1 \in \mathscr{A}$. Translating it into the language of operators in $L(H)$, $\Phi(\mathscr{A})$ is a commutative, weakly closed subalgebra of $L(H)$ containing $I = \Phi(1)$. Moreover, $\Phi(\mathscr{A})$ consists of normal operators in H. Now we apply Sarason's theorem ([8], Theorem 9.21) to infer that $\Phi(\mathscr{A})$ is reflexive. By Corollary 2, $\Phi(\mathscr{A})$ admits the antisymmetric decomposition. Let $\mathfrak{M}$ be the set of all maximal antisymmetric projections for $\Phi(\mathscr{A})$. By Proposition 3, $\mathfrak{M} \subset W^*(\Phi(\mathscr{A})) \subset M$.

Therefore each $F \in \mathfrak{M}$ has the form $Ff = \chi_\sigma f$, $f \in H$, with some measurable set $\sigma \subset X$. Since different elements of $\mathfrak{M}$ are pairwise orthogonal, there is a partition $\{\Delta_F: F \in \mathfrak{M}\} \cup \{\Delta_\mathfrak{N}\}$ of X, where Δ_F $(F \in \mathfrak{M})$, $\Delta_\mathfrak{N}$ are measurable subsets of X and $\Delta_\mathfrak{N}$ corresponds to the part $\Phi(\mathscr{A})_\mathfrak{N}$ of $\Phi(\mathscr{A})$. Since μ is a finite measure, the set $\{\Delta_F: F \in \mathfrak{M}\}$ must be at most countable, moreover, the part $\Phi(\mathscr{A})_\mathfrak{N}$ of $\Phi(\mathscr{A})$ corresponds exactly to the pseudosymmetric part $\mathscr{A}|_{\Delta_0}$ of $\mathscr{A}$ in Theorem C–O, by the definition of the set $\mathfrak{N}$. It is plain that $\mathscr{A}|_{\Delta F}$ $(F \in \mathfrak{M})$ are antisymmetric subalgebras of $L^\infty(\Delta_F, \mu)$. By remarks after the definition of antisymmetric projections in 2.1; every $F \in \mathfrak{M}$ is a minimal projection in $\Phi(\mathscr{A})$. Now Theorem C–O follows from Corollary 2.

4. APPLICATIONS AND EXAMPLES

In this section $\mathrm{sp}(T)$ denotes the spectrum of an operator $T \in L(H)$. If X is a compact Hausdorff space, then $C(X)$ stands for the algebra of all continuous complex functions on X with the uniform norm. If X is a subset of the complex plane C, then $P(X)$ is the $C(X)$-closure of the algebra of restrictions of all polynomials to X. R denotes the real line.

4.1. Examples concerning the antisymmetric decomposition.

Example 1. Let T be a self-adjoint operator in a separable Hilbert space H, let $\{z_1, z_2, \ldots\}$ be the point spectrum of T and let E_i be the projection onto the eigenspace $\ker(T - z_i I)$ $(i = 1, 2, \ldots)$. It is clear that the only maximal antisymmetric projections for the algebra $\mathscr{A} = \overline{\mathscr{A}(T)}^s$ are precisely E_i and $\mathscr{A}_\mathfrak{N} = \mathscr{A}_{E_0}$, where $E_0 = I - (\bigoplus_{i=1}^{\infty} E_i)$. Since $E_1 \in \mathscr{A}$ $(i = 1, 2, \ldots)$, $\mathscr{A}$ admits the antisymmetric decomposition, by Proposition 4.

Example 2. Now we present a non-commutative example of the antisymmetric decomposition. In a three dimensional Hilbert space consider two operators:

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n a fixed orthonormal basis. Clearly, $TS \neq ST$. Let $\mathscr{A}$ be the algebra with I of all polynomials in two non-commuting variables T, S. The projection $F = T^2$ is the only maximal antisymmetric projection for $\mathscr{A}$ and $\mathscr{A}_\mathfrak{N} = \mathscr{A}_{I-F}$. Since $F \in \mathscr{A}$, $\mathscr{A}$ admits the antisymmetric decomposition, by Proposition 4.

4.2. Antisymmetric projections for algebras of normal operators and the Bishop decomposition. Now we want to give applications and interpretations of theorems proved in previous sections to algebras generated by normal operators. Let $T \in L(H)$ be a normal operator with the spectrum $X \subset C$ and the spectral measure E. For brevity, we will write simply: "a normal operator (T, H, X, E)". It is known that the closed support of E denoted by $\operatorname{supp} E$ is equal to X and the projections $E(Y)$ belong to the commutative von Neumann algebra $W^*(T)$ for every Borel set $Y \subset X$. For $x \in H$ we denote by m_x the corresponding positive scalar measure of the form $m_x(Y) = (E(Y)x, x)$ (Y is a Borel set, $Y \subset X$). If $Y \subset X$ is a closed set, we denote by E_Y the restriction of E to Y and by T_Y the normal operator $T|_{E(Y)H}$. The inclusion $\operatorname{supp} E_Y = \operatorname{sp}(T_Y) \subset Y$ is known ([8], Theorem 1.13) and it may be proper, as we will see also in one of examples. The Gelfand–Naimark theorem yields a $*$-isomorphism φ: $C^*(T) \to C(X)$ of the C^*-algebra generated by T and I onto $C(X)$, such that $\varphi(I) = 1$ and $\varphi(T) = z_X$ – the identity function on X. Hence $\varphi(\overline{\mathscr{A}(T)}) = P(X)$ and $\overline{\mathscr{A}(T)}$ is antisymmetric if and only if $P(X)$ is. Let now $\mathscr{K}$ be the Bishop decomposition for $P(X)$, i.e. $\mathscr{K}$ consists of all maximal antisymmetric sets for $P(X)$. Elements of $\mathscr{K}$ are pairwise disjoint and closed, moreover, X is the union of all the members of $\mathscr{K}$. If $Y \in \mathscr{K}$, then Y is a (weak) peak set for $P(X)$, hence $P(X)|_Y = \{f|_Y: f \in P(X)\} \subset C(Y)$ is closed in $C(Y)$ and, consequently, $P(X)|_Y = P(Y)$ (see [5]). Now it is natural to ask, if, for given $Y \in \mathscr{K}$, $E(Y)$ is a maximal antisymmetric projection for $\overline{\mathscr{A}(T)}$ and conversely, if every maximal antisymmetric projection for $\overline{\mathscr{A}(T)}$ arises in this way. We will show, that under a certain natural assumption the answer is "yes". But generally, the solution is negative and the reason is, roughly speaking, that $\mathscr{K}$ depends essentially only of the nature of $P(X)$ and it does not depend on a special choice of a measure on X. To be more precise, let us recall, that every cyclic normal operator T with $\operatorname{sp}(T) = X$ is unitarily equivalent to the multiplication L_z by z_X in the Hilbert space $L^2(\mu)$ with a suitable finite, positive measure μ such that $X = \operatorname{supp} \mu$ – the closed support of μ. Then $\operatorname{sp}(L_z) = X$, the spectral measure of L_z has the form $E(Y)f = \chi_Y f$ (for any Borel set $Y \subset X$) and corresponding to E scalar measures are $m_f = |f|^2 \mu$, $f \in L^2(\mu)$. Now we show an example of a normal operator, for which our problem has a negative solution:

EXAMPLE 3. Let $\Gamma \subset C$ be the unit circle. Choose a bounded sequence $z_n \in C$ such that $|z_n| > 1$ for all n, z_n has no cluster point z_0 satisfying $|z_0| > 1$ and every point of Γ is a cluster point of z_n. Let K be the closed interval $[0, 1/2] \subset R$. Define the compact $X = \Gamma \cup \{z_n: n = 1, 2, \ldots\} \cup \cup K$. The maximal antisymmetric sets for $P(X)$ are exactly $\{z_1\}$, $\{z_2\}$, ... and $Y = \Gamma \cup K$. Let δ_n be the point mass at z_n and let m_K be the linear

Lebesgue measure restricted to K. Define the positive, finite Borel measure $\mu = m_K + \sum_{m=1}^{\infty} 2^{-n}\delta_n$. It is clear that $\operatorname{supp}\mu = X$ and $\mu(\Gamma) = 0$. Hence $\mu(Y) = \mu(K)$. Put $T = L_z$ in $L^2(\mu)$. Since $\operatorname{supp}\mu = \operatorname{supp}E$, we have $\operatorname{sp}(T) = X$. We claim that $E(Y)$ is not any antisymmetric projection for $\overline{\mathscr{A}(T)}$. Indeed, since $E(Y) = E(K)$, we have for all $f \in L^2(\mu)$:

$$\overline{(TE(Y)f, f)} = (TE(K)f, f) = \int_K z|f|^2 d\mu = \int_K z|f|^2 dm_K.$$

Thus $TE(Y)$ is a self-adjoint operator, but $TE(Y)$ can not be any scalar multiple of $E(Y)$. Let us point out, that $\operatorname{sp}(T_Y) = \operatorname{supp}E_Y = K$ is properly contained in Y.

Now we will prove the following lemma:

LEMMA. *Let (T, H, X, E) be a normal operator.*

(a) *If $Y \subset X$ is a closed set such that $Z = \operatorname{sp}(T_Y)$ is a maximal antisymmetric set for $P(X)$, then $E(Y)$ is an antisymmetric projection for $\overline{\mathscr{A}(T)}$.*

(b) *If $P \in \mathscr{A}(T)'$ is an antisymmetric projection for $\overline{\mathscr{A}(T)}$, then $V = \operatorname{sp}(T|_{PH})$ is an antisymmetric set for $P(X)$. Moreover, $P \leqslant E(V)$.*

Proof. First we prove (a). Since $Z = \operatorname{sp}(T_Y) = \operatorname{supp}E_Y$, we have $E(Y) = E(Z)$ thus $T_Y = T_Z$ and the algebras $C^*(T_Y)$ and $C(Z)$ are $*$-isomorphic, by the Gelfand–Naimark theorem. By the assumptions and remarks at the beginning of 4.2, $P(Z) = P(X)|_Z$ is antisymmetric, hence $\overline{\mathscr{A}(T)}|_{E(Y)H} = \overline{\mathscr{A}(T_Y)}$ is antisymmetric, thus $E(Y)$ is an antisymmetric projection for $\overline{\mathscr{A}(T)}$. To prove (b) denote by E_1 the spectral measure of $T|_{PH}$. Then we have for all Borel sets $\sigma \subset X$: $E_1(\sigma) = E(\sigma)|_{PH}$ and $\operatorname{supp}E_1 = V = \operatorname{sp}(T|_{PH})$. If $f \in C(X)$, we get:

$$f(T)|_{PH} = \int f dE_1 = \int_V f dE_1 = f(T|_{PH}).$$

Suppose that $f \in P(X)$ is real on V. The above equality implies, that $f(T)|_{PH} \in \overline{\mathscr{A}(T)}|_{PH}$ is a self-adjoint operator. By the antisymmetry of P, $f(T)x = rx$ with some $r \in R$, for all $x \in PH$. Using again the Gelfand–Naimark theorem to identify $C^*(T|_{PH})$ and $C(V)$ we conclude that $f = r$ on V. Since $E_1(V)$ is the identity operator in PH, the inequality $P \leqslant E(V)$ is clear and the proof is finished.

The following theorem gives a positive solution of the problem considered in this section:

THEOREM 3. *Let (T, H, X, E) be a normal operator and let $\mathscr{K}$ be the Bishop decomposition for $P(X)$. Suppose that:*

$(**)$ *for every $Y \in \mathscr{K}$ if $E(Y) \neq 0$, then $Y = \operatorname{sp}(T_Y)$.*

(a) *If* $Y \in \mathscr{K}$ *and* $E(Y) \neq 0$, *then* $E(Y)$ *is a maximal antisymmetric projection for* $\overline{\mathscr{A}(T)}$.

(b) *If* $P \neq 0$, $P \in \mathscr{A}(T)'$ *is a maximal antisymmetric projection for* $\overline{\mathscr{A}(T)}$, *then there is* $Y \in \mathscr{K}$ *such that* $E(Y) = P$.

Proof. To prove (a) take $Y \in \mathscr{K}$ such that $E(Y) \neq 0$. Since $Y = \mathrm{sp}(T_Y)$, we have, by Lemma (a), that $E(Y)$ is an antisymmetric projection for $\overline{\mathscr{A}(T)}$. Let F be the maximal antisymmetric projection for $\overline{\mathscr{A}(T)}$ such that $E(Y) \leqslant F$ and let $Z_1 = \mathrm{sp}(T|_{FH})$. By Lemma (b), Z_1 is an antisymmetric set for $P(X)$. Then there is $Z \in \mathscr{K}$ such that $Z_1 \subset Z$. Hence $E(Y) \leqslant F \leqslant E(Z_1) \leqslant E(Z)$. We have now $E(Y \cap Z) = E(Y)E(Z) = E(Y) \neq 0$ and this implies $Y = Z$, because $Y, Z \in \mathscr{K}$ and different elements of $\mathscr{K}$ are disjoint. Now we prove (b). Let $P \neq 0$ be a maximal antisymmetric projection for $\overline{\mathscr{A}(T)}$. Using again (b) of Lemma we have, that $Z = \mathrm{sp}(T|_{PH})$ is an antisymmetric set for $P(X)$. Now we find $Y \in \mathscr{K}$ such that $Z \subset Y$ and we get $P \leqslant E(Z) \leqslant E(Y)$. By (a) in lemma and by (**), $E(Y)$ is an antisymmetric projection for $\overline{\mathscr{A}(T)}$. Since P is maximal antisymmetric, $P = E(Y)$ and our theorem is now completely proved.

This theorem establishes one-to-one correspondence between the set of all maximal antisymmetric sets Y for $P(X)$, which satisfy $E(Y) \neq 0$ and the set of all non-zero maximal antisymmetric projections for $\overline{\mathscr{A}(T)}$. Comparing this result with Theorem 1 we see, that under assumptions of Theorem 3 the projection E_1 of Theorem 1 must be zero and $I = \bigoplus_{Y \in \mathscr{K}} E(Y)$. To see that assumption (**) is satisfied in many cases consider the following example:

EXAMPLE 4. Take in the complex plane C two disjoint closed discs X_1, X_2 and put $X = X_1 \cup X_2$. Let (T, H, X, E) be a normal operator. Clearly, X_1, X_2 are the only maximal antisymmetric sets for $P(X)$ and, since $\mathrm{supp}\, E_{X_i} = X_i$ $(i = 1, 2)$, $E(X_1)$, $E(X_2)$ are the only maximal antisymmetric projections for $\overline{\mathscr{A}(T)}$.

It may occur that if (T, H, X, E) is a normal operator and $Y \subset X$ is a maximal antisymmetric set for $P(X)$ (even infinite), then $E(Y) = 0$. One can easily see this, modifying Example 3.

4.3. Antisymmetric projections and dilatable representations of function algebras. To finish this paper let us lock at Theorem 3 as at a theorem concerning representations of function algebras. Namely, if (T, H, X, E) is a normal operator, then the algebra $C(X)$ has a $*$-representation φ: $C(X) \to L(H)$ given by the spectral integral $\varphi(f) = \int f dE$, $f \in C(X)$. In particular, the restriction of φ to $P(X)$ gives a representation (i.e. an algebra homomorphism) of $P(X)$ into $L(H)$, which has a spectral measure. Assume now that X is an arbitrary compact Hausdorff space and $A \subset C(X)$ is a function algebra. Let $\mathscr{K}$ be the Bishop decomposition for A. Consider a representation $\varphi\colon A \to L(H)$ and suppose that there is a semi-spectral measure F on X such that $\varphi(u) = \int\limits_X u dF$, $u \in A$. By the properties of semi-spectral integrals, this representation is continuous and $\|\varphi(u)\| \leqslant \|u\|$, $u \in A$, $\varphi(I) = I$. Now one can ask, what is the connection between $\mathscr{K}$ and the set of all maximal antisymmetric projections for $\overline{\varphi(A)}$ (the norm closure). It follows from a construction due to Mlak (in a more general setting) [7], that if $Y \in \mathscr{K}$, then $F(Y)$ is a projection in the center of the von Neumann algebra $W^*(\varphi(A))$. It follows from Example 3, that in general, for $Y \in \mathscr{K}$, $F(Y)$ need not be antisymmetric for $\overline{\varphi(A)}$. But the following theorem holds true:

THEOREM 4. *Let $A \subset C(X)$ be a function algebra and let $\varphi\colon A \to L(H)$ be a representation of A of the form $\varphi(u) = \int\limits_X u dF$, $u \in A$, with some semi-spectral measure F. If $P \in \varphi(A)'$ is an antisymmetric projection for $\overline{\varphi(A)}$, then there is a maximal antisymmetric set $Y \subset X$ for A such that $P \leqslant F(Y)$.*

Proof. For $x \in H$ we denote by m_x the positive scalar measure associated with F of the form $m_x(\sigma) = (F(\sigma)x, x)$ (σ are Borel subsets of X). Let Z denote the union of all $\operatorname{supp} m_{Px}$, where x runs over H. We will prove that Z is an antisymmetric set for A. Let $f \in A$ be a function real on Z. Then, for every $x \in H$, f is real an $\operatorname{supp} m_x$. Since the semi-spectral integral preserves the involution, we have

$$\big((\varphi(f)^* - \varphi(f))Px, x\big) = \int (f - \bar{f})\, dm_{Px} = 0.$$

Hence $\varphi(f)P = \varphi(f)^*P$ and, by the antisymmetry of P, there is $r \in R$ such that $\varphi(f)P = rP$. Since $\varphi(1) = I$, we have $\varphi(f-r)P = 0$ and, by the multiplicativity of φ, $\varphi\big((f-r)^2\big)P = 0$. Now, for $x \in H$:

$$\int (f-r)^2\, dm_{Px} = \big(\varphi\big((f-r)^2\big)Px, x\big) = 0$$

and $f = r$ on $\operatorname{supp} m_{Px}$, because m_{Px} is positive and f is real on $\operatorname{supp} m_{Px}$. Hence $f = r$ on Z. Now we can find a maximal antisymmetric set $Y \subset X$ for A such that $Z \subset Y$. It remains only to prove $P \leqslant F(Y)$. Let us recall, that $F(Y)$ is a central projection in $W^*(\varphi(A))$. For every $x \in H$ we have

$$\big(F(Y)Px, x\big) = m_{Px}(Y) = m_{Px}(X) = (Px, x),$$

because $\operatorname{supp} m_{Px} \subset Y$ and Y is closed. The proof is now complete.

Acknowledgements. I would like to express my thanks to Professor A. Guichardet for very useful remarks and discussions concerning the final version of this paper.

References

[1] J. B. Conway, R. F. Olin, *A functional calculus for subnormal operators*, Mem. Amer. Math. Soc. 10. 184 (1977).

[2] — — *A functional calculus for subnormal operators*, Bull. Amer. Math. Soc. 82 (1976), p. 259–261.

[3] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Paris 1969.

[4] T. W. Gamelin, *Uniform algebras*, Prentice Hall INC., Englewood Cliffs N. J., 1969.

[5] I. Glicksberg, *Measures orthogonal to algebras and sets of antisymmetry*, Trans. Amer. Math. Soc. 105 (1962), p. 415–435.

[6] P. R. Halmos, *A Hilbert space problem book*, D. Van Nostrand Co., Princeton 1967.

[7] W. Mlak, *Decompositions of operator-valued representations of function algebras*, Studia Math. 36 (1970), p. 111–123.

[8] H. Radjavi, P. Rosenthal, *Invariant subspaces*, Springer Verlag Inc., New York 1973.

[9] W. Szymański, *Decompositions of operator-valued functions in Hilbert spaces*, Studia Math. 50 (1974), p. 265–280.

[10] — *Antisymmetric operator algebras, I*, this fascicule, p. 263–274.

[11] D. M. Topping, *Lectures on von Neumann algebras*, Van Nostrand Reinhold Co., London 1971.

INSTITUTE OF MATHEMATIC, POLISH ACADEMY OF SCIENCES
KRAKÓW, POLAND

Reçu par la Rédaction le 15. 9. 1977