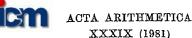
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On the theorem of Jarník and Besicovitch

Ъу

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1. Let $\alpha > 0$ be fixed in all that follows and $E(\alpha)$ be the set of real numbers x such that the inequality $||nx|| \leq n^{-1-\alpha}$ has arbitrarily large integer solutions n. (As usual ||t|| is the distance from t to the nearest integer.) We recall the classical theorem of Jarník [3] and Besicovitch [1].

I. E(a) has Hausdorff dimension $2(2+a)^{-1}$.

We shall obtain for E(a) a stronger property:

II. There exists a positive measure μ whose support is a compact subset of $E(\alpha)$, whose Fourier-Stieltjes transform obeys the relation

$$\hat{\mu}(u) = o(\log|u|) |u|^{-1/2+\alpha}, \quad |u| \to +\infty.$$

By a theorem of Beurling [3, III], the closed support of μ (or any Borel set of positive μ -measure) must have dimension at least $2(2+\alpha)^{-1}$; however the property of $E(\alpha)$ is not shared by certain sets of positive Lebesgue measure, so that II could not be deduced from I (see [4], p. 351).

2. In this paragraph we define some auxiliary functions, and obtain an inequality on Fourier coefficients, more or less equivalent to the main result II. First of all we construct the function $F_R(x)$, 0 < R < 1/4,

$$F_R(x) = egin{cases} 35 \, (32)^{-1} R^{-7} (R^2 - x^2)^3, & |x| \leqslant R, \ 0, & R \leqslant |x| \leqslant 1/2. \end{cases}$$

Then we extend \mathcal{F}_{R} to a periodic function and expand it in a Fourier series

$$F_R(x) = \sum a_m^{(R)} e^{2\pi i m x},$$
 $a_0^{(R)} = 1, \quad |a_m^{(R)}| \leqslant 1, \quad |a_m^{(R)}| \leqslant m^{-2}R^{-2}.$

In the construction below M is a large positive integer and $R=(4M)^{-1-a}$; we write $q_m(x)=\sum_p^*F_R(px)$, where \sum_p^* means that the sum is extended over primes p in the interval $M\leqslant p\leqslant 2M$. We also

write C_M for the number of these primes, $C_M = \pi(2M) - \pi(M-1) \cong M/\log M$. Then we have the expansion

$$q_M(x) - C_M = \sum_m \sum_p a_m^{(R)} e^{2\pi i m p x}.$$

From this formula we can derive upper bounds for the Fourier coefficients \hat{q}_M . Because $|a_m^{(R)}| \leq 1$, $\hat{q}_M(k)$ does not exceed the number of solutions of the equation k = mp. Hence

$$\hat{q}_{\mathcal{M}}(k) = 0, \quad 1 \leqslant |k| < M,$$

$$|\hat{q}_M(k)| \leq 2\log|k|/\log M$$
, always.

The last bound can be improved when $|k| > 2MR^{-1}$. In this case all the solutions mp = k satisfy $|m| \ge |k|/|p| > |k|/2M$, so $|a_m^{(R)}| \le m^{-2}R^{-2} \le 4k^{-2}M^2R^{-2}$ and then

$$|\hat{q}_M(k)| \leq 4k^{-2}M^2R^{-2}\log|k|/\log M$$
.

Let us now set $g_M(x) = C_M^{-1} q_M(x)$. From the previous inequalities we find

$$|\hat{g}_M(k)| \leqslant A \log M/M, \quad 1 \leqslant |k| \leqslant 2MR^{-1},$$

(a)

$$|\hat{g}_M(k)| \leqslant Ak^{-2}MR^{-2}\log|k|, \quad |k| > 2MR^{-1}.$$

Recalling that $R = (4M)^{-1-\alpha}$ we get

(b)
$$|\hat{g}_M(k)| \leq A|k|^{-1/2+\alpha} \log |k|, \quad |k| > 2MR^{-1}.$$

In (a) and (b), A depends only on α .

Let now ψ be a function of class C^2 and compact support. The next lemma expresses in compact form the basic step in the construction of μ .

LEMMA. For the Fourier transform of (ψg_M) we have the inequalities

$$|(\psi g_M)^{\hat{}}(u) - \hat{\psi}(u)| \leq B \log(2 + |u|)(|u| + 1)^{-1/2 + \alpha},$$

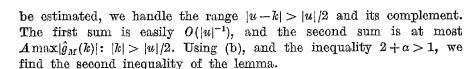
$$|(\psi g_M)\hat{}(u) - \hat{\psi}(u)| \leq B \log M/M$$

where B depends only on a and ψ .

Proof. The difference in question is a sum $\sum_{k} \hat{\psi}(u-k)\hat{g}_{M}(k)$; because ψ is C^{2} , we can majorize this by

$$A\sum_{k}'(1+|u-k|)^{-2}|\hat{g}_{M}(k)|.$$

Clearly this sum is most $A \max |\hat{g}_M(k)|$ (maximum for $k \neq 0$) and this, by (a) and (b) gives the second inequality. The second is stronger than the first when $|u| < 4MR^{-1}$, so we can assume $|u| \ge 4MR^{-1}$. In the sum to



3. Before passing to the proof of II, we write a more refined estimate than given in the lemma. We fix ψ as above and $\varepsilon > 0$, and attempt to find a function G (like $g_{\mathcal{M}}$) so that

$$|(\psi G)(u) - \hat{\psi}(u)| \leq \varepsilon \log(2 + |u|)(1 + |u|)^{-1/2 + \alpha}$$

For this function G we cannot choose g_M (unless our method of estimation is improved) but we can choose $N^{-1}(g_{M_1}+\ldots+g_{M_N})$, with N large and $M_1<\ldots< M_N$ an appropriate sequence. To see this we have only to observe that each $g_M\psi$ is C^2 , so its Fourier transform is $O(u^{-2})$.

Finally, we remark that if G(x) > 0 then $||px|| < p^{-1-\alpha}$ for a certain prime p in the interval $[M_1, 4M_N]$.

4. Let $\psi_0 \geqslant 0$, $\int \psi_0 = 1$, ψ_0 of class C^2 and compact support. By the method just described we choose G_1, \ldots, G_k, \ldots and $H_k = G_1 \ldots G_k$ $(H_0 = 1)$, so that

$$|\psi_0 H_k^{\hat{}} - \psi_0 H_{k+1}^{\hat{}}| \leq 2^{-k-1} \log(2+|u|) (1+|u|)^{-1/2+a}.$$

Then the $w^* = \text{limit}$ of the measures $\psi_0(x)H_k(x)dx$, a positive measure μ , certainly fulfills the relation $\hat{\mu}(u) = o(\log|u|)|u|^{-1/2+\alpha}$ as $|u| \to \infty$. For each k, the closed support of μ is contained in the closure F_k of the set $\{\psi_0(x)G_k(x)>0\}$. Clearly $\bigcap F_k \subset E(\alpha)$ and our main theorem is proved.

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