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On Čech cohomology and weakly confluent mappings into ANR's

by

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Abstract. In this paper it is proved that weakly confluent mappings of a compact Hausdorff space onto an n-manifold, $n>1$, or a Q-manifold induce monomorphisms in first Čech cohomology. This result is not true if the image space is not nice enough. It is also proved that for locally connected compacta being contractible with respect to any one-dimensional ANR is equivalent to being contractible with respect to S^1. Finally, it is proved that any mapping from a compact Hausdorff space of dimension at least three into any compact, simply connected ANR is homotopic to a weakly confluent onto mapping.

1. Introduction. In 1935, Eilenberg showed in [7] that open mappings or monotone mappings of compact metric spaces induce monomorphisms between the first Čech cohomology groups. This implies that contractibility with respect to S^1 of compact metric spaces is an invariant under monotone mappings and open mappings. In 1966, Lelek extended in [16] Eilenberg's results to the class of confluent mappings. In [10] and [12] the author together with E. D. Tymchatyn generalized Lelek's result to confluent mappings of compact Hausdorff spaces as well as to semi-confluent mappings of compact Hausdorff spaces onto hereditarily unicoherent continua, and to contractibility with respect to any connected one-dimensional ANR. There are examples to show that more general mappings, like pseudo-confluent or even weakly confluent mappings, do not preserve contractibility with respect to S^1 (see [17, p. 99]).

In [24] Wilson constructed a monotone open mapping of the Menger universal curve onto the Hilbert cube Q, and in [25] he constructed monotone open mappings of any compact, connected triangulated m-manifold M, with $m \geqslant 3$, onto any cell. Using techniques from [24] and [25] Walsh proved in [22] and [23] that any mapping of M into a compact, connected ANR Y is homotopic to a monotone open mapping of M onto Y if and only if $f_*\colon \pi_1(M) \to \pi_1(Y)$ is surjective.

In this paper it is proved that pseudo-confluent mappings of compact Hausdorff spaces induce monomorphisms between the first Čech cohomology groups, and that

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contractibility with respect to any one-dimensional ANR is an invariant under pseudo-confluent mappings provided that the image space has a certain property. (In particular if it is an n-manifold, $n\geqslant 2$, or a Hilbert cube manifold). It is also proved that for locally connected metric continua "being contractible with respect to S^1" is equivalent to "being contractible with respect to any non-simply connected graph". Finally, it is proved that if X is a compact Hausdorff space of dimension $\geqslant 3$ and Y is an arbitrary compact, simply connected ANR, then any mapping of X into Y is homotopic to a weakly confluent mapping of X onto Y. Two unsolved problems are also posed.

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2. Preliminaries. A *compactum* is a compact metric space, and a *continuum* is a compact, connected Hausdorff space. A *mapping* is always a continuous function. By an ANR we mean a metric absolute neighbourhood retract [1]. A mapping $f\colon X\to Y$ is said to be *essential* provided that f is not homotopic to a constant mapping. We write f non $\simeq 1$. If $f\colon X\to Y$ is homotopic to a constant mapping we write $f\simeq 1$. We write f irr non $\simeq 1$ provided that f is essential, but $f|K\simeq 1$ for every proper closed subset K of X. A continuum X is said to be *unicoherent* provided that $X\neq P\cup Q$, where P and Q are subcontinua of X such that $P\cap Q$ is not connected. A space X is said to be *contractible with respect to* a space Y provided that every mapping of X into Y is homotopic to a constant. By S^n we denote the standard n-sphere.

By a *simple closed curve* we always mean a homeomorphic copy of S^1. If X is a subset of Y, then by $\mathrm{Cl}(X)$, $\mathrm{Bd}(X)$ and $\mathrm{Int}(X)$ we denote the closure of X in Y, the boundary of X in Y, and the interior of X in Y, respectively. We say that a simple closed curve S in a space X is *approximated by a spiral* provided that X contains a homeomorphic copy C of $[0,+\infty)$ such that $S\subset\mathrm{Cl}(C)$ and $S\cup C$ is homeomorphic to the planar continuum

$$A=\{(\varrho,\theta)|\ \varrho=1,\text{ or }\varrho=\frac{2+e^\theta}{1+e^\theta}\text{ and }\theta\geqslant 0\},$$

where (ϱ,θ) denotes a point of the plane in polar coordinates. We call A the *standard spiral.*

A mapping $f\colon X\to Y$ of a compact Hausdorff space X onto a Hausdorff space Y is said to be *monotone* provided that $f^{-1}(y)$ is connected for each $y\in Y$. The mapping f is said to be *open* provided that $f(U)$ is open in Y for each open set U in X. We say that f is *weakly confluent* [17] (resp. *pseudo-confluent* [18]) provided that for each subcontinuum K (resp. each irreducible subcontinuum K) of Y there exists some component C of $f^{-1}(K)$ such that $f(C)=K$. (For the definitions of confluent mappings and semi-confluent mappings see [5] and [19]). It is known that open mappings and monotone mappings are confluent, confluent mappings are semi-confluent, semi-confluent mappings are weakly confluent, and weakly confluent mappings are pseudo-confluent.

3. Contractibility with respect to graphs. In this section we prove that contractibility with respect to any one-dimensional ANR is an invariant under pseudo-confluent mappings if the image space belongs to a certain class of spaces. The first proposition is due to Borsuk [2, p. 184] and Eilenberg [8, p. 70]:

3.1. PROPOSITION. *Let Y be a locally connected metric continuum. Then the following are equivalent:*

(i) *Y is not unicoherent;*

(ii) *Y is not contractible with respect to S^1;*

(iii) *Y contains a simple closed curve which is a retract of Y;*

(iv) *if $g\colon Y\to S^1$ is an essential mapping, then Y contains a simple closed curve S such that $g|S$ non $\simeq 1$.*

Proof. The equivalence of (i) and (ii) is established in [8, p. 70], and the equivalence of (i) and (iii) is established in [2, p. 184]. It is also obvious that (iv) implies (ii). The fact that (ii) implies (iv) can be proved by using the proof of the result in [2, p. 184] (see also [15, p. 439]). For this let $g\colon Y\to S^1$ be the essential mapping, and let S^1_+ and S^1_- be the upper-half and the lower-half of S^1, respectively. Let also $a=(1,0)$ and $b=(1,\pi)$. Consider the set $K_0=g^{-1}(S^1_+)$ and $K_1=g^{-1}(S^1_-)$. As in [15, Theorem 7, p. 432] there exist two locally connected continua C_0 and C_1 such that $K_0\subset C_0$. $K_1\subset C_1$, $Y=C_0\cup C_1$, and $g|C_j\simeq 1$ for $j\in\{0,1\}$. By [15, Theorem 3, p. 417], $C_0\cap C_1$ is not connected. Moreover, C_0 and C_1 can be constructed so that $C_0\cap C_1=F_0\cup F_1$, where F_0 and F_1 are closed, non-empty, disjoint subsets of Y with

$$g^{-1}(a)\subset F_0\quad\text{and}\quad g^{-1}(b)\subset F_1\,. \tag{1}$$

Let A_0 be an arc contained in C_0 and irreducibly connected between F_0 and F_1. Put $A_0\cap F_j=\{p_j\}$ for $j\in\{0,1\}$, and let A_1 be an arc p_0p_1 contained in C_1. It follows that $A_0\cup A_1=S$ is a simple closed curve. The proof of Theorem 4, p. 439 in [15] shows that $A_0\cup A_1$ is a retract of Y. Notice that the only difference between this construction and the construction in [15, p. 439] is that C_0 and C_1 are chosen in such a way that (1) is also satisfied. It is not difficult, now, to prove that the retraction $r\colon Y\to S$ can be taken so that

$$|g_1\circ r(y)-g(y)|<2$$

for each $y\in Y$, where $g_1=g|S$ This implies that $g_1\circ r$ is homotopic to g, and hence, g_1 non $\simeq 1$.

We shall use the following formulation of Fort's Lemma (see [9, page 542]) (for the definition of a locally trivial fiber see [20, p. 328]).

3.2. LEMMA (M. K. Fort [9]). *Let (E,B,p) be a locally trivial fiber space such that for each $b\in B$, the fibre $p^{-1}(b)$ is totally disconnected, and such that E is not arcwise connected. If $f\colon K\to E$ is a mapping of a connected space K onto E, then $p\circ f$ non $\simeq 1$.*

Consider the following continua in the plane

$$B = \left\{(\varrho, \theta) \mid \varrho = 1,\ \varrho = 2, \text{ or } \varrho = \frac{2+e^\theta}{1+e^\theta} \text{ and } -\infty<\theta<+\infty\right\},$$

and

$$A = \left\{(\varrho, \theta) \mid \varrho = 1, \text{ or } \varrho = \frac{2+e^\theta}{1+e^\theta} \text{ and } 0\leqslant\theta<+\infty\right\}.$$

It can be seen that B is the one-point union of two homeomorphic copies of A, say A_1 and A_2. Let $\{a\} = A_1 \cap A_2$. Consider the mapping $h\colon B \to S^1$ defined by $h(\varrho, \theta) = (1, \theta)$ for each $(\varrho, \theta) \in B$, and let $h_1\colon A \to S^1$ be the restriction of h on A.

For each non-negative integer n, let $g_n\colon S^1 \to S^1$ be the mapping of S^1 onto S^1 defined by $g_n(1, \theta) = (1, n\cdot\theta)$ (here consider S^1 as the set of all points in the plane having polar coordinates $(1, \theta)$ with $0\leqslant\theta\leqslant 2\pi$). It is known that each mapping of S^1 onto S^1 is homotopic to g_n for some n.

Let $g\colon A \to S^1$ be an essential mapping of A onto S^1. It is easy to see that $g \simeq g_n \circ h_1$ for some n. Let $f\colon K \to A$ be a mapping of a continuum K onto A. If $g \circ f \simeq 1$, then since $g \simeq g_n \circ h_1$, we have that $g_n \circ h_1 \circ f \simeq 1$. Thus, by [8, p. 68] there exists a mapping $\varphi\colon K \to \boldsymbol{R}$ such that $g_n \circ h_1 \circ f = \psi \circ \varphi$, where $\psi\colon \boldsymbol{R} \to S^1$ is the mapping defined by $\psi(t) = e^{2\pi i t}$ for each $t \in \boldsymbol{R}$.

We introduce the following notation: If D is a compact Hausdorff space and $d \in D$, let

$$U(D, d) = D \times \{0, 1\}/\{(d, 0), (d, 1)\},$$

that is $U(D, d)$ is the one-point union of two homeomorphic copies of D. Let $\varphi_D\colon D \times \{0, 1\} \to U(D, d)$ be the quotient mapping of $D \times \{0, 1\}$ onto $U(D, d)$. Let $f\colon D \to E$ be a mapping of D onto a Hausdorff space E. Then by

$$f \vee f\colon U(D, d) \to U(E, f(d))$$

we denote the mapping which is induced by f, that is the mapping defined by

$$(f \vee f)[\varphi_D(d, i)] = \varphi_E(f(d), i)$$

for each point $(d, i) \in D \times \{0, 1\}$.

Choose, now, a point $x_0 \in f^{-1}(a)$, and consider the mappings

$$f \vee f\colon U(K, x_0) \to U(A, a),$$

$$\varphi \vee \varphi\colon U(K, x_0) \to U(\varphi(K), \varphi(x_0)), \quad (g_n \circ h_1) \vee (g_n \circ h_1)\colon U(A, a) \to S^1 \vee S^1$$

($S^1 \vee S^1$ denotes the one point union of two S^1) and

$$\psi \vee \psi\colon U(\varphi(K), \varphi(x_0)) \to S^1 \vee S^1.$$

Then we have that

$$[(g_n \circ h_1) \vee (g_n \circ h_1)] \circ (f \vee f) = (\psi \vee \psi) \circ (\varphi \vee \varphi). \tag{1}$$

Define a mapping $\gamma\colon S^1 \vee S^1 \to S^1$ such that γ maps each copy of S^1 (in $S^1 \vee S^1$) homeomorphically onto S. It is very easy to see that γ can be taken so that

$$\gamma \circ [(g_n \circ h_1) \vee (g_n \circ h_1)] = g_n \circ h \tag{2}$$

(notice that $U(A, a) = B$). By (1) and (2) we obtain that

$$(g_n \circ h) \circ (f \vee f) = \gamma \circ [(g_n \circ h_1) \vee (g_n \circ h_1)] \circ (f \vee f) = \gamma \circ (\psi \vee \psi) \circ (\varphi \vee \varphi),$$

and since $U(\varphi(K), \varphi(x_0))$ is contractible ($U(\varphi(K), \varphi(x_0))$ is the one-point union of two arcs), we have that

$$(g_n \circ h) \circ (f \vee f) \simeq 1. \tag{3}$$

Consider the triple $(B, S^1, g_n \circ h)$. This is a locally trivial fiber space such that for each $s \in S^1$, the fibre $(g_n \circ h)^{-1}(s)$ is totally disconnected, and such that B is not arcwise connected. By Lemma 3.2, $(g_n \circ h) \circ (f \vee f)$ non$\simeq 1$. This contradicts (3), and hence, we have proved the following:

3.3. Lemma. *If $g\colon A \to S^1$ is an essential mapping of A onto S^1, and if $f\colon K \to A$ is a mapping of a continuum K onto A, then $g \circ f$ is an essential mapping of K onto S^1.*

Let $\mathfrak{F}$ denote the class of all compact metric spaces Y with the property that if $f\colon Y \to S^1$ is a mapping such that f non$\simeq 1$, then there exists a subcontinuum B of Y which is homeomorphic to the standard spiral A and such that $f|B$ non$\simeq 1$.

It can be proved that the class $\mathfrak{F}$ contains all compact n-manifolds, $n>1$, and all compact Hilbert cube manifolds [4].

To see that compact n-manifolds ($n>1$) are in class $\mathfrak{F}$ let M be an n-manifold ($n>1$) and let $f\colon M \to S^1$ be an essential mapping. By Proposition 3.1, there exists a simple closed curve S in M such that $f|S$ non$\simeq 1$. Let $h\colon S^1 \to M$ be an embedding of S^1 into M such that $h(S^1) = S$. It is easy to see that given $\varepsilon>0$ there exists an embedding $h'\colon S^1 \to M$ of S^1 into M such that $d(h, h')<\varepsilon$ (here by $d(h, h')$ we denote the number $\sup\{\varrho(h(x), h'(x)) \mid x \in S^1\}$ where ϱ denotes the metric in M), and such that $h'(S^1) \cap \partial M = \emptyset$. Moreover, we can take h' so that $h'(S^1)$ is a tame simple closed curve. Since M is an ANR, we can choose ε sufficiently small so that h' is homotopic to h. Then we have that $f|S'$ non$\simeq 1$, where $S' = h'(S^1)$. Consider, now, a tubular neighbourhood U of S' in M such that

$$\mathrm{Cl}(U) \cap \partial M = \emptyset.$$

Now it is obvious that we can construct a homeomorphic copy T of the half-line in U so that $S' \cup T$ is homeomorphic to the standard spiral A. Moreover, $f|S' \cup T$ non$\simeq 1$.

By using the same method one can show that Hilbert cube manifolds are also in class $\mathfrak{F}$.

3.4. Theorem. *Let $f\colon X \to Y$ be a pseudo-confluent mapping of a compact Hausdorff space X onto a space Y in $\mathfrak{F}$. If $g\colon Y \to S^1$ is a mapping such that $g \circ f \simeq 1$, then $g \simeq 1$.*

Proof. Suppose, on the contrary, that g non$\simeq 1$. By hypothesis, there exists a continuum B in Y which is homeomorphic to the standard spiral A and such that $(g|B)$ non$\simeq 1$. Since B is irreducible, there exists a subcontinuum K in X such that $f(K) = B$. By Lemma 3.3, $(g|B)\circ(f|K)$ non$\simeq 1$. Thus, $g\circ f$ non$\simeq 1$. This contradiction proves the theorem.

By $H^1(X)$ we denote the first Čech cohomology group of a compact Hausdorff space X based on arbitrary open coverings and with integer coefficients. By the Bruschlinsky theorem (see [6, 8.1]) we have that $H^1(X)$ is isomorphic to the group of the homotopy classes of mappings of X into S^1. Therefore, a compact Hausdorff space is contractible with respect to S^1 if and only if $H^1(X)$ is the trivial group.

The following results follows from Theorem 3.4:

3.5. Corollary. *Let $f\colon X\to Y$ be a pseudo-confluent mapping of a compact, Hausdorff space X onto some space $Y\in\mathfrak{F}$. Then the induced mapping*

$$f^*\colon H^1(Y)\to H^1(X)$$

is a monomorphism.

3.6. Corollary. *Let $f\colon X\to Y$ be a pseudo-confluent mapping of a compact, Hausdorff space X, which is contractible with respect to S^1, onto a space $Y\in\mathfrak{F}$. Then Y is contractible with respect to S^1.*

Next, we shall prove that Corollary 3.6 holds true for locally connected members of $\mathfrak{F}$ if instead of S^1 we have any graph G (i.e., compact, connected one-dimensional polyhedron). We need first to show that for locally connected compact metric spaces "being contractible with respect to S^1" is equivalent to "being contractible with respect to any non-simply connected graph". It is known that this is not true for non-locally connected continua. For example the Case-Chamberlin continuum [3] is a one-dimensional continuum which is contractible with respect to S^1 but it is notcontractible with respect to the one-point union of two circles $S^1\vee S^1$ (sometimes called a "figure 8").

It is known that a compact metric space X is contractible with respect to a graph G if and only if each component of X is contractible with respect to G. The proof of this is identical with the proof in [8, p. 66] where G is taken to be the 1-sphere S^1. Consequently, in order to prove that for locally connected, compact, metric spaces "being contractible with respect to S^1" is equivalent to "being contractible with respect to any non-simply connected graph", it suffices to prove the following:

3.7. Proposition. *Let X be a locally connected metric continuum. Then the following are equivalent*:

(i) *X is contractible with respect to S^1*;

(ii) *X is contractible with respect to a non-simply connected graph G.*

Proof. (ii) implies (i): Since G is not simply connected, it contains a homeomorphic copy of S^1. Therefore, every mapping of X into S^1 is homotopic to a constant mapping, since X is contractible with respect to G.

(i) implies (ii): Let X be a continuum which is contractible with respect to S^1. To show that X is contractible with respect to a non-simply connected graph G, let $f\colon X\to G$ be a mapping. We shall prove that $f\simeq 1$. Since every subcontinuum of a graph is a graph, we may assume without loss of generality, that f is surjective. There exist two simply connnected graphs A_1 and A_2 such that $G = A_1\cup A_2$ (see [1, p. 156]). Let $F_1 = f^{-1}(A_1)$ and $F_2 = f^{-1}(A_2)$. Then $X = F_1\cup F_2$, $f|F_1\simeq 1$ and $f|F_2\simeq 1$. We shall prove, by using the same method as in [15, Theorem 6, p. 431], that there exist two locally connected continua X_1 and X_2 in X such that $F_1\subset X_1$, $F_2\subset X_2$, $f|X_1\simeq 1$ and $f|X_2\simeq 1$.

First, notice that since F_1 is a compact subset of X with $f|F_1\simeq 1$, by [12, Proposition 4.4] there exists an open subset U of X containing F_1 and such that $f|\mathrm{Cl}(U)\simeq 1$. Since X is locally connected, we can take U so that $\mathrm{Cl}(U)$ has a finite number of components each one being a locally connected continuum. Thus, without loss of generality, we may assume that $F_1 = C_1\cup\ldots\cup C_m$, where the sets $C_1,\ldots,C_m$ are disjoint, locally connected continua.

We use induction. If $m = 1$, then the claim is true, that is, there exists a locally connected continuum X_1 containing $F_1 = C_1$ and such that $f|X_1\simeq 1$. Assume that it holds for the number $m-1$. Let L be an arc ab such that $L\cap F_1$ consists of exactly two points a and b, such that $a\in C_{m-1}$ and $b\in C_m$. Since $f|C_1\cup\ldots\cup C_{m-1}\simeq 1$, $f|L\simeq 1$ and $(C_1\cup\ldots\cup C_{m-1})\cap L = \{a\}$, it follows easily (see the proof of Proposition 2.2 in [9]) that $f|C_1\cup\ldots\cup C_{m-1}\cup L\simeq 1$. Thus, $f|C_1\cup\ldots\cup C_{m-1}\cup L\cup C_m\simeq 1$, since $(C_1\cup\ldots\cup C_{m-1}\cup L)\cap C_m = \{b\}$ and $f|C_m\simeq 1$. By the inductive hypothesis, there exists a locally connected continuum X_1 in X such that $F_1\subset X_1$ and $f|X_1\simeq 1$. Similarly, we obtain a locally connected continuum X_2 in X such that $F_2\subset X_2$ and $f|X_2\simeq 1$.

We now have that $X = X_1\cup X_2$, where X_1 and X_2 are subcontinua of X. Since X is contractible with respect to S^1, Proposition 3.1 implies that X is unicoherent. Thus, $X_1\cap X_2$ is connected. Let $p\colon J\to G$ be the universal covering projection for G. Since $f|X_1\simeq 1$ and $f|X_2\simeq 1$, there exist mappings $\varphi\colon X_1\to J$ and $\psi\colon X_2\to J$ such that $f|X_1 = p\circ\varphi$ and $f|X_2 = p\circ\psi$. Moreover, φ and ψ can be taken so that $\varphi(a) = \psi(a)$ for some point $a\in X_1\cap X_2$. It follows that $\varphi(x) = \psi(x)$ for each $x\in X_1\cap X_2$, and hence, the mapping $g\colon X\to J$ defined by

$$g(x) = \begin{cases}\varphi(x), & \text{if } x\in X_1,\\ \psi(x), & \text{if } x\in X_2\end{cases}$$

is such that $f = p\circ g$. This proves (compare with the proof of [10, Proposition 2.2]) that $f\simeq 1$.

Proposition 3.7 is related to a theorem of Krasinkiewicz [14, p. 237] who proved that the result of Proposition 3.7 is true for a class of one-dimensional metric continua, namely the class of pointed movable one-dimensional metric continua (for the definition see [14]).

3.8. Theorem. *Let $f\colon X\to Y$ be a pseudo-confluent mapping of a compact Hausdorff space X, which is contractible with respect to a graph G, onto a locally connected space Y in $\mathfrak{F}$. Then Y is contractible with respect to G.*

Proof. If G is a simply connected graph, then the theorem is obvious. So assume that G is not simply connected. Then X is contractible with respect to S^1, and hence, by Corollary 3.6, Y is contractible with respect to S^1. The theorem now follows from Proposition 3.7.

Let M be a one-dimensional connected ANR. Then M contains at most finitely many simple closed curves. Hence, there exists a graph $G \subset M$ with fundamental group $\pi_1(M) = \pi_1(G)$ and such that there exists a monotone retraction of M onto G. As a consequence we have the following:

3.9. COROLLARY. *Let f: $X \to Y$ be a pseudo-confluent mapping of a compact Hausdorff space X, which is contractible with respect to a one-dimensional connected* ANR *M, onto a locally connected space Y in $\mathfrak{F}$. Then Y is contractible with respect to M.*

It is clear that contractibility with respect to higher dimensional ANR's is not preserved. For example, for each $n \geqslant 2$ there exists a monotone mapping of I^n onto S^n.

4. Mappings into simply connected ANR's. Let Y be an ANR and let B be a subset of Y. We say that B *is contractible in* Y provided that there exists a homotopy F: $B \times I \to Y$ (I is the unit interval $[0, 1]$) such that $F(b, 0) = b$ and $F(b, 1) = b_0$ for some point $b_0 \in B$ and for each $b \in B$. It is known that ANR's are locally contractible, that is, if Y is an ANR and $y \in Y$, then there exists an open neighbourhood U of y in Y such that U is contractible in Y (see [1, p. 87]). A connected ANR Y is said to be *simply connected* provided that the fundamental group $\pi_1(Y)$ is trivial.

4.1. THEOREM. *Let X be a compact Hausdorff space with* $\dim X \geqslant 3$ *and let Y be a compact, simply connected* ANR. *Then any mapping of X into Y is homotopic to a weakly confluent mapping of X onto Y.*

Proof. Let f: $X \to Y$ be a mapping of X into Y. Since $\dim X \geqslant 3$, there exists a point $x_0 \in X$ such that if U is an open neighbourhood of x_0 in X, then $\dim \mathrm{Cl}(U) \geqslant 3$. Let V be an open set in X with $U \subset \mathrm{Cl}(U) \subset V$. Since Y is an ANR, Y is locally contractible, and hence, we can take V so that $f[\mathrm{Cl}(V)]$ is contractible in Y. Since $\dim \mathrm{Cl}(U) \geqslant 3$, by [12, Theorem 4.3] there exists a weakly confluent mapping g_1': $\mathrm{Cl}(U) \to I^3$ of $\mathrm{Cl}(U)$ onto the 3-cell I^3. Since I^3 is an AR, there exists a mapping g_1: $\mathrm{Cl}(V) \to I^3$ which extends g_1'. Then g_1 is a weakly confluent mapping of $\mathrm{Cl}(V)$ onto I^3.

Consider now the 3-cube I^3. Since $\pi_1(I^3) = 0$, by Walsh's result [22, Proposition 2.0], there exists a monotone mapping p: $I^3 \to Y$ of I^3 onto Y. Put

$$\begin{tikzcd} & & & I^3 \\ \mathrm{Cl}(U) \arrow[urrr, "g_1'"] & \subset & \mathrm{Cl}(V) \arrow[ur, "g_1"] \arrow[r, "f'"'] & Y \arrow[u, "p"'] \end{tikzcd}$$

$f' = f|\mathrm{Cl}(V)$. It is clear, now, that since I^3 is contractible, we have that $p \circ g_1$: $\mathrm{Cl}(V) \to Y$ is homotopic to a constant mapping. Also, since $f[\mathrm{Cl}(V)]$ is contractible in Y, the mapping f': $\mathrm{Cl}(V) \to Y$ is homotopic to a constant mapping. Therefore, there exists a homotopy

$$f_t\colon \mathrm{Cl}(V) \times I \to Y$$

such that $f_1(x) = f'(x)$ and $f_0(x) = p \circ g_1(x)$ for each $x \in \mathrm{Cl}(V)$. By using this homotopy we shall define a weakly confluent mapping from X onto Y which is homotopic to f.

For this let φ: $\mathrm{Cl}(V) \to [0, 1]$ be an Uryshon mapping such that $\varphi[\mathrm{Cl}(U)] = 0$ and $\varphi[\mathrm{Bd}(V)] = 1$. Define a mapping g': $\mathrm{Cl}(V) \to Y$ by putting

$$g'(x) = f_{\varphi(x)}(x)$$

for each $x \in \mathrm{Cl}(V)$. Then for each $x \in \mathrm{Cl}(U)$ we have that $g'(x) = f_{\varphi(x)}(x) = f_0(x) = p \circ g_1(x)$, and for each $x \in \mathrm{Bd}(V)$ we have that $g'(x) = f_{\varphi(x)}(x) = f_1(x) = f'(x) = f(x)$. We finally define a function g: $X \to Y$

$$g(x) = \begin{cases} g'(x), & \text{if } x \in \mathrm{Cl}(V), \\ f(x), & \text{if } x \in X \setminus \mathrm{Cl}(V). \end{cases}$$

Since $g'|\mathrm{Bd}(V) = f|\mathrm{Bd}(V)$, we have that g is continuous. In order to see that g is a weakly confluent mapping of X onto Y, notice that $g|\mathrm{Cl}(U) = p \circ g_1'$. Since p is monotone, it is weakly confluent, and by [18, 1.5], $p \circ g_1'$ is a weakly confluent mapping of $\mathrm{Cl}(U)$ onto Y. It is now apparent that $f \simeq g$, since they are both homotopic to a constant mapping in $\mathrm{Cl}(V)$ and they coincide on $X \setminus \mathrm{Cl}(V)$. This completes the proof of the theorem.

We saw in the proof of Theorem 4.1 that every simply connected, compact, connected ANR Y is the weakly confluent image of any given compact Hausdorff space with dimension at least three. By the lifting theorem [21, p. 67], for every mapping f: $Y \to S^1$ there exists a mapping φ: $Y \to \boldsymbol{R}$ such that $f(y) = e^{2\pi i \varphi(y)}$ for each $y \in Y$. Hence, Y is contractible with respect to S^1. The converse is not true. The projective plane P_2 is an example of an ANR which is contractible with respect to S^1 but which is not simply connected. The following problems can be posed:

PROBLEM 1. Let X be a compact connected Hausdorff space with dimension at least three, and let Y be a compact, connected ANR which is contractible with respect to S^1. Is there a weakly confluent mapping of X onto Y?

PROBLEM 2. Let X be a compact connected Cantor manifold with dimension at least three, and let Y be a compact connected ANR. Is it true that any mapping f: $X \to Y$ is homotopic to a weakly confluent mapping of X onto Y if and only if f^*: $H^1(Y) \to H^1(X)$ is a monomorphism?

In general Theorem 4.1 is not true if we do not assume that Y is a simply connected ANR. Corollary 3.6 guarantees that there is no weakly confluent mapping of I^3 onto the torus $S^1 \times S^1$.

In [11, Example 4.2] a two-dimensional continuum was constructed which admits an essential mapping onto S^2, but which does not admit any semi-confluent mapping onto S^2. By using exactly the same technique one can construct n-dimensional continua ($n \geqslant 2$) which admit no semi-confluent mapping onto S^n. Thus, in Theorem 4.1 weakly confluent mappings cannot be replaced by semi-confluent ones.

Added in proof. Problems 1 and 2 have been answered in the negative by J. Grispolakis and E. D. Tymchatyn.

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