Integrating intuitionistic and classical theories

by

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Abstract. A completeness theorem for $\mathbf{HAS}_\omega$ (the intuitionistic theory of species + the ω-rules) is given using Beth models. The completeness theorem is then used to show that $\mathbf{HAS}_\omega$ has the disjunction property. A conservative extension of **HAS**, called **UHAS**, is also defined and it has the property that $\mathbf{HAS}_\omega$, $\mathbf{UHAS}_\omega$ and $\mathbf{UHAS}_{\omega^c}$ (**UHAS** + the recursively restricted ω-rules) are all equivalent.

Contents

§ 0. Introduction. The principal aim of this note is to give a completeness theorem for $\mathbf{HAS}_\omega$, the intuitionistic theory of species with ω-rules. The models used are, except for minor modifications, the structures originally introduced by E. W. Beth.

The completeness theorem is then used to show that some properties of the consequence relation of $\mathbf{HAS}_\omega$ can be derived by model-theoretical methods; we only consider the disjunction property, however it is clear that the method is applicable in other instances.

Since it is known that $\mathbf{HAS}_\omega$ is not equivalent to $\mathbf{HAS}_{\omega^c}$ (**HAS** + the recursively restricted ω-rules) we introduce the system **UHAS**, a conservative extension of **HAS**, which has the property that the infinitary extensions $\mathbf{UHAS}_\omega$ and $\mathbf{UHAS}_{\omega^c}$ are equivalent conservative extensions of $\mathbf{HAS}_\omega$.

It may be of some interest to observe both $\mathbf{HAS}_\omega$ and **UHAS** involve a mixture of classical and intuitionistic notions. $\mathbf{HAS}_\omega$ yields all the classically true arithmetical statements; **UHAS** has, in addition to the intuitionistic disjunction, a very weak kind of classical disjunction (see Prawitz [5] for similar considerations).

§ 1. **The system HAS.** To all intent and purposes by the system **HAS** we understand the (two sorted) impredicative theory of species given in Troelstra [9].

One of the minor modifications will be that we shall only use unary predicate variables (and called them *species* variables). Also instead of writing "Xx" we shall often write "$x \in X$".

We shall distinguish between the logical system and its axioms; furthermore the axioms will always be sentences (i.e. formulae having no number or species parameters). Thus **HAS** is the intuitionistic system of impredicative analysis, and HAS is the set of axioms of HAS.

Observe that the following axiom of extensionality

$$\forall X \forall x \forall y [x \in X \wedge x = y \supset y \in X]$$

is assumed to be in HAS.

We shall use the symbol "$\vdash$" for derivability in (intuitionistic) second-order logic. In particular a theorem of **HAS** is any formula A such that HAS $\vdash A$.

The following theorem will be needed later on and since it is not usually included in the developments of the intuitionistic theory of species we include it here.

1.1. Theorem. *For any formula $A(P, S, y)$ of* **HAS**:

$$\mathrm{HAS} \vdash \forall X \forall P \forall Q \forall y (\forall x (x \in P \equiv x \in Q)) \supset (A(P, S, y) \equiv A(Q, S, y)).$$

Proof. By induction on the length of the formula A.

§ 2. **The unfaithful intuitionistic theory of species.** The extension **UHAS** of **HAS** will be obtained by adding a very restricted classical disjunction, and because the classical disjunction is not allowed to interfere with the other connectives we obtain that **UHAS** is a conservative extension of **HAS**.

2.1. *The formulae of* **UHAS** are defined so that:

(1) every formula of **HAS** is a formula of **UHAS**,

(2) if $\Phi_1, \ldots, \Phi_n$ are formulae of **UHAS** then $\Phi_1 \oplus \ldots \oplus \Phi_n$ is a formula of **UHAS**.

2.2. *The axioms of* **UHAS** are formulae of **UHAS** the form $\Phi_1 \oplus A \oplus \Phi_2$ where (one or both of Φ_1, Φ_2 may be absent and) A is an axiom of **HAS**.

2.3. *The rules of inference of* **UHAS** are such that if $|\!\!\rightarrow$ is the derivability relation in **UHAS**, Θ, Φ_1, Φ_2 are formulae of **UHAS** then:

(1) $\Phi_1 \mid\!\!\rightarrow \Phi_1 \oplus \Phi_2$,

(2) $\Phi_2 \mid\!\!\rightarrow \Phi_1 \oplus \Phi_2$,

(3) if $\Phi_1 \mid\!\!\rightarrow \Theta$ and $\Phi_2 \mid\!\!\rightarrow \Theta$ then $\Phi_1 \oplus \Phi_2 \mid\!\!\rightarrow \Theta$;

(4) if B is a consequence of $A_0, \ldots, A_i$ by an application of a rule of inference of **HAS**, then for all **UHAS** formulae Φ

$$\Phi \oplus A_0, \ldots, \Phi \oplus A_i \mid\!\!\rightarrow \Phi \oplus B.$$

Although we are not particularly interested in how one obtains the derivability relation $|\!\!\rightarrow$, one way is to first introduce sequents

$$\Theta_0, \ldots, \Theta_{i-1} \Rightarrow \Theta$$

of **UHAS** formulae and then the appropriate sequent-calculus is developed. Then $T \mid\!\!\rightarrow \Theta$ could stand for " there are finitely many **UHAS** formulae $\Theta_0, \ldots, \Theta_{i-1}$ in T such that the sequent $\Theta_0, \ldots, \Theta_{i-1} \Rightarrow \Theta$ is derivable in the calculus.

The following lemma gives some easily verified properties.

2.4. Lemma.

(1) $\Theta_0 \oplus \Theta \oplus \Theta_1 \oplus \Theta \mid\!\!\rightarrow \Theta_0 \oplus \Theta \oplus \Theta_1$,

(2) $\Theta_0 \oplus \Theta_1 \oplus \Theta_2 \mid\!\!\rightarrow \Theta_2 \oplus \Theta_1 \oplus \Theta_0$.

The principal theorem of this section is the following:

2.5. Theorem. **UHAS** *is conservative over* **HAS**.

Proof. Let $\Gamma \cup \{A\}$ be a finite set of formulae of **HAS**. Then it has to be shown that the following two conditions are equivalent:

(1) $\Gamma \vdash A$,

(2) $\Gamma \mid\!\!\rightarrow A$.

That (1) $\Rightarrow$ (2) is immediate since **HAS** is a subsystem of **UHAS**. To prove that (2) $\Rightarrow$ (1) we need to analyze in some detail the $|\!\!\rightarrow$ relation. So let us assume that $|\!\!\rightarrow$ arose out of a sequent calculus. Thus to assume (2) is equivalent to having a proof tree $\mathcal{T}$ whose end-sequent is

$$B_1, \ldots, B_r \Rightarrow A$$

where $\Gamma = \{B_1, \ldots, B_r\}$. Suppose that Φ is an **UHAS** formula of the form $\Phi_1 \oplus F \oplus \Phi_2$ (where one, or both of Φ_1, Φ_2 may be absent). Then let us agree to call the **HAS** formula F an **HAS**-subformula of Φ. Next suppose we are given a sequent $\mathfrak{S}$ $\Phi_0, \ldots, \Phi_{i-1} \Rightarrow \Theta$ of **UHAS** formulae. Then a sequent $F_0, \ldots, F_{i-1} \Rightarrow G$ is an **HAS**-*subsequent of* $\mathfrak{S}$ if and only if F_0 is an **HAS**-subformula of $\Phi_0, \ldots, F_{i-1}$ is an **HAS**-subformula of Φ_{i-1} and G is an **HAS**-subformula of Θ. Finally given a proof tree $\mathcal{T}$ of **UHAS** sequents then a tree $\mathcal{T}'$ of **HAS** sequents is called an **HAS**-*restriction of* $\mathcal{T}$ if and only if it has exactly the same tree structure and the sequent occurring at a node of $\mathcal{T}'$ is an **HAS**-subsequent of the sequent occurring at the homologous node of $\mathcal{T}$.

Now if $\mathcal{T}$ is a proof tree of the sequent $B_1, \ldots, B_r \Rightarrow A$ then every **HAS**-restriction of $\mathcal{T}$ will have the same endsequent. The proof that (2) $\Rightarrow$ (1) is completed by showing that every proof tree in **UHAS** has an **HAS**-restriction consisting entirely of **HAS** provable sequents. The latter is done by induction on the length of the proof tree.

In view of the above theorem we shall not bother to distinguish between $\vdash$ and $|\!\!\rightarrow$.

§ 3. The infinitary extensions $\mathbf{HAS}_\omega$ and $\mathbf{UHAS}_\omega$. The way by which we shall inject some classical reasoning into **HAS** is by adding to the rules of inference of **HAS** the ω-rules.

$$(\omega_\forall)\qquad \frac{\text{From}\quad C\supset A(0^{(n)})\quad \text{for } n=0,1,2,\ldots}{\text{To conclude}\quad C\supset \forall x\, A(x)}$$

and

$$(\omega_\exists)\qquad \frac{\text{From}\quad A(0^{(n)})\supset C\quad \text{for } n=0,1,2,\ldots}{\text{To conclude}\quad \exists x A(x)\supset C\,.}$$

The resulting system shall be denoted by $\mathbf{HAS}_\omega$. Note that $\mathbf{HAS}_\omega$ and **HAS** have the same axioms; however, the derivations in $\mathbf{HAS}_\omega$ are no longer finite objects. If Γ is a finite set of formulae of **HAS** then "$\Gamma \vdash_\omega A$" is to be understood as an abbreviation for the statement that there is a derivation from Γ of A using the logical rules of **HAS** and the ω-rules.

If Δ is an infinite set then we set $\Delta \vdash_\omega A$ if and only if for some finite subset $\Gamma\subseteq\Delta$, $\Gamma\vdash_\omega A$.

Since we do not require that the derivations of the premisses of the ω-rules be given effectively it is not surprising that the arithmetical part of $\mathbf{HAS}_\omega$ coincides with classical truth, or more precisely:

3.1. Theorem. *If A is an arithmetical sentence of* **HAS** (*i.e. a sentence without any species parameters or variables*) *then*:

(i) *if A is* (*classically*) *true then* $\mathbf{HAS}\vdash_\omega A$,

(ii) *if A is* (*classically*) *false then* $\mathbf{HAS}\vdash_\omega \neg A$.

Proof. A trivial induction on the complexity of the arithmetical sentence A.

Note however the addition of the unrestricted ω-rules does not make $\mathbf{HAS}_\omega$ a completely classical system. For example, we shall show that the sentence $\forall X\forall x(x\in X\vee\neg x\in X)$ is not provable in $\mathbf{HAS}_\omega$ (see Section 5).

We can of course, add the corresponding ω-rules to **UHAS** to obtain the system $\mathbf{UHAS}_\omega$ which turns out to be a conservative extension of $\mathbf{HAS}_\omega$.

The ω-rules for $\mathbf{UHAS}_\omega$ are:

$$(\omega_\forall)\qquad \frac{\text{From}\quad \Phi_1\oplus(C\supset A(0^{(n)}))\oplus\Phi_2\quad \text{for } n=0,1,2,3,\ldots}{\text{To conclude}\quad \Phi_1\oplus(C\supset\forall x A(x))\oplus\Phi_2}$$

$$(\omega_\exists)\qquad \frac{\text{From}\quad \Phi_1\oplus(A(0^{(n)})\supset C)\oplus\Phi_2\quad \text{for } n=0,1,2,\ldots}{\text{To conclude}\quad \Phi_1\oplus(\exists x A(x)\supset C)\oplus\Phi_2}$$

where Φ_1, Φ_2 are formulae of **UHAS**.

By essentially the same method as that used in the proof of Theorem 2.5 (except that some form of transfinite induction has to be used) we obtain

3.2. Theorem. *If $\Delta\cup\{A\}$ is a set of formulae of* **HAS** *then the following two conditions are equivalent*:

(1) $\Delta\vdash_\omega A$,

(2) $\Delta \mid\!\!\rightarrow_\omega A$.

3.3. Corollary. *For any formula A of* **HAS**, *A is a theorem of* $\mathbf{HAS}_\omega$ *if and only if A is a theorem of* $\mathbf{UHAS}_\omega$.

§ 4. The recursively restricted versions of $\mathbf{HAS}_\omega$ and $\mathbf{UHAS}^\omega$. One of the possible criticisms of the system $\mathbf{HAS}_\omega$, and of its conservative extension $\mathbf{UHAS}_\omega$, is that the derivations are no longer finite objects and thus do not belong to "Proof-theory". Now it is well known that one cannot have the power of systems like $\mathbf{HAS}_\omega$ and still insist on finite derivations. A compromise (originated for classical systems by Shoenfield [6] and Novikov [4]) is to have infinite but recursively given derivations. That is, by suitable codings a derivation in $\mathbf{HAS}_\omega$ can be viewed as a number theoretic function, and so the compromise is in effect to restrict oneself to derivations given by recursive functions. An alternate, but equivalent way is to restrict the applications of the ω-rules to those instances where there is a recursive function giving the (codes of the) derivations of the premisses.

Let $\mathbf{HAS}_{\omega^c}$ and $\mathbf{UHAS}_{\omega^c}$ be the subsystems of $\mathbf{HAS}_\omega$ and $\mathbf{UHAS}_\omega$ respectively obtained by applying such recursive restrictions on the ω-rule.

Using the appropriate form of realizability (and probably also normal form theorems) it can be shown that:

$\mathbf{HAS}_\omega$ *is not a conservative extension of* $\mathbf{HAS}_{\omega^c}$.

On the other hand, because in $\mathbf{UHAS}_\omega$ we have available the (weak) classical disjunction $\oplus$, the method introduced by Takahashi [8] (see also López-Escobar [3]) can be applied to show that

$\mathbf{UHAS}_\omega$ *is a conservative extension of* $\mathbf{UHAS}_{\omega^c}$.

Hence combining the above with Corollary 3.3 to Theorem 3.2 we obtain that:

as far as formulae of **HAS** *are concerned the systems* $\mathbf{HAS}_\omega$ *and* $\mathbf{UHAS}_{\omega^c}$ *are equivalent.*

Each of the systems $\mathbf{HAS}_\omega$ and $\mathbf{UHAS}_{\omega^c}$ has its own merits (and demerits?): $\mathbf{HAS}_\omega$ is much more suitable for model-theoretic discussion or when one is interested in obtaining a derivation of some formula and is not particularly interested in the form of the derivation (for example, in completeness results); $\mathbf{UHAS}_{\omega^c}$ is more suitable to pacify one's constructive (and formalist) conscience. It also shows that it is possible to mix classical and intuitionistic concepts within one formal system without trivializing the system.

§ 5. A Beth semantics for $\mathbf{HAS}_\omega$. It is already common practice to require the domain of individuals of a Beth structure to be the set N of natural numbers. So it is a natural step to require that the domain of individuals for the Beth models suitable for interpreting $\mathbf{HAS}_\omega$ to be N *together with all the usual arithmetical functions.* In order to have a way of interpreting the species variables we make the following definition:

5.1. Definition. Suppose $\mathfrak{M} = (M, \preccurlyeq)$ is a partially ordered set. Then by an $\mathfrak{M}$-*species* we understand a function h such that for each node $n \in M$, $h(n) \subseteq N$ and such that $h(n_1) \subseteq h(n_2)$ whenever $n_1 \preccurlyeq n_2$.

If S is an $\mathfrak{M}$-species then instead of writing "$S(n)$" we shall write "$S \restriction n$" (and read it: *S restricted to the node n*).

5.2. Definition. A *Beth ω-structure* is a system $\mathfrak{B} = (M, \preccurlyeq, D)$ such that $(M, \preccurlyeq)$ is a partial ordering and D is a set of $(M, \preccurlyeq)$-species.

In order to define satisfaction in the Beth ω-structure $\mathfrak{B} = (M, \preccurlyeq, D)$ it is convenient to assume that we have a species parameter S for each element of D. Then the formula A is forced (or true) at k in $\mathfrak{B}$; $\mathfrak{B}, k \models A$ (or simply: $k \models A$) if and only if one of the following conditions holds:

(1) A is a true atomic number-theoretic sentence,

(2) A is the atomic formula $0^{(n)} \in S$ and there is a bar B for k (i.e. $B \subseteq M$ and each path in M through k meets B) such that for all $k' \in B$, $n \in S \restriction k'$,

(3) A is $B \wedge C$ and $k \models B$, $k \models C$.

(4) A is $B \vee C$ and there is a bar B for k such that for all $k' \in B$, either $k' \models B$, or $k' \models C$,

(5) A is $B \supset C$ and for all $k' \succcurlyeq k$, if $k' \models B$ then $k' \models C$,

(6) A is $\neg B$ and for all $k' \succcurlyeq k$, k' not $\models B$,

(7) A is $\forall x B(x)$ and for all $n \in N$, $k \models B(0^{(n)})$,

(8) A is $\exists x B(x)$ and there is a bar B for k such that for all $k' \in B$ there is an $n \in N$ such that $k \models B(0^{(n)})$,

(9) A is $\forall X B(X)$ and for all $S \in D$, $k \models B(S)$,

(10) A is $\exists X B(X)$ and there is a bar B for k such that for all $k' \in B$ there is an $S \in D$ such that $k' \models B(S)$.

It is straightforward to verify that if A is a classically true arithmetical sentence and $\mathfrak{B}$ a Beth ω-structure then A is satisfied at every node of $\mathfrak{B}$. In fact more can be said.

5.3. Definition. HAS_0 is the set of sentences obtained by deleting from HAS all the instances of comprehension.

5.4. Theorem. *If $\mathfrak{B} = (M, \preccurlyeq, D)$ is a Beth ω-structure and A a sentence such that $\mathrm{HAS}_0 \vdash_\omega A$ then for all nodes k of M, $\mathfrak{B}, k \models A$.*

Proof. In view of what is known about satisfaction in Beth structures we immediately conclude that the theorem holds for the intuitionistic propositional axioms. Similarly it carries through modus ponens. The ω-rule causes no problem. For $\forall X B(X)$ it suffices to observe that if $\mathrm{HAS}_0 \vdash_\omega B(S)$ then for any other parameter S', $\mathrm{HAS}_0 \vdash_\omega B(S')$.

5.5. Definition. A sentence A is *true in a Beth ω-structure* $\mathfrak{B} = (M, \preccurlyeq, D)$ if and only if for all $k \in M$, $\mathfrak{B}, k \models A$. We express it in symbols: $\mathfrak{B} \models A$, and then call $\mathfrak{B}$ an *ω-model* of A.

If Γ is a set of sentences then $\mathfrak{B}$ is an ω-model of Γ if and only if for each sentence A of Γ, $\mathfrak{B} \models A$. In particular, Theorem 5.4 tells us that every Beth ω-structure $\mathfrak{B}$ is an ω-model of HAS_0. However, our interest lies more in the ω-models of HAS.

The following give us a way of obtaining ω-models for HAS.

5.6. Definition. The full ω-structure on a partial ordering $(M, \preccurlyeq)$ is the *Beth ω-structure* $\mathfrak{M} = (M, \preccurlyeq, D^*)$ where D^* consists of all $(M, \preccurlyeq)$-species.

The following lemmas will be useful in the proof of Theorem 5.9 (and also in other places).

5.7. Lemma. *If $\mathfrak{B}$ is a Beth ω-structure A a formula of* **HAS**, *k, l nodes of $\mathfrak{B}$ $k \preccurlyeq l$, and $\mathfrak{B}, k \models A$ then $\mathfrak{B}, l \models A$.*

5.8. Lemma. *If $\mathfrak{B}$ is a Beth ω-structure and A a formula of* **HAS**, *then the following conditions are equivalent:*

(i) $\mathfrak{B}, k \models A$,

(ii) *there is a bar B for k such that for all $k' \in B$ $\mathfrak{B}, k' \models A$.*

5.9. Theorem. *The full ω-structure on a partial ordering $(M, \preccurlyeq)$ is an ω-model of* HAS.

Proof. In view of Theorem 5.4 it only remains to verify that (all instances of) the comprehension axiom are true in the full model. Let us consider a typical example

$$\forall X \exists Y \forall x (x \in Y \equiv A(X, x))$$

where $A(X, x)$ is some formula of **HAS** in which Y does not occur. Let k be a node of $(M, \preccurlyeq)$; we will show that for all $S \in D^*$

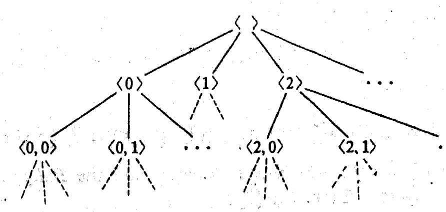
$$\mathfrak{M}, k \models \exists Y \forall x (x \in Y \equiv A(S, x)) .$$

Let, for each mode l of $(M, \preccurlyeq)$, $T_l = \{n \mid \mathfrak{M}, l \models A(S, 0^{(n)})\}$. Lemma 5.7 then tells us that $\{T_l \colon l \in M\}$ is an $(M, \preccurlyeq)$-species. From the fullness of the model we get that $\{T_l \colon l \in M\}$ belongs to D^* and since it satisfies the formula

$$\forall x (x \in Y \equiv A(S, x))$$

at the node k we have completed the proof.

5.10. Definition. The *universal spread* $\mathfrak{U}$ is the spread of all finite sequences of natural numbers with the initial-segment ordering



5.11. DEFINITION. The *van Dalen ω-model* $\mathfrak{D}$ is the full ω-structure on the universal spread.

5.12. COROLLARY. $\mathfrak{D}$ *is an ω-model of* HAS.

5.13. COROLLARIES.

(1) *If S is a sentence such that $\mathfrak{D}$ not $\models S$, then* HAS *not* $\vdash_\omega S$.

(2) *If S is a sentence such that $\mathfrak{D} \models S$, then* HAS *not* $\vdash_\omega \neg S$.

The full models are in certain sense maximal ω-models. In the case of classical analysis H. Friedman has shown that there are no minimal ω-models. Consequently, for some partial orderings (e.g. consisting of exactly one node), there will not be minimal Beth ω-models.

PROBLEM. What is the precise situation concerning minimal Beth ω-models of HAS?

In the 1975 article van Dalen showed that $\mathfrak{D}$ was (was not) a model of various intuitionistic meaningful formulae and thus obtained consistency results for **HAS**. Using Corollary 5.13 we obtain analogous results about the system $\mathbf{HAS}_\omega$.

5.14. THEOREM. $\mathfrak{D}$ *is not an ω-model of* $\forall x \forall X(x \in X \vee x \notin X)$.

Proof. Suppose on the contrary that $\mathfrak{D}$ were a model of $\forall X \forall x(x \in X \vee x \notin X)$. Then in particular we would obtain that for all S

$$\mathfrak{D}, \langle\ \rangle \models (0 \in S \vee 0 \notin S).$$

A contradiction is obtained by considering the $\mathfrak{U}$-species S such that

$$S \upharpoonright \langle\ \rangle = \varnothing,$$

$$S \upharpoonright \langle a_0, \ldots, a_n\rangle = \begin{cases} \varnothing & \text{if all } a_i = 0,\ i = 0, 1, \ldots, n, \\ \{0\} & \text{otherwise.}\end{cases}$$

5.15. COROLLARY. $\mathbf{HAS}_\omega$ *is not the same as classical analysis.*

To show that $\mathbf{HAS}_\omega$ is not the same as classical analysis (although in view of Theorem 3.1 $\mathbf{HAS}_\omega$ contains the arithmetical part) we could also have used the following theorem of classical logic:

$$\forall X \forall Y[\forall x(0 \in Y \vee x \in X) \supset 0 \in Y \vee \forall x(x \in X)].$$

5.16. DEFINITIONS.

(1) MP is the sentence:

$$\forall X\{\forall x(x \in X \vee x \notin X) \supset (\neg\neg\exists x(x \in X) \supset \exists x(x \in X))\},$$

(2) IP_0 is the sentence

$$\forall X \forall Y\{\forall x(x \in X \vee x \notin X) \supset [(\forall x(x \in X) \supset \exists y(y \in Y)) \supset \exists y(\forall x(x \in X) \supset y \in Y)]\}.$$

MP and IP_0 are often called *Markov's Principle* and the *Independence of Premiss Principle* respectively (see Troelstra [9]).

5.17. DEFINITIONS. UP!, $\mathrm{UP^{cl}}$ and UP are the following schemas:

UP! $\quad \forall X \exists! x A(X, x) \supset \exists! x \forall X A(X, x)$,

$\mathrm{UP^{cl}}$ $\quad \forall X \exists x A(X, x) \supset \exists x \forall X A(X, x)$, where A contains only X and x free,

UP $\quad \forall X \exists x A(X, x) \supset \exists x \forall X A(X, x)$.

5.18. THEOREM (van Dalen [1]). (1) $\mathfrak{D}$ not $\models$ MP, (2) $\mathfrak{D}$ not $\models \mathrm{IP}_0$, (3) $\mathfrak{D} \models$ UP!, (4) $\mathfrak{D} \models \mathrm{UP^{cl}}$, (5) $\mathfrak{D}$ not $\models$ UP.

5.19. COROLLARY. *Neither* MP, UP, *nor* IP_0 *are theorems of* $\mathbf{HAS}_\omega$.

By similar methods it can be shown that

$$\forall X(\forall x \neg\neg x \in X \supset \neg\neg \forall x\, x \in X)$$

is not a theorem of $\mathbf{HAS}_\omega$. On the other hand, in view of Theorem 3.1 we have that for arithmetic $A(x)$

$$\forall x \neg\neg A(x) \supset \neg\neg \forall x A(x)$$

is a theorem of $\mathbf{HAS}_\omega$.

§ 6. A completeness theorem.

In this section we shall show how to associate with each (consistent) set Γ of sentences a Beth ω-structure $\mathfrak{B}_\Gamma$ such that for all sentences A

$$\mathfrak{B}_\Gamma \models A \quad \text{iff} \quad \Gamma \vdash_\omega A.$$

But first we need to get organized. So we make the following arrangements.

$P_0, P_1, \ldots$ is an enumeration, without repetitions, of all the species parameters.

$\mathrm{Sp}(A)$ is the set species parameters occurring in A.

$\mathrm{Sp}(\Delta) = U_{A \in \Delta} \mathrm{Sp}(A)$ if Δ is a set of formulae.

$F_0, F_1, \ldots$ is an enumeration, without repetition, of all the formulae of **HAS**.

$\langle x, y, z\rangle$ is a (1-1) function from $N \times N \times \{0, 1\}$ onto N such that $\langle x, y, 1\rangle = \langle x, y, 0\rangle + 1$.

We shall first define a (classical) spread of finite sequences of natural numbers. At each node we shall attach a set of formulae and a (finite) set of species parameters. The definition of the spread (and the attached sets) proceeds inductively.

Basis step. $\Delta_{\langle\ \rangle} = \Gamma$, $S_{\langle\ \rangle} = \varnothing$.

Inductive step. Suppose $\Delta_{\langle a_0, \ldots, a_i\rangle}$ and $S_{\langle a_0, \ldots, a_i\rangle}$ have already been defined. We now proceed to define $\Delta_{\langle a_0, \ldots, a_i, j\rangle}$ and $S_{\langle a_0, \ldots, a_i, j\rangle}$ for $j = 0, 1, \ldots$

First of all determine the unique numbers p, q, r such that:

$$i = \langle p, q, r\rangle.$$

Case 1. $r = 0$. Let $F = F_p$, $k = \langle a_0, \ldots, a_i\rangle$.

Subcase 1.1. $\mathrm{Sp}(F_p) \subseteq \mathrm{Sp}(\Delta_k)$ and $\Delta_k \vdash_\omega F_p$. Then set

$$\Delta_{k^\frown\langle 1\rangle} = \Delta_k \cup \{F\},$$
$$S_{k^\frown\langle 1\rangle} = S_k \cup \{P_i\}$$

and let $\Delta_{k^\frown\langle j\rangle}$, $S_{k^\frown\langle j\rangle}$ be undefined otherwise.

Subcase 1.2. $\mathrm{Sp}(F_p) \subseteq \mathrm{Sp}(\Delta_k)$, Subcase 1.1 does not apply and

$$\Delta_k \cup \{F\} \text{ not } \vdash_\omega 0 = 1.$$

Then set

$$\Delta_{k^\frown\langle 4\rangle} = \Delta_k, \qquad \Delta_{k^\frown\langle 1\rangle} = \Delta_k \cup \{F\},$$
$$S_{k^\frown\langle 0\rangle} = S_k \cup \{P_i\}, \qquad S_{k^\frown\langle 1\rangle} = S_k \cup \{P_i\}$$

and undefined otherwise.

Subcase 1.3. Neither 1.1 nor 1.2 applies then set

$$\Delta_{k^\frown\langle 0\rangle} = \Delta_k,$$
$$S_{k^\frown\langle 0\rangle} = S_k \cup \{P_i\}$$

and undefined otherwise.

Case 2. $r = 1$. Then $i = \langle p, q, 1\rangle = \langle p, q, 0\rangle + 1$. And thus Case 1 has just been applied to F. We proceed by cases depending on the syntactical form of F.

Subcase 2.1. F is not of the forms $A \vee B$, $\exists x A$ nor $\exists X A$. Then we set

$$\Delta_{k^\frown\langle 0\rangle} = \Delta_k,$$
$$S_{k^\frown\langle 0\rangle} = S_k$$

and undefined otherwise.

Subcase 2.2. F is of the form $A_1 \vee A_2$. If F was not added at the previous stage then proceed as in Subcase 2.1. Now if F was added at the previous stage then $\Delta_{\langle a_0,\dots,a_{i-1}\rangle} \cup \{F\}$ not $\vdash_\omega 0 = 1$. It follows then that for some $u \leqslant 2$, $\Delta_{\langle a_0,\dots,a_{i-1}\rangle} \cup \cup \{F, A_u\}$ not $\vdash_\omega 0 = 1$. For the sake of argument let us suppose that $u = 1, 2$. Then we set

$$\Delta_{k^\frown\langle 1\rangle} = \Delta_k \cup \{A_1\}, \qquad \Delta_{k^\frown\langle 2\rangle} = \Delta_k \cup \{A_2\},$$
$$S_{k^\frown\langle 1\rangle} = S_k, \qquad S_{k^\frown\langle 2\rangle} = S_k.$$

If $\Delta_k \cup \{A_u\} \vdash_\omega 0 = 1$ then we would have left $\Delta_{k^\frown\langle u\rangle}$ and $S_{k^\frown\langle u\rangle}$ undefined.

Subcase 2.3. $F = \exists x A$ and F had been added at the previous stage. Let $n_0, n_1, \dots$ be the set of natural numbers such that

$$\Delta_k \cup \{A(O^{(n_u)})\} \text{ not } \vdash_\omega 0 = 1.$$

Then for $u = 0, 1, \dots$ we define

$$\Delta_{k^\frown\langle n_u+1\rangle} = \Delta_k \cup \{A(O^{n_u})\},$$
$$S_{k^\frown\langle n_u+1\rangle} = S_k.$$

Subcase 2.4. $F = \exists X A$ and F had been added at the previous stage. Let m be the least natural number such that $P_m \notin S_k$. Then set

$$\Delta_{k^\frown\langle 1\rangle} = \Delta_k \cup \{A(P_m)\},$$
$$S_{k^\frown\langle 1\rangle} = S_k \cup \{P_m\}.$$

Observe that $\Delta_{k^\frown\langle 1\rangle}$ not $\vdash_\omega 0 = 1$.

End of the definition of the spread, Δ and S.

6.1. DEFINITIONS.

(i) $M_\Gamma = \{\langle a_0, \dots, a_{i-1}\rangle : \Delta_{\langle a_0,\dots,a_{i-1}\rangle}$ is defined$\}$.

(ii) $k \prec_\Gamma k'$ iff $k, k' \in M_\Gamma$ and k is a proper initial segment of k'.

(iii) For any species parameter P and $k \in M_\Gamma$ let $P \restriction k = \{n : \Delta_k \vdash_\omega (0^{(n)} \in P)\}$.

(iv) $\mathfrak{B}_\Gamma = (M_\Gamma, \preccurlyeq_\Gamma, \{\{P \restriction k : k \in M_\Gamma\} : P$ a species parameter$\})$.

It is fairly evident that $\mathfrak{B}_\Gamma$ is a Beth ω-structure. In order to prove that $\mathfrak{B}_\Gamma$ is an ω-model of Γ we need to know some further facts about the Δ_k, $k \in M_\Gamma$. For the remainder of this section we shall follow the convention that $\alpha, \beta, \dots$ are paths in M_Γ, i.e. for all n; $\bar{\alpha}n, \bar{\beta}n, \dots \in M_\Gamma$. We will also abbreviate $\exists n(k = \bar{\alpha}n)$ by $k \in \alpha$.

6.2. LEMMA. *If* $\mathrm{Sp}(A \supset B) \subseteq S_k$ *then the following two conditions are equivalent*:

(1) $\Delta_k \vdash_\omega (A \supset B)$,

(2) $\forall \alpha_{k \in \alpha} \forall r [A \in \Delta_{\bar{\alpha}r} \to \forall \beta_{\bar{\alpha}r \in \beta} \exists t (B \in \Delta_{\bar{\beta}t})]$.

Proof. (1) $\Rightarrow$ (2). Suppose $\Delta_k \vdash_\omega (A \supset B)$ and $k' \succcurlyeq k$ and $A \in \Delta_{k'}$. Then $\Delta_{k'} \vdash_\omega B$. Now consider a path α such that $k' \in \alpha$. Determine the unique p such that $B = F_p$. Consider then a $t > \mathrm{lh}(k')$ such that $t = \langle p_0, q, 0\rangle$. Then Subcase 1.1 of the definition of Δ will apply and $B \in \Delta_{\bar{\alpha}(t+1)}$.

(2) $\Rightarrow$ (1). Assume (2). Let $i = \mathrm{lh}(k)$. Determine the unique p such that $A = F_p$. Next consider the subspread T of M_Γ which

(a) contains k and all its initial segments,

(b) the nodes below k are obtained by modifying the definition of Δ so that in Subcase 1.2 we choose the right side when $F = A$ otherwise we take the left path.

If $\Delta_k \cup \{A\} \vdash_\omega 0 = 1$ then (1) clearly holds. Thus we might as well assume that $\Delta_k \cup \{A\}$ not $\vdash_\omega 0 = 1$. It then follows that for all $k' \in T$

$$\Delta_{k'} \cup \{A\} \text{ not } \vdash_\omega 0 = 1.$$

Next suppose that α is a path through T such that $k \in \alpha$. Then there is a smallest r such that $r \geqslant \mathrm{lh}(k)$ and $A \in \Delta_{\bar{\alpha}r}$. Then

$$\forall \beta_{\bar{\alpha}r \in \beta \in T} \exists t (B \in \Delta_{\bar{\beta}t}).$$

Thus

$$\forall \beta_{\bar{\alpha}r \in \beta \in T} \exists t (\Delta_{\bar{\beta}t} \vdash_\omega B).$$

Consider those $\Delta_{\bar{\beta}t} \vdash_\omega B$. Each $\Delta_{\bar{\beta}t}$ will be of the form $\Delta_k \cup \{C_0, \dots, C_s, A\}$ so that we have

$$\Delta_k \cup \{C_0, \dots, C_s, A\} \vdash_\omega B.$$

Using Bar induction we see that $C_0, \dots, C_s$ are superfluous and finally we conclude

$$\Delta_k \cup \{A\} \vdash_\omega B$$

from which $\Delta_k \vdash_\omega (A \supset B)$ follows.

Using similar methods we obtain the following:

6.3. LEMMA. *If* $\mathrm{SP}(A) \subseteq S_k$ *then the following two conditions are equivalent*:

(1) $\Delta_k \vdash_{\dot{\omega}} A$,

(2) $\forall \alpha_{k \in \alpha} \exists t (A \in \Delta_{\bar{\alpha}t})$.

An immediate consequence of Lemma 6.3 and of the definition $\Delta_{\langle\,\rangle}^{n} = \Gamma$ is that for sentences A

$$\Gamma \vdash_{\omega} A \quad \text{iff} \quad \forall \alpha \exists t (A \in \Delta_{\bar{\alpha}t}).$$

Transfering the above lemmata (and those corresponding to the other logical connectives which we have not written down) to the Beth structure we obtain:

6.4. THEOREM. *If* $\mathrm{Sp}(A) \subseteq S_k$ *then the following two conditions are equivalent*:

(1) $\Delta_k \vdash_{\omega} A$,

(2) $\mathfrak{B}_{\Gamma}, k \models A$.

6.5. COROLLARY. *For any sentence A the following are equivalent*:

(1) $\mathrm{HAS} \vdash_{\omega} A$,

(2) $\mathfrak{B}_{\mathrm{HAS}} \models A$.

6.6. DEFINITION. $\mathrm{Val}^{\omega}_{\mathrm{HAS}}$ is the set of sentences A such that for all Beth ω-structures $\mathfrak{M}$ such that $\mathfrak{M} \models \mathrm{HAS}$, $\mathfrak{M} \models A$. We read $A \in \mathrm{Val}^{\omega}_{\mathrm{HAS}}$: "$A$ is ω-valid".

Combining the above theorems we obtain:

6.7. THEOREM. *For any sentence A the following conditions are equivalent*:

(1) *A is ω-valid,*

(2) *A is provable in* $\mathbf{HAS}_{\omega}$,

(3) *A is provable in* $\mathbf{UHAS}_{\omega}$,

(4) *A is provable in the recursively restricted* $\mathbf{UHAS}_{\omega^c}$.

§ 7. The Smorynski calculus of models. Beth wrote, in one of his earlier articles, that he hoped that his models would be as fruitful for intuitionism as the Tarskian models were for classical mathematics. Unfortunately most model theoretic discussion of intuitionistic formal theories involve Kripke models rather than Beth models. That was certainly the case in Chapter V of Troelstra 1973 (written by Smorynski,) which is one of the first (if not the first) instances where model theory is used to obtain results, beyond mere completeness, about formal intuitionistic theories.

Smorynski's method was, loosely speaking, to consider various mappings of Kripke models to Kripke models such that the theory in question (say **HA**) was invariant under them. In this section we shall outline the beginnings of a similar calculus for Beth ω-models.

7.1. *A normalization procedure for Beth ω-structures.* As already mentioned, the proof of the completeness theorem tells us that we may restrict ourselves to those Beth ω-structures $\mathfrak{B} = \langle T, \preccurlyeq, D\rangle$ in which T is a subspread of the universal spread. In particular we can assume that the top node of T is the empty sequence and that as we progress downward in T the nodes of T are finite sequences of natural numbers.

Now suppose S is a $\langle T, \preccurlyeq\rangle$-species (so that for each $t \in T$, $S \upharpoonright t$ is a set of natural numbers). Then by $\hat{S}$ we understand the $\langle T, \preccurlyeq\rangle$-species such that for each $t \in T$

$$\hat{S} \upharpoonright t = \{n \mid n < \mathrm{length}(t) \wedge n \in S \upharpoonright t\}.$$

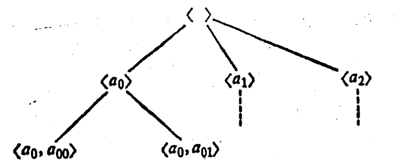
Note that for each $t \in T$, $\hat{S} \upharpoonright t$ is a finite set.

Finally by the *normalized* $\mathfrak{B}$ *we understand the* ω-structure $\hat{\mathfrak{B}} = \langle T, \preccurlyeq, \hat{D}\rangle$ where $\hat{D} = \{\hat{S} \mid S \in D\}$.

7.2. THEOREM. *For any formula $A(X_1, \ldots, X_v, x_1, \ldots, x_u)$ and Beth ω-structure $\mathfrak{B} = \langle T, \preccurlyeq, D\rangle$ the following two conditions are equivalent for all $n_1, \ldots, n_u \in N$, $P_1, \ldots, P_v \in D$ and $k \in T$*

(1) $\mathfrak{B}, k \models A(P_1, \ldots, P_v, n_1, \ldots, n_u)$,

(2) $\hat{\mathfrak{B}}, k \models A(\hat{P}_1, \ldots, \hat{P}_v, n_1, \ldots, n_u)$.

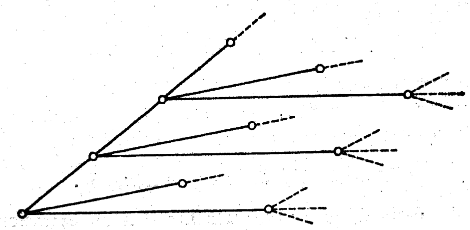


Proof. By induction on the complexity of the formula A. The only case of interest is the atomic formula $x \in X$, and it follows from the fact that for $\mathfrak{B}, k \models 0^{(n)} \in P$ it is not required that $n \in P \upharpoonright k$ but rather that there be a bar B for k such that for all $k' \in B$, $n \in P \upharpoonright k'$.

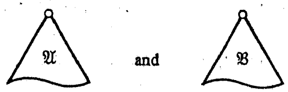
7.3. COROLLARY. $\mathfrak{B}$ *is an ω-model of* HAS *if and only if* $\hat{\mathfrak{B}}$ *is an ω-model of* HAS.

7.4. *Alternating sums of ω-structures.* Given two Beth ω-structures $\mathfrak{A}$, $\mathfrak{B}$ we shall now form the Beth ω-structure $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$, which we shall call the alternating sum of $\mathfrak{A}$ and $\mathfrak{B}$ (relative to $\hat{\mathfrak{D}}$). The construction of $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$ is best explained using diagrams.

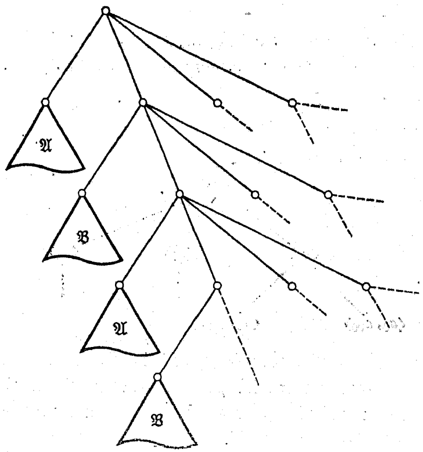
Let us represent $\hat{\mathfrak{D}}$ (the normalization of the van Dalen model) by the following diagram:



and let us represent $\mathfrak{A}$ and $\mathfrak{B}$ by



respectively. Then $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$ may be represented by:



The underlying spread (ordering) of $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$ is clear from the diagram and let us denote it by $\mathcal{T} = \langle T, \preccurlyeq\rangle$. The $\mathcal{T}$-species are obtained by extending the corresponding species of $\mathfrak{A}$, $\mathfrak{B}$ and $\hat{\mathfrak{D}}$ to all of $\mathcal{T}$. The species of $\mathfrak{A}\,[\mathfrak{B}]$ are extended to all of $\mathcal{T}$ by letting them be $\emptyset$ for all the nodes outside of $\mathfrak{A}\,[\mathfrak{B}]$. The species of $\hat{\mathfrak{D}}$ are extended to all of $\mathcal{T}$ by letting them be the value they had at the last node of $\hat{\mathfrak{D}}$.

7.5. THEOREM. *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are* ω*-models of* HAS *then so is* $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$.

Proof. From the construction it follows that $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$ is an ω-structure. Thus it follows from Theorem 5.4 that it suffices to verify that $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$ satisfies all instances of the comprehension schema. Let us consider a typical example

$$(*) \qquad \forall X \exists Y \forall x (x \in X \equiv A(x, X)).$$

In order to show that (*) is satisfied in $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$ we need to show that to every $\mathcal{T}$-species P of $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$ there corresponds a bar B to the top node such that to each node $k' \in B$ there corresponds *a* $\mathcal{T}$-species Q such that

$$(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}, k' \vDash \forall x (x \in Q \equiv A(x, P)).$$

We proceed by cases depending on whether P is an extension of a species of $\mathfrak{A}$, $\mathfrak{B}$, or $\hat{\mathfrak{D}}$.

Case 1. P is an extension of a species of $\mathfrak{A}$. Then P behaves like the constantly $\emptyset$ apecies on $\mathfrak{B}$ and $\hat{\mathfrak{D}}$. Since the constantly $\emptyset$ species is a species of $\mathfrak{B}$ and $\hat{\mathfrak{D}}$ (because $\mathfrak{B}$ snd $\hat{\mathfrak{D}}$ are both models of HAS) we have that in $\mathfrak{B}$ and $\hat{\mathfrak{D}}$ the formula

$$\exists Y \forall x (x \in Y \equiv A(x, \emptyset))$$

is satisfied. Also in $\mathfrak{A}$ we have the satisfaction of

$$\exists Y \forall x (x \in Y \equiv A(x, P))$$

(because $\mathfrak{A}$ is also a model of HAS). Hence we get appropriate bars in $\mathfrak{A}$, $\mathfrak{B}$ and $\hat{\mathfrak{D}}$ which can then be used to construct a bar in $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$.

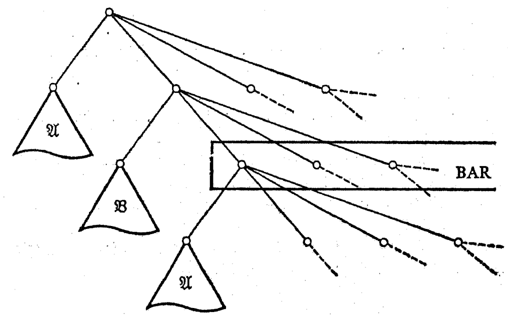
Case 2. P is an extension of a species of $\mathfrak{B}$.

Analogous to Case 1.

Case 3. P is an extension of a species of $\hat{\mathfrak{D}}$.

First determine the bar on $\hat{\mathfrak{D}}$ which forces the satisfaction of the formula

$$\exists Y \forall x (x \in Y \equiv A(x, P)).$$



Then consider those nodes on the left most branch of $\hat{\mathfrak{D}}$ which are either on the bar or lie above it. P restricted to those nodes is a finite set. Hence the values of P in the $\mathfrak{A}$'s and $\mathfrak{B}$'s will be of fixed finite values. Now $\mathfrak{A}$ and $\mathfrak{B}$ being models of HAS will contain such finite species. Hence we can find bars in $\mathfrak{A}$ and $\mathfrak{B}$ which will force in $\mathfrak{A}$ and $\mathfrak{B}$ respectively the formula

$$\exists Y \forall x (x \in Y \equiv A(x, P)).$$

(Theorem 1.1 being helpful in this situation.)

7.6. THEOREM. *The system* $\mathbf{HAS}_\omega$ *has the disjunction property, i.e. if* A, B *are sentences of* **HAS** *and* $\mathrm{HAS} \vdash_\omega (A \vee B)$ *then either* $\mathrm{HAS} \vdash_\omega A$ *or* $\mathrm{HAS} \vdash_\omega B$.

Proof. Suppose that $\mathrm{HAS} \vdash_\omega (A \vee B)$ and that neither $\mathrm{HAS} \vdash_\omega A$ nor $\mathrm{HAS} \vdash_\omega B$. Then, by the completeness theorem, there would exist ω-models $\mathfrak{A}$, $\mathfrak{B}$ of HAS such that $\mathfrak{A}$ not $\models A$ and $\mathfrak{B}$ not $\models B$. Consider then the ω-model $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$. Since it is an ω-model of HAS we have that

$$(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}} \models (A \vee B).$$

However using Lemma 5.7 we see that it is not possible to find a bar B in $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$ such that for all nodes $k \in B$ either A or B is satisfied at k. But then $(A \vee B)$ is not true in $(\mathfrak{A}+\mathfrak{B})/\hat{\mathfrak{D}}$.

Remark. It should be clear that similar methods could be applied to obtain other common closure properties of intuitionistic systems.

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Set-valued mappings on metric spaces

by

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Abstract. In this paper we consider a mapping F of a complete metric space (X, d) into the class $B(X)$ of nonempty, bounded subsets of X. For A in $B(X)$ we define $FA = \bigcup_{a \in A} Fa$ and for A, B in $B(X)$ we define $\delta(A, B) = \sup\{d(a, b)\colon a \in A, b \in B\}$. It is proved that if F maps $B(X)$ into $B(X)$ and satisfies the inequality

$$\delta(Fx, Fy) \leqslant c.\max\{\delta(x, Fx), \delta(y, Fy), \delta(x, Fy), \delta(y, Fx), d(x, y)\}$$

for all x, y in X, where $0 \leqslant c < 1$, then there exists a unique point z in X such that $z \in Fz$ and further $Fz = \{z\}$.

In a paper by Kaulgud and Pai, see [3], they consider mappings F of a metric space (X, d) into either $b(X)$, the class of nonempty, closed and bounded subsets of X, or $\mathrm{Cpt}(X)$, the class of nonempty, compact subsets of X, or 2^X, the class of nonempty, closed subsets of X. The classes $b(X)$ and $\mathrm{Cpt}(X)$ are given the Hausdorff metric D induced by the metric d. With F satisfying various conditions, they prove a number of fixed point theorems for F, a fixed point being defined as a point z in X for which z is in the set Fz. For example, they prove the following theorem in which $d(x, A)$ with x in X and A in $\mathrm{Cpt}(X)$ is defined by

$$d(x, A) = \inf\{d(x, A)\colon a \in A\}.$$

THEOREM 1. *Let F be a mapping of a complete metric space (X, d) into $\mathrm{Cpt}(X)$ satisfying the inequality*

$$D(Fx, Fy) \leqslant a_1 d(x, Fx) + a_2 d(y, Fy) + a_3 d(x, Fy) + a_4 d(y, Fx) + a_5 d(x, y)$$

for all x, y in X, where $a_1, \ldots, a_5 \geqslant 0$ and $a_1 + \ldots + a_5 < 1$. Then F has a fixed point in X.

In the following we consider a mapping F of a metric space (X, d) into $B(X)$, the class of all nonempty, bounded subsets of X. We define the function $\delta(A, B)$ with A, B in $B(X)$ by

$$\delta(A, B) = \sup\{d(a, b)\colon a \in A, b \in B\}.$$

If the set A consists of a single point a we write

$$\delta(A, B) = \delta(a, B)$$