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On some Banach ideals of operators

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Abstract. For each lattice E of measurable functions we define an operator ideal $RN_{E'}$. It is shown that for every ideal space E of measurable functions with a non-discrete dual E' the class $RN_{E'}$ coincides with the class of Radon–Nikodym operators introduced by W. Linde and by the author. Other related characterizations of the Radon-Nikodym operators are also proved.

In [4] we found a number of equivalent characteristic properties of Radon–Nikodym operators in the language of certain mappings acting between Banach spaces and Banach lattices of measurable functions of minimal type. In the present paper we obtain further characterizations of RN operators in those terms with no redundant assumptions on the lattices of measurable functions except for the natural assumption that the lattices in question are ideal spaces.

When discussing lattices of measurable functions we will follow the terminology of [1].

1. Preliminaries. Throughout the paper (Ω, Σ, μ) is a finite non-negative measure space. $L^0(\mu) = L^0(\Omega, \Sigma, \mu)$ denotes the lattice of all (equivalence classes of) scalar-valued μ-measurable functions on Ω. For $f \in L^0(\mu)$ we put $\operatorname{supp} f = \{\omega \in \Omega \mid f(\omega) \neq 0\}$; if $F \subset L^0(\mu)$, then $\operatorname{supp} F$ stands for the smallest set which belongs to Σ and contains $\operatorname{supp} f$ for each $f \in F$.

Let E be a vector sublattice of $L^0(\mu)$. We shall say that E is *discrete* if the restriction of μ to the σ-field

$$\Sigma_0 = \Sigma \mid \operatorname{supp} E = \{A \cap \operatorname{supp} E \mid A \in \Sigma\}$$

is a purely atomic measure.

E is said to be *an ideal space* if $f \in L^0(\mu)$, $g \in E$ and $|f| \leqslant |g|$ implies $f \in E$; if, in addition, E is a Banach sublattice, then E is called *the Banach ideal space* (we shall write briefly IS and BIS, respectively).

For a vector sublattice $E \subset L^0(\mu)$ we define *the dual* IS E' as the space

$$E' = \{g \in L^0(\mu) \mid \operatorname{supp} g \subset \operatorname{supp} E,\ gf \in L^1(\mu) \text{ for each } f \in E\}.$$

If X is a Banach space, then $E(X)$ denotes the vector space of all (Bochner) μ-measurable functions $\bar{f}\colon \Omega \to X$ such that the functions $\|\bar{f}(\cdot)\|$ belong to E. If E is a Banach sublattice of $L^0(\mu)$, $E(X)$ is a Banach space when equipped with the norm $\|\bar{f}\|_{E(X)} = \|\|\bar{f}(\cdot)\|_X\|_E$.

Let E and F be the vector sublattices of $L^0(\mu)$. We shall write $E \dashv F$ if $\operatorname{supp} E \subset \operatorname{supp} F$ and if there is a sequence $\{\Omega_n\}_{n=1}^{\infty}$ of sets $\Omega_n \in \Sigma$ such that $\bigcup_{n=1}^{\infty} \Omega_n = \operatorname{supp} E$ and $f\chi_{\Omega_n} \in F$ for each $f \in E$ and every $n = 1, 2, \ldots$ (i.e. $E|\Omega_n \subset F|\Omega_n$).

The simple proof of the following proposition consists of standard measure-theoretic arguments, and so we omit it.

Proposition 1.1. *Let $E \subset L^0(\mu)$ be an IS. Then*

(1) *if $\operatorname{supp} E = \Omega$, then $L^\infty(\mu) \dashv E$;*

(2) *the set $X = \{f \in E \mid f = \sum_{k=1}^{\infty} a_k \chi_{A_k},\ A_k \cap A_j = \emptyset \text{ if } k \neq j\}$ is dense in E for the topology of "uniform convergence a.e.", i.e. for each $g \in E$ and each $\varepsilon > 0$ there is a function $f \in X$ such that $|f(\omega) - g(\omega)| < \varepsilon$ a.e. on Ω.*

Let X be a Banach space and let E be a vector sublattice of $L^0(\mu)$. We shall denote by $\mathfrak{M}(E, X)$ the set of all X-valued measures $\bar{m}\colon \Sigma \to X$ of bounded variation possessing the following property: there exists a function $g \in E'$ such that $\int |f|\, dV(\bar{m}) \leqslant \int |f| g\, d\mu$ for each $f \in E$ where $V(\bar{m})$ is the variation of $\bar{m}$. $\mathfrak{M}_0(E, X)$ denotes the subset of the set $\mathfrak{M}(E, X)$ which consist of those measures $\bar{m}$ for which there exist functions $\bar{g}_{\bar{m}} \in E'(X)$ with $\int f\, d\bar{m} = \int f \bar{g}_{\bar{m}}\, d\mu$ for each $f \in E$. If $E = L^\infty(\mu)$, then we get exactly the set of all μ-continuous X-valued measures of bounded variation, (or the set of all such measures having derivatives with respect to μ).

If E is a vector lattice, then $P(X, E)$ denotes the vector space of linear mappings from X to E which map the unit ball of X into an order bounded subset of E. If E is a sublattice of $L^0(\mu)$, then we define the vector space $\bar{S}(E, X)$ as follows: the linear transformation $T\colon E \to X$ is in $\bar{S}(E, X)$ iff there is a function $e' \in E'$ such that $\|Te\| \leqslant \int |e| e'\, d\mu$ for each $e \in E$. $\bar{S}_0(E, X)$ denotes the vector subspace of $\bar{S}(E, X)$ which consists of the mappings T such that there exist functions $\bar{g}_T \in E'(X)$ with $Te = \int e \bar{g}_T\, d\mu$ for each $e \in E$. If E is a Banach sublattice in $L^0(\mu)$, then the spaces $P(X, E)$ and $\bar{S}(E, X)$ are Banach spaces when equipped with the norms

$$p(T) = \inf\{\|\gamma\|_E \mid \gamma \in E,\ |Tx| \leqslant \|x\|\gamma \text{ for each } x \in X\}$$

and, respectively,

$$\bar{s}(T) = \inf\{\|e'\|_{E^*} \mid e' \in E',\ \|Te\| \leqslant \int |e| e'\, d\mu, \quad \text{for each} \quad e \in E\}.$$

Note that $\bar{S}(L^1(\mu), X) = L(L^1(\mu), X)$ is the space of all bounded linear operators from $L^1(\mu)$ to X.

The following statement is very simple and we omit the proof.

Proposition 1.2. *If $E \subset L^0(\mu)$ is an IS whose dual space E' is discrete, then for every Banach space X we have*

$$\mathfrak{M}(E, X) = \mathfrak{M}_0(E, X) \quad \textit{and} \quad \bar{S}(E, X) = \bar{S}_0(E, X).$$

We conclude this section with the following proposition, which will be useful in the next section.

Proposition 1.3. *Suppose $E \subset L^0(\mu)$ is an IS such that $L^\infty(\mu) \subset E \subset L^1(\mu)$. Then there exists a bijection π from $\bar{S}(E, X)$ onto $\mathfrak{M}(E, X)$ such that if $U \in \bar{S}(E, X)$, then $Uf = \int f\, d\pi(U)$ for each $f \in E$ and $\pi(\bar{S}_0(E, X)) = \mathfrak{M}_0(E, X)$.*

Proof. Let $U \in \bar{S}(E, X)$ be given. Then there is a function $g \in E'$ such that $\|Uf\| \leqslant \int |f| g\, d\mu$ for each $f \in E$. The set function $\bar{m}\colon \Sigma \to X$ defined by $\bar{m}(A) = U(\chi_A)$ for $A \in \Sigma$ is countably additive: if $\{A_n\}_{n=1}^{\infty} \subset \Sigma$, $A_n \cap A_k = \emptyset$, $n \neq k$, then $\bar{m}\big(\bigcup_{n=1}^{\infty} A_n\big) = U(\chi_{\bigcup_{n=1}^{\infty} A_n}) = \sum_{n=1}^{\infty} U(\chi_{A_n})$ (the last equality follows from the fact that

$$\Big\|U(\chi_{\bigcup_{n=1}^{\infty} A_n}) - \sum_{n=1}^{\infty} U(\chi_{A_n})\Big\| \leqslant \int_{\bigcup_{n=m+1}^{\infty} A_n} g\, d\mu \to 0$$

if $m \to +\infty$ and

$$\sum_{n=1}^{\infty} \|U(\chi_{A_n})\| \leqslant \int_{\bigcup_{n=1}^{\infty} A_n} g\, d\mu < +\infty$$

since $g \in E' \subset L^1(\mu)$). It is clear that $\bar{m}$ has a bounded variation and $\int |f|\, dV(\bar{m}) \leqslant \int |f| g\, d\mu$ for each $f \in E$. Hence $\bar{m} \in \mathfrak{M}(E, X)$. To show that $Uf = \int f\, d\bar{m}$ for each $f \in E$ it is sufficient (by Proposition 1.1.2) to verify this equality on the functions of type $\sum_{k=1}^{\infty} a_k \chi_{A_k}$ where $\{A_k\}_{k=1}^{\infty}$ is a sequence of pairwise disjoint members of Σ. Let f be such a function and let $\varepsilon > 0$ be given. Then we have

$$\Big\|Uf - U\Big(\sum_{k=1}^{m-1} a_k \chi_{A_k}\Big)\Big\| = \Big\|U\Big(\sum_{k=m}^{\infty} a_k \chi_{A_k}\Big)\Big\| \leqslant \int_{\bigcup_{k=m}^{\infty} A_k} |f| g\, d\mu,$$

and if m is large enough, then it follows from the absolute continuity of the integral that the number on the right is less than ε. Consequently

$Uf = \sum_{k=1}^{\infty} a_k \overline{m}(A_k)$ (the series is convergent in X since $U \in \overline{S}(E, X)$). Of course $\int f d\overline{m} = \sum_{k=1}^{\infty} a_k \overline{m}(A_k)$. To complete the proof we only need to notice that $U \in \overline{S}_0(E, X)$ iff $\overline{m} \in \mathfrak{M}_0(E, X)$.

2. The spaces $\mathrm{RN}_{E'}(X, Y)$. Let X and Y be Banach spaces and let $L(X, Y)$ be the space of all bounded linear operators from X to Y. An operator $T \in L(X, Y)$ is said to be *a Radon-Nikodym operator* (see [2], [3]) if it takes each X-valued measure $\overline{m}$ of bounded variation $V(\overline{m})$ to a Y-valued measure having the derivative with respect to $V(\overline{m})$. We denote by $\mathrm{RN}(X, Y)$ the set of all Radon–Nikodym operators from X to Y. It is well known and easily seen that the class $\mathrm{RN} = \{\mathrm{RN}(X, Y) \mid X$ and Y are Banach spaces$\}$ equipped with the usual operator norm is an injective Banach ideal of operators in the sence of A. Pietsch.

Let (Ω, Σ, μ) be a finite non-negative measure space, let $E \subset L^0(\mu)$ be an IS and let $T \in L(X, Y)$ be given. The operator T is called *an operator of type* $\mathrm{RN}_{E'}$ if for each measure $\overline{m} \in \mathfrak{M}(E, X)$ the measure $T\overline{m}$ belongs to the set $\mathfrak{M}_0(E, Y)$. Thus the operators of type $\mathrm{RN}_{E'}$ from X to Y are exactly those linear bounded mappings for which the associated mappings $\mathfrak{M}(E, X) \to \mathfrak{M}(E, Y)$ take their values in the set $\mathfrak{M}_0(E, Y)$. The collection of all operators of type $\mathrm{RN}_{E'}$ from X to Y will be denoted by $\mathrm{RN}_{E'}(X, Y)$.

Our first statement about the properties of such operators is elementary and follows immediately from Propositions 1.2 and 1.3 and from Proposition 1 of [3].

PROPOSITION 2.1. (1) *If $E \subset L^0(\mu)$ is an* IS *with the discrete dual E', then* $\mathrm{RN}_{E'}(X, Y) = L(X, Y)$;

(2) *If the measure μ is not purely atomic, then* $\mathrm{RN}_{(L^\infty(\mu))'}(X, Y) = \mathrm{RN}(X, Y)$.

The main result of this section is the following theorem, which asserts (in view of part (1) of the previous proposition) that, given any ideal space $E \subset L^0(\mu)$, the operator class $\mathrm{RN}_{E'}$ is either the ideal of all bounded linear operators or the ideal of all operators of type RN.

THEOREM 2.1. *Suppose $E \subset L^0(\mu)$ is an* IS *whose dual space E' is not discrete. Then for every pair of Banach spaces X and Y the equality* $\mathrm{RN}(X, Y) = \mathrm{RN}_{E'}(X, Y)$ *holds. Thus the class* $\mathrm{RN}_{E'}$ *equipped with the usual operator norm is an injective Banach ideal of operators.*

As a special case of the above theorem we get

COROLLARY 2.1. *Let $E \subset L^0(\mu)$ be an* IS *with a non-discrete dual E', and let X be a Banach space. The space X possesses the Radon–Nikodym property iff X possesses the* $\mathrm{RN}_{E'}$ *property (i.e. the identity map $X \to X$ is of type* $\mathrm{RN}_{E'}$). *Thus if X has the* RN *property, then X has the* $\mathrm{RN}_{E'}$ *property for every* IS E.

THEOREM 2.1 is an easy consequence of part (2) of Proposition 2.1 and the following three statements, which also give other characterizations of RN operators.

LEMMA 2.1. *Let E and F be two ideal spaces on a finite non-negative measure space (Ω, Σ, μ). Suppose $E \dashv F$ and* $\operatorname{supp} E' = \operatorname{supp} F'$ (see Section 1 for the notation). *Then* $\mathrm{RN}_{E'}(X, Y) \subset \mathrm{RN}_{F'}(X, Y)$.

Proof. Suppose $T \in \mathrm{RN}_{E'}(X, Y)$ and $\overline{m} \in \mathfrak{M}(F, X)$. Let $g \in F'$ be a function such that $\int |f| dV(\overline{m}) \leqslant \int |f| g d\mu$ for each $f \in F$. We must show that $T\overline{m} \in \mathfrak{M}_0(F, Y)$. Since $E \dashv F$, by Proposition 1.1 there is a sequence of pairwise disjoint sets $\Omega_n \in \Sigma$ such that $\bigcup_{n=1}^{\infty} \Omega_n = \operatorname{supp} E$ and $L^\infty(\mu) | \Omega_n \subset E | \Omega_n \subset F | \Omega_n$ for each n. Since $g \in F'$ and $E | \Omega_n \subset F | \Omega_n$, we have $g\chi_{\Omega_n} \in E'$. Hence if $\overline{m}_n(A) = \overline{m}(A \cap \Omega_n)$ for $A \in \Sigma$, then $\overline{m}_n \in \mathfrak{M}(E, X)$. By hypothesis there is a function $\overline{g}_n \in E'(Y)$ such that $\int f dT\overline{m}_n = \int f \overline{g}_n d\mu$ for each $f \in E$. Now, the inclusion $L^\infty(\mu) | \Omega_n \subset E | \Omega_n$ implies that $\|\overline{g}_n\| \leqslant \|T\| g$ a.e. on Ω_n. Let us put $\overline{g}(\omega) = \overline{g}_n(\omega)$ if $\omega \in \Omega_n$ and $\overline{g}(\omega) = 0$ if $\omega \in \Omega \setminus \bigcup_{n=1}^{\infty} \Omega_n$. Since $\|\overline{g}\| \leqslant \|T\| g$ a.e. on Ω and $g \in F'$, we have $\overline{g} \in F'(Y)$. It follows from the equality $\operatorname{supp} E' = \operatorname{supp} F'$ and from the inclusion $\operatorname{supp} E' \subset \operatorname{supp} E = \bigcup_{n=1}^{\infty} \Omega_n$ that $\overline{m}(A) = 0$ for every set $A \in \Sigma$ with $A \subset \Omega \setminus \bigcup_{n=1}^{\infty} \Omega_n$. Therefore, for each $f \in F$ we have

$$\int_{\Omega} f dT\overline{m} = \sum_{n=1}^{\infty} \int_{\Omega_n} f dT\overline{m}_n = \sum_{n=1}^{\infty} \int_{\Omega_n} f \overline{g}_n d\mu = \int_{\Omega} f \overline{g} d\mu.$$

Thus we have shown that $T\overline{m} \in \mathfrak{M}_0(F, Y)$.

LEMMA 2.2 *Let $E \subset L^0(\mu)$ be an* IS *and let T be an operator from X to Y. The following assertions are equivalent*:

(1) $T \in \mathrm{RN}_{E'}(X, Y)$;

(2) *for every mapping $U \in \overline{S}(E, X)$ there is a function $\overline{g} \in E'(Y)$ such that $TUf = \int f \overline{g} d\mu$ for each $f \in E$; moreover, if $g \in E'$ is a function such that $\|Ue\| \leqslant \int |e| g d\mu$ for each $e \in E$, then $\|\overline{g}\| \leqslant \|T\| g$ a.e. on Ω.*

Proof. That (2) implies (1) is evident. To prove the converse suppose that $U \in \overline{S}(E, X)$ and $g \in E'$ is a function for which $\|Ue\| \leqslant \int |e| g d\mu$ if $e \in E$. To show that there is a function $\overline{g} \in E'(Y)$ such that $TUf = \int f \overline{g} d\mu$ for each $f \in E$ we may assume that $\operatorname{supp} E = \operatorname{supp} E'$ and $L^\infty(\mu) \dashv E$ (by Proposition 1.1). Now (again by Proposition 1.1) it follows that $L^\infty(\mu) \dashv E'$, hence $E \dashv L^1(\mu)$. Thus we have $L^\infty(\mu) \dashv E \dashv L^1(\mu)$. Let $\{\Omega_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint members of Σ for which $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$ and $L^\infty(\mu) | \Omega_n \subset E | \Omega_n \subset L^1(\mu) | \Omega_n$ $(n = 1, 2, \ldots)$. Fix $n = 1, 2, \ldots$ Since

$\|Ue\| \leqslant \int_{\Omega_n} |e| g\, d\mu$ for each $e \in E$ with $e|(\Omega \setminus \Omega_n) = 0$, by Proposition 1.3 there is a measure $\overline{m}_n \in \mathfrak{M}(E, X)$ such that $Ue = \int e\, d\overline{m}_n$ for every function $e \in E$ whose support is in Ω_n. Therefore for every $n = 1, 2, \ldots$ there exists a function $\overline{g}_n \in E'(Y)$ such that $Ue = \int e \overline{g}_n\, d\mu$ for each $e \in E$ with $\operatorname{supp} e \subset \Omega_n$. Since $L^\infty(\mu)|\Omega_n \subset E|\Omega_n$, we have $\|\overline{g}_n\| \leqslant \|T\| g$ a.e. on Ω_n. To complete the proof it is now enough to put $\overline{g}(\omega) = \overline{g}_n(\omega)$ if $\omega \in \Omega_n$.

Now we are ready to prove

THEOREM 2.2. *For a linear bounded operator T from X to Y, the following four assertions are equivalent:*

(1) $T \in \mathrm{RN}(X, Y)$;

(2) *for each finite non-negative measure space (Ω, Σ, μ), each* IS *$E \subseteq L^0(\mu)$ and every mapping $U \in \overline{S}(E, X)$ the map TU belongs to the space $\overline{S}_0(E, Y)$;*

(3) *for each finite non-negative measure space (Ω, Σ, μ), each* BIS *$E \subset L^0(\mu)$ and every mapping $U \in \overline{S}(E, X)$ there exists a function $\overline{g} \in E'(Y)$ such that $TUe = \int e\overline{g}\, d\mu$, $e \in E$, and $\|\overline{g}\|_{E'(Y)} \leqslant \|T\| \overline{s}(U)$;*

(4) *there exist a finite positive non-purely atomic measure space (Ω, Σ, μ) and an* IS *$E \subset L^0(\mu)$ with a non-discrete dual E' such that for each mapping $U \in \overline{S}(E, X)$ the map TU is in the space $\overline{S}_0(E, Y)$.*

Proof. If $T \in \mathrm{RN}(X, Y)$, then $T \in \mathrm{RN}_{(L^\infty(\mu))'}(X, Y)$ for every space $L^\infty(\mu)$. Now it is easily seen that $T \in \mathrm{RN}_{E'}(X, Y)$ for any IS E mentioned in assertions (2) and (3) (Proposition 1.1 and Lemma 2.1). To prove that (1) implies (2) it is now enough to utilize Lemma 2.2. That (2) implies (3) and (3) implies (4) is trivial. Thus it is only necessary to show that (4) implies (1).

Let T satisfy (4). Of course, without loss of generality we may assume that $\operatorname{supp} E = \operatorname{supp} E'$. In this case $L^\infty(\mu)|\operatorname{supp} E' \dashv E'$ (Proposition 1.1) and therefore $E \dashv L^1(\mu)|\operatorname{supp} E$. By Lemma 2.2 we get $T \in RN_{E'}(X, Y)$, and so Lemma 2.1 now implies that $T \in \mathrm{RN}_{F'}(X, Y)$ where $F = L^1(\mu)|\operatorname{supp} E$. It follows from Theorem 1 of [3] and Lemma 2.2 that T is of type RN.

Let us conclude the present section with an application of the above theorem. The following result is slightly stronger than the one obtained in Theorem 4 of [3], implication (3) $\Rightarrow$ (1), since every $L^\infty(\mu)$-space constructed on the non-purely atomic measure space is isomorphic to a space $C(K)$ for some non-dispersed compact set K. Note that the converse of the following theorem is also valid.

THEOREM 2.3. *Let $T \in L(X, Y)$ be given. If there is a non-dispersed compact set K such that for each integral operator $U\colon C(K) \to X$ the operator TU is nuclear, then T is of type* RN.

Proof. Let μ be a non-purely atomic Radon measure on K. We shall

show that T satisfies (4) of Theorem 2.2 for the IS $E = L^\infty(\mu)$. Suppose $U \in \overline{S}(L^\infty(\mu), X)$ and let g be a function such that $g \geqslant 1$ a.e. and $\|Uf\| \leqslant \int |f| g\, d\mu$ for each $f \in L^\infty(\mu)$. Then there is a map $U_2 \in L(L^1(\mu), X)$ such that $U = U_2 U_1$ where $U_1 f = fg$ for each $f \in L^\infty(\mu)$. Since $|U_1 f| \leqslant \|f\| g$ a.e., U_1 is an integral operator from $L^\infty(\mu)$ into $L^1(\mu)$, and so U is also an integral operator. By hypothesis TUj is a nuclear map where $j\colon C(K) \to L^\infty(\mu)$ is the natural injection. Let $TUj = \sum \langle \cdot, \mu_n \rangle y_n$ be a representation of this nuclear map ($\mu_n \in C^*(K)$, $y_n \in Y$, $\sum \|\mu_n\| \|y_n\| < +\infty$). Let P be a canonical projection from $C^*(K)$ onto $L^1(\mu)$ and let $i\colon L^1(\mu) \to C^*(K)$ be the injection $i(\varphi) = \varphi\mu$ for $\varphi \in L^1(\mu)$. If $\tilde{\mu}_n = f_n \mu = P\mu_n$ ($f_n \in L^1(\mu)$), then $(TUj)^*(Y^*) \subset i(L^1(\mu))$ and for $f \in C(K)$ we have $TUj(f) = \sum \langle f, \tilde{\mu}_n \rangle y_n$. Since $g \geqslant 1$ a.e., the set $U_1 j(C(K))$ is dense in $L^1(\mu)$ and therefore $TUf = \sum \langle f, f_n \rangle y_n$ for each $f \in L^\infty(\mu)$ where $\sum \|f_n\| \|y_n\| < \infty$. Thus $TUf = \int f \overline{g}\, d\mu$ where $\overline{g} = \sum f_n \otimes y_n \in L^1(\mu; Y)$, and the operator T satisfies condition (4) of Theorem 2.2. Hence it is a Radon–Nikodym operator.

3. Some other characterizations of RN **operators.** Using the above results, we shall obtain in this section some other necessary and sufficient conditions for a linear bounded operator to be of type RN. As we have seen, RN operators, when composed with some linear mappings, improve that mappings in a certain sense. The following theorems show that this is always the case when we consider the compositions of RN operators both with the scalarly measurable vector-valued functions and with the "order bounded" mappings. Throughout the section X and Y are Banach spaces.

THEOREM 3.1. *For an operator $T \in L(X, Y)$ the following three statements are equivalent:*

(1) *T is of type* RN;

(2) *if (Ω, Σ, μ) is a finite non-negative measure space, $E \subset L^0(\mu)$ is an* IS, *$\overline{f}\colon \Omega \to X^{**}$ is a X^*-scalarly measurable function such that $\int f \overline{f}\, d\mu \in X$ for each $f \in E$ and $|\langle \overline{f}, x' \rangle| \leqslant \varphi \in E'$ for each $x' \in X^*$ a.e. on Ω, then there is a function $\overline{g} \in E'(Y)$ such that the functions $\overline{g}$ and $T^{**}\overline{f}$ are Y^*-scalarly equivalent;*

(3) *there exist a finite non-negative measure space (Ω, Σ, μ) and an* IS *$E \subset L^0(\mu)$ with a non-discrete dual E' such that for any function $\overline{f}$ mentioned in (2) one can find a function $\overline{g} \in E'(Y)$ which is Y^*-scalarly equivalent to $T^{**}\overline{f}$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $T \in \mathrm{RN}(X, Y)$ and let an IS $E \subset L^0(\mu)$ be given. Without loss of generality we may assume that $\operatorname{supp} E = \Omega$. Let $\overline{f}$ and φ be as in (2). By Proposition 1.1 there exists a sequence $\{\Omega_n\}_{n=1}^\infty$ of pairwise disjoint members of Σ such that $\bigcup_{n=1}^\infty \Omega_n = \Omega$, $L^\infty(\mu)|\Omega_n \subset E|\Omega_n$

and φ is bounded on every set Ω_n. Let us put $\bar{f}_n = \bar{f}\chi_{\Omega_n}$ and define an operator $U \in L(L^1(\mu), X)$ by $Uf = \int f\bar{f}_n d\mu$. It follows from Theorem 2.2 that $TUf = \int f\bar{g}_n d\mu$ where $\bar{g}_n \in L^\infty(\mu; Y)$. If $y' \in Y^*$, then $\langle T^{**}\bar{f}_n, y'\rangle = \langle \bar{g}_n, y'\rangle$ a.e. on Ω_n. Further, $\|\bar{g}_n\| \leqslant \|T\|\varphi$ a.e. on Ω_n. Now, it is sufficient to put $\bar{g} = \sum_{n=1}^{\infty} \bar{g}_n\chi_{\Omega_n}$.

(3) $\Rightarrow$ (1). Let (Ω, Σ, μ) be a finite positive non-purely atomic measure space and let $E \subset L^0(\mu)$ be an IS with a non-discrete dual E'. By Proposition 1.1 we can find a set $\Omega_0 \in \Sigma$ such that the restriction of μ on Ω_0 is a purely non-atomic measure μ_0, $L^\infty(\mu_0) = L^\infty(\mu)|\Omega_0 \subset E'|\Omega_0$ and $E|\Omega_0 \subset L^1(\mu_0)$. Assume that (3) holds for E. If $U \in L(L^1(\mu_0), X)$, then there is a X^*-scalarly measurable function $\bar{f}_0$: $\Omega_0 \to X^{**}$ such that $Uf = \int f\bar{f}_0 d\mu_0$ for each $f \in L^1(\mu_0)$. Let us put $\bar{f} = \bar{f}_0\chi_{\Omega_0}$. Since $L^\infty(\mu_0) \subset E'|\Omega_0$ and $E|\Omega_0 \subset L^1(\mu_0)$, it is easily seen that the function $\bar{f}$ possesses the properties mentioned in (2). Hence there is a function $\bar{g} \in E'(Y)$ such that $\langle T^{**}\bar{f}, y'\rangle = \langle \bar{g}, y'\rangle$ a.e. on Ω for each $y' \in Y^*$. It now follows that $TUf = \int f\bar{g}d\mu$ for every function $f \in L^1(\mu)$ with $\operatorname{supp} f \subset \Omega_0$. By Theorem 1 of [3] the operator T is of type RN.

THEOREM 3.2. *For an operator $T \in L(X, Y)$, the following three statements are equivalent:*

(1) *T^* is of type* RN;

(2) *for every finite non-negative measure space (Ω, Σ, μ) and for each* IS *$E \subset L^0(\mu)$, for every map $U \in P(Y, E)$ there exists a function $\bar{g} \in E(X^*)$ such that $UTx = \langle x, \bar{g}\rangle$ for each $x \in X$; if E is a* BIS, *then $\|\bar{g}\|_{E(X^*)} \leqslant \|T\|p(U)$;*

(3) *there exist a finite positive measure space (Ω, E, μ) and a non-discrete* IS *$E \subset L^0(\mu)$ such that for every map $U \in P(Y, E)$ there is a function $\bar{g} \in E(X^*)$ which satisfies $UTx = \langle \bar{x}, \bar{g}\rangle$ for each $x \in X$.*

Proof. (1) $\Rightarrow$ (2). It is easily seen (by the lifting theorem) that if $U \in P(Y, E)$, then there exists a Y-scalarly measurable function $\bar{f}$: $\Omega \to Y^*$ such that $Uy = \langle y, \bar{f}\rangle$ for each $y \in Y$. Hence $UTx = \langle x, T^*\bar{f}\rangle$ for each $x \in X$. If T^* is of type RN, then using Theorem 3.1 we can find a measurable function $\bar{g}$: $\Omega \to X^*$ for which $UTx = \langle x, \bar{g}\rangle$. It is clear that $\bar{g} \in E(X^*)$ and if E is a BIS, then $\|\bar{g}\|_{E(X^*)} \leqslant \|T\|p(U)$. That (2) implies (3) is trivial. Clearly, to prove that (3) implies (1) it is enough to prove that (3) implies that

(3a) *for some finite non-purely atomic measure space $(\Omega_0, \Sigma_0, \mu_0)$ and for each operator $U \in L(Y, L^\infty(\mu_0))$ there is a function $\bar{g} \in L^\infty(\mu_0; X^*)$ such that $UTx = \langle x, \bar{g}\rangle$ for each $x \in X$.*

Suppose T satisfies (3). Since E is not discrete, the restriction of μ on $\operatorname{supp} E$ is not a purely atomic measure. It follows from Proposition 1.1 that there is a set $\Omega_0 \subseteq \operatorname{supp} E$, $\Omega_0 \in \Sigma$ for which $\mu_0 = \mu|\Omega_0$ is not a purely atomic measure and $L^\infty(\mu_0) \subset E|\Omega_0$. Now if $U \in L(Y, L^\infty(\mu_0))$ and j: $L^\infty(\mu_0) \to E$ is a natural injection, then we have $jU \in P(Y, E)$ and consequently $jUTx = \langle x, \bar{g}\rangle$, $x \in X$, where $\bar{g} \in E(X^*)$. Since jUT maps the unit ball of X into a bounded subset of $L^\infty(\mu_0)$, the function $\bar{g}|\Omega_0$ belongs to $L^\infty(\mu_0)$. This completes the proof.

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