Now we come to the proof of Theorem B. Suppose that $f \in L(\log^+ L)^{n-1}$ so that the set where $M_n \ldots M_1 f > 1$ has finite measure in virtue of Theorem A. If we assume that $M_n \ldots M_1 f$ is integrable over every set of finite measure, then

$$\begin{aligned}\infty > \int\limits_{\{M_n\ldots M_1 f>1\}} M_n \ldots M_1 f(x)\,dx &\geqslant \int_1^\infty m\{M_n \ldots M_1 f > t\}\,dt \\ &\geqslant \frac{2^{-n}}{(n-1)!}\int_1^\infty dt \int\limits_{|f|>t} \frac{|f|}{t}\left(\log\frac{|f|}{t}\right)^{n-1} dx \\ &= \frac{2^{-n}}{(n-1)!}\int dx\,|f|\int_1^{|f|}\left(\log\frac{|f|}{t}\right)^{n-1}\frac{dt}{t} = \frac{2^{-n}}{n!}\int |f|(\log^+|f|)^n\,dx.\end{aligned}$$

**3. An open problem.** Suppose that $f \in L(\log^+ L)^{n-1}$ so that $f^*$ is finite almost everywhere. We conjecture that $f^*$ is integrable over every set of finite measure if and only if $f \in L(\log^+ L)^n$.

By virtue of Theorems A and B this conjecture is reduced to proving (or disproving) that if $f^*$ is integrable over every set of finite measure, then so is $M_n \ldots M_1 f$; but this remains, as far as we know, an open question.

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# Zero-one laws for Gaussian measures on metric abelian groups

by

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**Abstract.** We prove zero-one laws for Gaussian measures on metric abelian groups. As a consequence, we derive the zero-one law for Gaussian processes with values in an LCA group $G$. The latter result contains, in particular, all the zero-one laws of Kallianpur and Jain as well as the zero-one law for the $G$-valued Wiener process.

In 1951 Cameron and Graves proved that every measurable rational subspace of $C[0, 1]$ has Wiener measure zero or one [7]. This result has been generalized by Kallianpur [16] to a large class of Gaussian processes and by Baker [1], Rajput [18] and others to Gaussian measures on Banach spaces or on Fréchet spaces.

The proofs presented by these authors depend heavily on methods of linear spaces as well as on the stability of Gaussian measures.

A completely different approach has been proposed in the proof of Theorem 2.2 in [4]. This proof, suggested by an algebraic definition of Gaussian measure, seems to be more natural and relatively simple. The arguments, being of group-theoretic nature, enable us to extend this result to Gaussian measures on measurable groups, as pointed out in [6].

The aim of this paper is to generalize the result of Theorem 2.2 in [4] in two directions. First, we consider Gaussian measures on arbitrary Hausdorff abelian groups; secondly, we prove the zero-one law for measurable subgroups (instead of rational subspaces, as in [4]).

Section 1 is preliminary. In Section 2 we prove that every symmetric Gaussian measure without idempotent factors on a Hausdorff abelian group $G$ can be embedded in a unique continuous semigroup of symmetric Gaussian measures under the assumption that $x \to 2x$ is a Borel automorphism of $G$. From this result it follows, in particular, that every symmetric Gaussian measure on $G$ is infinitely divisible.

In Section 3 we prove that every Gaussian measure on a metric abelian group $G$ is a translation of a symmetric Gaussian measure (under the assumption that $G$ has no non-zero elements of order 2). This property of Gaussian measures is well known if $G$ is a Banach space (or, more generally, a complete separable locally convex space) or if $G$ is an LCA group [13]. The proof of the general case (Theorem 3.1) is more difficult; it is based on a characterization of positive isometries between $L_p$ spaces, given in [14], Theorem 3.1 is in an essential way used in the next section.

In Section 4 we prove the zero-one law for Gaussian measures on metrizable abelian groups (Theorem 4.2). The proof is divided into two parts: the first part is stated as Theorem 4.1 and is idependent of the preceding sections. Theorem 4.1 contains, in particular, the zero-one law for Gaussian measures on complete separable locally convex spaces (a result proved earlier by Rajput [18]) or on separable Orlicz spaces.

In Section 5 we derive the zero-one law for Gaussian processes with values in a second countable LCA group $G$ (Theorem 5.2 and Corollary 5.1). This result contains Kallianpur's zero-one laws [16] as well as the zero-one law for the $G$-valued Wiener process.

**1. Preliminaries.** Let $(G, \mathscr{B})$ be a measurable space. By a *probability measure* on $(G, \mathscr{B})$ we mean a non-negative $\sigma$-additive measure $\mu$ such that $\mu(G) = 1$. Whenever a topological structure on $G$ appears, we assume that $\mathscr{B}$ is the Borel $\sigma$-field of $G$ and that $\mu$ is tight, i.e.

$$\mu(A) = \sup\{\mu(K), K \text{ compact} \subseteq A\} \quad \text{for every } A \in \mathscr{B}.$$

The measure concentrated at $x \in G$ will be denoted by the same symbol $x$.

Now, let $G$ be a Hausdorff abelian group. The space of all probability measures on $G$ will always be considered as a topological space with the weak topology: a net $\mu_a$ of probability measures converges to a probability measure $\mu$ iff

$$\lim \int f(x)\mu_a(dx) = \int f(x)\mu(dx),$$

for every $f \in C(G)$ (the space of all real-valued continuous bounded functions defined on $G$). Since we consider only tight measures, this topology is Hausdorff (see [22]).

A family $\Pi$ of probability measures is said to be *uniformly tight* if for every positive $\varepsilon > 0$ there exists a compact subset $K \subseteq G$ such that $\mu(K) > 1-\varepsilon$ for every $\mu \in \Pi$. Prohorov's theorem asserts that every uniformly tight family is conditionally compact.

By $\mathscr{B}\times\mathscr{B}$ we shall denote the product of the Borel $\sigma$-algebras on $G$ and by $(\mathscr{B}\times\mathscr{B})^{\mu\times\nu}$ the completion of $\mathscr{B}\times\mathscr{B}$ with respect to $\mu\times\nu$. It is known that if $\mu, \nu$ are two probability measures on $G$, then the mapping

$$(x, y) \to x+y$$

from $G\times G$ into $G$ is measurable with respect to $\mathscr{B}$ and $(\mathscr{B}\times\mathscr{B})^{\mu\times\nu}$ (see [21], p. 281). Thus the convolution of two probability measures $\mu, \nu$ can be defined by the following formula:

$$\mu*\nu(B) = \mu\times\nu(\{(x, y);\ x+y \in B\}) = \int \mu(B-y)\nu(dy).$$



It is well known that $\mu*\nu$ is a (tight) probability measure and that the convolution is associative and jointly continuous.

The following lemmas are modifications of Theorems 2.1 and 2.2, III, in [17].

LEMMA 1.1. *Let $\{\mu_a\}_{a\in A}$, $\{\nu_a\}_{a\in A}$ be two families of probability measures on $G$. Assume that $\{\mu_a*\nu_a\}_{a\in A}$ is uniformly tight. Then there exists a family $\{x_a\}_{a\in A}$, $x_a \in G$ such that $\{\mu_a*x_a\}_{a\in A}$ and $\{(-x_a)*\mu_a\}_{a\in A}$ are uniformly tight.*

LEMMA 1.2. *Let $\{\mu_a\}_{a\in A}$, $\{\nu_a\}_{a\in A}$ be two families of probability measures. Assume that $\{\mu_a\}_{a\in A}$ and $\{\mu_a*\nu_a\}_{a\in A}$ are uniformly tight. Then $\{\nu_a\}_{a\in A}$ is uniformly tight.*

If $\mu$ is a probability measure, then by the support of $\mu$ we will mean the set $C(\mu)$ of all $x \in G$ having the following property: for every open neighbourhood $U$ of $x$, $\mu(U)$ is strictly positive. It is well known that $C(\mu)$ is equal to the smallest closed subset $D$ of $G$ such that $\mu(D) = 1$. Also, the following holds:

$$C(\mu*\nu) = \overline{C(\mu)+C(\nu)}. \tag{1.1}$$

We say that a probability measure $\mu$ *has no idempotent factors* if the equality $\mu = \mu*\lambda$, where $\lambda$ is an idempotent (that is $\lambda*\lambda = \lambda$), implies $\lambda = 0$. The following lemma can be found in [21]:

LEMMA 1.3. *Let $\mu$ be a probability measure. Assume that*

$$\mu = \mu*\nu$$

*for a probability measure $\nu$. Then the closed subgroup generated by $C(\nu)$ is compact, and if $\lambda$ is the normed Haar measure concentrated on this subgroup, then*

$$\mu = \mu*\lambda.$$

*In particular, $\mu$ has no idempotent factors iff the equality*

$$\mu = \mu*\nu$$

*implies $\nu = 0$.*

From this lemma it follows that a probability measure $\lambda$ is an idempotent iff $\lambda$ is the normed Haar measure concentrated on a compact subgroup.

If $\mu$ is a probability measure on $G$ and $\{\lambda_i;\ i \in I\}$ is the family of all idempotent factors of $\mu$, then this family is directed by a partial ordering:

$$\lambda_i < \lambda_j \quad \text{iff} \quad \lambda_i*\lambda_j = \lambda_j.$$

It is easy to verify that there exists a unique maximal element $\lambda$ in this family. $\lambda$ will be called the *maximal idempotent factor of* $\mu$ and can be characterized by the following property:

(1.2) if $\mu = \mu*\nu$ for a probability measure $\nu$, then $C(\nu) \subseteq C(\lambda)$.

Now, a family $(\mu_t)_{t>0}$ is called a *semigroup of probability measures* if $\mu_t * \mu_s = \mu_{t+s}$ for every $t, s > 0$; it is called a *continuous semigroup* if the mapping $t \to \mu_t$ is continuous. $(\mu_t)_{t>0}$ is called *0-continuous* if $\lim_{t\to 0+} \mu_t = 0$. It is known [20] that a semigroup $(\mu_t)_{t>0}$ is continuous if and only if $\lim_{t\to 0+} \mu_t$ exists. If this limit exists, it is the identity of this semigroup (hence idempotent).

A probability measure $\mu$ will be called *embeddable* if there exists a continuous semigroup $(\mu_t)_{t>0}$ of probability measures such that $\mu_1 = \mu$.

**2. Embedding of Gaussian measures.** Throughout this section $G$ will denote a Hausdorff abelian group.

DEFINITION 2.1. A probability measure $\mu$ on $G$ is called *Gaussian* if there are probability measures $\nu_1, \nu_2$ such that

$$\psi(\mu\times\mu) = \nu_1\times\nu_2,$$

that is,

$$\mu\times\mu\big(\psi^{-1}(E)\big) = \nu_1\times\nu_2(E) \tag{2.1}$$

for every $E \in \mathscr{B}\times\mathscr{B}$, where $\psi\colon G\times G\to G\times G$ is defined by

$$\psi(x, y) = (x+y, x-y). \tag{2.2}$$

As has been mentioned in the Preliminaries, the mappings

$$(x, y)\to x\pm y$$

are measurable with respect to $\mathscr{B}$ and the completion of $\mathscr{B}\times\mathscr{B}$ with respect to the product of arbitrary probability measures. Hence $\psi^{-1}(E) \in (\mathscr{B}\times\mathscr{B})^{\mu\times\nu}$ for every $E\in\mathscr{B}\times\mathscr{B}$, and so (2.1) makes sense.

The measures $\nu_1, \nu_2$ are uniquely determined by $\mu$:

$$\nu_1(A) = \nu_1\times\nu_2(A\times G) = \mu\times\mu\big(\{(x, y);\ x+y\in A\}\big) = \mu*\mu(A),$$

for every $A\in\mathscr{B}$, so $\nu_1 = \mu*\mu$. Analogously, $\nu_2 = \mu*\bar{\mu}$, where $\bar{\mu}(B) = \mu(-B)$, for $B\in\mathscr{B}$.

Observe also that

$$\psi(\nu_1\times\nu_2) = \psi^2(\mu\times\mu),$$

that is,

$$\nu_1\times\nu_2\big(\psi^{-1}(E)\big) = \mu\times\mu\big(\{(x, y);\ (2x, 2y)\in E\}\big) \tag{2.3}$$

for $E\in\mathscr{B}\times\mathscr{B}$. This follows from the fact that

$$\psi^{-1}(E)\in(\mathscr{B}\times\mathscr{B})^{\nu_1\times\nu_2};$$

hence there are $C, D\in\mathscr{B}\times\mathscr{B}$ such that

$$C\subseteq\psi^{-1}(E)\subseteq D$$



and

$$\nu_1\times\nu_2(D\setminus C) = 0.$$

Then

$$\psi(\mu\times\mu)(D\setminus C) = \mu\times\mu\big(\psi^{-1}(D\setminus C)\big) = \nu_1\times\nu_2(D\setminus C) = 0$$

and

$$\psi^{-1}(C)\subseteq\psi^{-1}\big(\psi^{-1}(E)\big) = (\psi^2)^{-1}(E)\subseteq\psi^{-1}(D).$$

Hence

$$\mu\times\mu\big((\psi^2)^{-1}(E)\big) = \mu\times\mu\big(\psi^{-1}(D)\big) = \nu_1\times\nu_2(D) = \nu_1\times\nu_2\big(\psi^{-1}(E)\big),$$

which establishes (2.3).

Since every probability measure on $G$ is, by definition, tight, hence $\tau$-regular (see e.g. [22]), and $G$ is completely regular, Definition 2.1 can be stated equivalently:

$\mu$ is Gaussian iff there are probability measures $\nu_1, \nu_2$ such that

$$\int\int f(x+y)g(x-y)\mu(dx)\mu(dy) = \int\int f(x)g(y)\nu_1(dx)\nu_2(dy) \tag{2.4}$$

for every $f, g\in C(G)$.

Remark 2.1. Definition 2.1, suggested by Bernstein's characterization of Gaussian distributions on the real line, has been used by Fréchet in the case of Banach spaces [11] and by Corvin in the case of LCA groups [9]. If $G$ is a real vector space such that the continuous linear functionals generate $\mathscr{B}$, then this definition is equivalent to the usual one.

LEMMA 2.1. *Let $\mu_1, \mu_2$ be Gaussian measures on $G$. Then $\mu_1*\mu_2$ and $\bar{\mu}_1$, $\bar{\mu}_2$ are Gaussian. Moreover, the set of all Gaussian measures is closed in the weak topology.*

The proof of this lemma easily follows from the above equivalent form Definition 2.1 and from the standard properties of convolution.

A $\mathscr{B}$-measurable mapping $\varphi\colon G\to G$ will be called *bi-measurable* if $\varphi(A)\in\mathscr{B}$ whenever $A\in\mathscr{B}$.

LEMMA 2.2. *Let $\mu$ be a Gaussian measure on $G$. Assume that $\theta(x) = 2x$ is one-to-one and bi-measurable. Then $\mu$ is concentrated on a coset of the subgroup $G_0 = \bigcap_{n=0}^{\infty} 2^nG$. If $\mu$ is additionally symmetric, then it is concentrated on $G_0$ and there exists a unique symmetric Gaussian measure $\mu_{1/2}$ such that*

$$\mu_{1/2}*\mu_{1/2} = \mu.$$

Proof. Let $\theta(x) = 2x$ for $x\in G$. Putting in (2.3) $E = A\times G$, where $A\in\mathscr{B}$, we have

$$\theta(\mu)(A) = \mu(\{x;\ 2x \in A\}) = \psi^2(\mu\times\mu)(A\times G)$$
$$= \psi(\nu_1\times\nu_2)(A\times G) = \nu_1 * \nu_2(A).$$

Since $\theta(G) = 2G \in \mathscr{B}$, $\theta(\mu)$ is concentrated on $2G$. Thus $\nu_1 * \nu_2$ is concentrated on $2G$, and hence

$$\int \nu_1(2G-x)\,\nu_2(dx) = 1,$$

which implies that

$$\nu_1(2G-x) = 1 \quad \nu_2-\text{a.e.}$$

Hence we infer that there exists an $x_0 \in G$ such that $\nu_1(2G-x_0) = 1$. By repeating this argument we obtain $\mu(2G-x_1) = 1$ for an $x_1 \in G$. Hence $\nu_2 = \mu * \bar{\mu}$ is concentrated on $(2G-x_1)+(2G+x_1) = 2G$.

Now, let $\mu$ be symmetric. Let us denote $\nu = \nu_1 = \nu_2$. Let us define

$$\mu_{1/2}(A) = \nu(2A), \qquad A \in \mathscr{B}. \tag{2.5}$$

Since $\nu$ is concentrated on $2G$, $\mu_{1/2}$ is a symmetric probability measure. From (2.5) and from the standard formula for a change of variables we obtain

$$\psi(\mu_{1/2}\times\mu_{1/2})(A\times B) = \psi(\nu\times\nu)(2A\times 2B).$$

By (2.3) we have

$$\psi(\nu\times\nu)(2A\times 2B) = \mu\times\mu\big(\{(x, y);\ (2x, 2y) \in 2A\times 2B\}\big) = \mu\times\mu(A\times B)$$

for $A, B \in \mathscr{B}$. Hence we have

$$\psi(\mu_{1/2}\times\mu_{1/2}) = \mu\times\mu, \tag{2.6}$$

which means that $\mu_{1/2}$ is Gaussian. Moreover, (2.6) implies that

$$\mu_{1/2} * \mu_{1/2} = \mu.$$

If $\gamma$ is another symmetric Gaussian measure satisfying $\gamma * \gamma = \mu$, then also

$$\psi(\gamma\times\gamma) = \mu\times\mu.$$

By this formula along with (2.3) and (2.6) we obtain

$$\mu_{1/2}(\{x;\ 2x \in A\}) = \gamma(\{x;\ 2x \in A\})$$

for all $A \in \mathscr{B}$. Putting $A = 2B$, we obtain $\mu_{1/2}(B) = \gamma(B)$ for all $B \in \mathscr{B}$. Next, since $\mu_{1/2}$ is symmetric and Gaussian, we have $\mu_{1/2} * \mu_{1/2} = \mu_{1/2} * \bar{\mu}_{1/2} = \mu$; hence $\mu$ is concentrated on $2G$. By repeating this reasoning we infer that $\mu$ is concentrated on $G_0 = \bigcap_{n=0}^{\infty} 2^n G$.

In the general case, $\nu_2$ is Gaussian and symmetric, and so it is concentrated on $G_0$. By the previous arguments $\mu$ is concentrated on a coset of $G_0$.

From this lemma we infer that every symmetric Gaussian measure has the symmetric Gaussian "roots" of order $2^n$, $n = 1, 2, \ldots$



Observe also that if $G$ is a metrizable abelian group and if $\mu$ is a Gaussian measure on $G$, then $\mu$ is concentrated on a $\sigma$-compact metrizable subgroup $G_1$ of $G$. By Urysohn's theorem, $G_1$ can be regarded as a Borel subset of the Hilbert cube. If $G$ has no non-zero elements of order two, then, by Kuratowski's theorem, the mapping $x \to 2x$ is bi-measurable on $G_1$. Using Lemma 2.2, we find that $\mu$ is concentrated on a coset of a Borel subgroup $G_0$ of $G_1$. $G_0$ is a Borel space, i.e. it is homeomorphic to a Borel subset of the Hilbert cube. Moreover, $x \to 2x$ is a Borel isomorphism of $G_0$. So, we have the following

PROPOSITION 2.1. *Let $\mu$ be a Gaussian measure on a metrizable abelian group $G$. Assume that $G$ has no non-zero elements of order two. Then there exists an element $x_0 \in G$ such that $\mu * x_0$ is concentrated on a Borel subgroup $G_0$ of $G$ which is a Borel space and has the property that $x \to 2x$ is a Borel automorphism of $G_0$.*

LEMMA 2.3. *Let $(\mu_n)_{n=0}^{\infty}$ be a sequence of symmetric probability measures on $G$. Assume that*

$$\mu_0 = \mu, \qquad \mu_n * \mu_n = \mu_{n-1}, \qquad n = 1, 2, \ldots$$

*Let $\lambda$ be the maximal idempotent factor of $\mu$. Assume that*

$$\lambda * \mu_n = \mu_n, \qquad n = 0, 1, \ldots$$

*Then there exists a unique continuous semigroup $(\nu_t)_{t>0}$ of symmetric probability measures such that $\nu_1 = \mu$, $\nu_{1/2^n} = \mu_n$. Moreover, $\lim_{t\to 0+} \nu_t = \lambda$.*

Proof. Let $D$ be the set of all $t \in (0, \infty)$ of the form

$$t = [t] + 1/2^{n_1} + 1/2^{n_2} + \ldots + 1/2^{n_k}, \tag{2.7}$$

where $0 < n_1 < n_2 < \ldots < n_k$ and $n_i$ are some positive integers, $i = 1, \ldots, k$, $k = 0, 1, \ldots$ For $t$ of the form (2.7) define

$$\nu_t = \mu_0^{*[t]} * \mu_{n_1} * \mu_{n_2} * \ldots * \mu_{n_k}.$$

Then $\nu_t * \nu_s = \nu_{t+s}$ and $\lambda * \nu_t = \nu_t$ for all $t, s \in D$.

Now we show that the family $\{\nu_t;\ t \in D \cap (0, 1]\}$ is uniformly tight. Let $\{t_n;\ n = 1, 2, \ldots\}$ be an enumeration of $D \cap (0, 1]$. Observe that

$$\nu_{t_n/2} * \nu_{(1-t_n/2)} = \mu.$$

In virtue of Lemma 1.1 there exists a sequence $(x_n)$, $x_n \in G$ such that

$$\{\nu_{t_n/2} * x_n\}$$

is uniformly tight. Hence $\{\nu_{t_n/2} * (-x_n)\}$ is also uniformly tight and so is

$$\{\nu_{t_n}\} = \{\nu_{t_n/2} * x_n * (-x_n) * \nu_{t_n/2}\}.$$

By Lemma 1.2 we infer that $\{\nu_{(1-t_n)}\}$ is also uniformly tight.

Next, let $(t_\alpha;\ \alpha \in A)$, $t_\alpha \in D \cap (0,1]$ be a net convergent to 0. Since $\{\nu_{t_\alpha};\ \alpha \in A\}$, $\{\nu_{(1-t_\alpha)};\ \alpha \in A\}$ are uniformly tight, they are conditionally compact. Let $\varepsilon$ be an accumulation point of $(\nu_{t_\alpha})_{\alpha \in A}$. Let $(t_\beta)$ be a subnet of $(t_\alpha)$ such that $\nu_{t_\beta}$ converges to $\varepsilon$ and $\nu_{(1-t_\beta)}$ converges to a probability measure $\gamma$. By the continuity of convolution we have $\mu = \varepsilon * \gamma$. Now, let $(s_\beta)$ be a subnet of $(t_\beta)$ such that $2s_\beta < t_\beta$. Then

$$\nu_{s_\beta} * \nu_{r_\beta} = \nu_{t_\beta},$$

where $r_\beta = t_\beta - s_\beta > s_\beta$. Observe that $\nu_{r_\beta}$ converges to $\lambda$: for, if $\varepsilon_0$ is an accumulation point of $(\nu_{r_\beta})$, then $\varepsilon * \varepsilon_0 = \varepsilon$, and hence

$$\varepsilon_0 * \mu = \varepsilon_0 * (\varepsilon * \gamma) = (\varepsilon_0 * \varepsilon) * \gamma = \varepsilon * \gamma = \mu,$$

which, in virtue of Lemma 1.3, the maximality of $\lambda$ and (1.2), gives

$$C(\varepsilon_0) \subseteq C(\lambda).$$

Hence $\varepsilon_0 * \lambda = \lambda$. On the other hand,

$$\nu_{r_\beta} * \lambda = \nu_{r_\beta};$$

hence $\varepsilon_0 * \lambda = \varepsilon_0$. Thus, we obtain $\varepsilon_0 = \varepsilon_0 * \lambda = \lambda$, and hence $\nu_{r_\beta}$ converges weakly to $\lambda$.

Next, $\nu_{r_\beta} = \nu_{s_\beta} * \nu_{u_\beta}$, where $u_\beta = t_\beta - 2s_\beta > 0$. If $\delta$ is an accumulation point of $\nu_{u_\beta}$, then $\lambda = \varepsilon * \delta$. Thus, in virtue of the symmetry of $\varepsilon$ and $\delta$, we obtain $C(\varepsilon) \subseteq C(\lambda)$, and so $\varepsilon * \lambda = \lambda$. Since $\lambda * \nu_{s_\beta} = \nu_{s_\beta}$, we have $\lambda * \varepsilon = \varepsilon$, which gives $\varepsilon = \lambda$.

Now, using Proposition 5.2 from [20], we infer that $(\nu_t)_{t \in D_0}$ is a continuous semigroup, where $D_0 = D \cup \{0\}$, $\nu_0 = \lambda$. Since $\{\nu_t;\ t \in D \cap (0,1]\}$ is uniformly tight, by Proposition 5.3 in [20] we find that there exists a unique continuous semigroup $(\nu_t)_{t>0}$ with the desired properties.

COROLLARY 2.1. *Let $(\mu_n)$ be a sequence of symmetric probability measures on $G$. Assume that $\mu_0 = \mu$, $\mu_n * \mu_n = \mu_{n-1}$. If $\mu$ has no idempotent factors, then there exists a unique continuous semigroup $(\nu_t)_{t>0}$ of symmetric measures such that $\nu_{1/2^n} = \mu_n$, $n = 0, 1, \ldots$ Moreover, $\lim_{t \to 0+} \nu_t = 0$.*

PROPOSITION 2.2. *Assume that $G$ has no non-zero elements of order two. Let $\mu$ be a symmetric Gaussian measure on $G$. Let $\lambda$ be the maximal idempotent factor of $\mu$. Then there exists a unique continuous semigroup of symmetric Gaussian measures $(\nu_t)_{t>0}$ such that $\nu_1 = \mu$. Moreover, $\lim_{t \to 0+} \nu_t = \lambda$.*

Proof. Let $\nu_{1/2}$ be a symmetric Gaussian measure such that

$$\nu_{1/2} * \nu_{1/2} = \mu.$$

From (2.5) we have

$$\nu_{1/2}(A) = \mu * \mu(2A).$$



Hence, if $x \in C(\lambda)$, then $x * \nu_{1/2} = \nu_{1/2}$, and so $\lambda * \nu_{1/2} = \nu_{1/2}$. By induction, we obtain $\nu_{1/2^n}$ such that $\nu_{1/2^n} * \nu_{1/2^n} = \nu_{1/2^{n-1}}$. By Lemma 2.3 we obtain the conclusion.

COROLLARY 2.2. *Assume that $G$ has no non-zero elements of order two. Then every symmetric Gaussian measure $\mu$ is infinitely divisible.*

**3. Essential symmetry of Gaussian measures.** Throughout this section $G$ denotes a metric abelian group having no non-zero elements of order two.

We prove that every Gaussian measure on $G$ is essentially symmetric, i.e., is a translation of a symmetric Gaussian measure. The main tool used in the proof of this fact is the theory of probability operators. We start with some definitions and basic facts concerning those operators.

By $UC(G)$ we will denote the space of all real-valued bounded functions defined on $G$ which are uniformly continuous with respect to a fixed translation-invariant metric on $G$. By $B(G)$ we will denote the space of all real-valued bounded functions defined on $G$ which are Borel measurable.

Now, given a probability measure $\mu$ on $G$, define $T_\mu\colon B(G) \to B(G)$ by the formula

$$T_\mu f(x) = \int f(x+y)\,\mu(dy). \tag{3.1}$$

It is easy to see that if $T_\mu = T_\nu$, then $\mu = \nu$ and that $T_{\mu*\nu} = T_\mu T_\nu$.

Now, let $\theta(x) = 2x$. $\theta$ induces an operator $f \to f \circ \theta$ from $B(G)$ into $B(G)$, where $(f \circ \theta)(x) = f(2x)$. If $G$ is such that $\theta$ is a Borel automorphism of $G$, then we can also define $\theta^{-1}(x) = x/2$. $\theta^{-1}$ induces, as above, an operator $f \to f \circ \theta^{-1}$. The operators induced on $B(G)$ by $\theta$, $\theta^{-1}$ will be denoted by the same symbols.

LEMMA 3.1. *Let $\mu$ be a Gaussian measure on $G$. Assume that $\theta$ is bimeasurable. Then*

$$T_\mu \theta = \theta T_{\mu^{*3} * \bar\mu} = \theta (T_\mu)^3 T_{\bar\mu}. \tag{3.2}$$

Proof. By the change of variables formula we have

$$\int f(y)\,\theta(\mu)(dy) = \int f \circ \theta(y)\,\mu(dy)$$

for every $f \in B(G)$. From (2.3) we infer that $\theta(\mu) = \mu^{*3} * \bar\mu$. Thus, by (3.1), we obtain

$$\begin{aligned}(T_\mu \theta f)(x) &= \int f(2x+2y)\,\mu(dy) = \int f(2x+y)\,\theta(\mu)(dy)\\ &= \int f(2x+y)(\mu^{*3} * \bar\mu)(dy) = (\theta T_{\mu^{*3}*\bar\mu} f)(x),\end{aligned}$$

which completes the proof.

THEOREM 3.1. *Assume that $G$ is a Borel space and that $\theta$ is a Borel* *isomorphism of $G$. Let $\mu$ be a Gaussian measure on $G$ without idempotent factors. Then there exists an $x_0 \in G$ such that*

$$\mu = (\mu*\bar{\mu})_{1/2}*x_0,$$

*where $(\mu*\bar{\mu})_{1/2}$ is the symmetric Gaussian square root of $\mu*\bar{\mu}$.*

Proof. Let $\mu_n = (\mu*\bar{\mu})_{1/2^n}$ be the symmetric Gaussian root of $\mu*\bar{\mu}$ of order $2^n$. By Proposition 2.2, $\lim \mu_n = 0$.

Now, since $\mu*\bar{\mu}$ is Gaussian, we obtain from (2.4) applied to $f, g \in B(G)$,

$$\begin{aligned} T_{\mu*\bar{\mu}}f(x)T_{\mu*\bar{\mu}}g(x) &= \iint f(x+x_1)g(x+x_2)(\mu*\bar{\mu})(dx_1)(\mu*\bar{\mu})(dx_2) \\ &= \iint f(x+x_1)g(x+x_2)\mu_1^{*2}(dx_1)\mu_1^{*2}(dx_2) \\ &= \iint f(x+x_1+x_2)g(x+x_1-x_2)\mu_1(dx_1)\mu_1(dx_2) \\ &= T_{\mu_1}\varphi(x), \end{aligned}$$

where

$$\varphi(y) = \int f(y+x_2)g(y-x_2)\mu_1(dx_2).$$

Analogously

$$T_{\mu*\mu_1}f(x)T_{\mu*\mu_1}g(x) = T_\mu\varphi(x).$$

Hence

$$\begin{aligned} T_{\mu_1}(T_{\mu*\bar{\mu}}fT_{\mu*\bar{\mu}}g) &= T_{\mu_1}(T_{\mu_1}\varphi) = T^{*2}_{\mu_1}\varphi \\ &= T_{\mu*\bar{\mu}}\varphi = T_{\bar{\mu}}(T_\mu\varphi) \\ &= T_{\bar{\mu}}(T_{\mu*\mu_1}fT_{\mu*\mu_1}g). \end{aligned}$$

Putting $f = g$, we obtain

$$T_{\mu_1}\big((T_{\mu*\bar{\mu}}f)^2\big) = T_{\bar{\mu}}\big((T_{\mu*\mu_1}f)^2\big). \tag{3.3}$$

Now, by Lemma 3.1,

$$\begin{aligned} T_{\mu_1}\theta &= \theta T_{(\mu*\bar{\mu})^{*2}}, \\ T_\mu\theta &= \theta T_{\mu^{*3}*\bar{\mu}}, \\ T_{\bar{\mu}}\theta &= \theta T_{\bar{\mu}^{*3}*\mu}. \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4), we have for $f \in B(G)$

$$\begin{aligned} T_{\mu_1}\big((T_{\mu_1}f)^2\big) &= T_{\mu_1}\big((T_{\mu_1}\theta(\theta^{-1}f))^2\big) = T_{\mu_1}\big(\theta(T_{(\mu*\bar{\mu})^{*2}}\theta^{-1}f)^2\big) \\ &= \theta T_{(\mu*\bar{\mu})^{*2}}\big((T_{(\mu*\bar{\mu})^{*2}}\theta^{-1}f)^2\big) = \theta T_{\mu_1^{*4}}\big((T_{\mu_1^{*4}}\theta^{-1}f)^2\big) \\ &= \theta T_{\mu_1^{*3}*\bar{\mu}}\big((T_{\mu_1^{*3}*\mu}\theta^{-1}f)^2\big) = \theta T_{\mu*\bar{\mu}^{*3}}\big((T_{\mu^{*3}*\bar{\mu}}\theta^{-1}f)^2\big) \\ &= T_{\bar{\mu}}\big((T_\mu f)^2\big). \end{aligned}$$

So, we have

$$T_{\mu_1}\big((T_{\mu_1}f)^2\big) = T_{\bar{\mu}}\big((T_\mu f)^2\big), \tag{3.5}$$



for $f \in B(G)$. Hence, if $T_{\mu_1}f = 0$ $\mu_1$-a.e., then $T_\mu f = 0$ $\bar{\mu}$-a.e. Thus, the formula

$$VT_{\mu_1}f = T_\mu f \tag{3.6}$$

defines on $\mathscr{R}(T_{\mu_1})$ (the range of $T_{\mu_1}$) a linear isometry $V$ from a subspace of $L_2(\mu_1)$ into $L_2(\bar{\mu})$.

We extend $V$ to $L_2(\mu_1)$. First of all, observe that

$$\theta V^2 T_{(\mu*\bar{\mu})^{*2}} = \theta T_{\mu^{*3}*\bar{\mu}} = T_\mu\theta = VT_{\mu_1}\theta = V\theta T_{(\mu*\bar{\mu})^{*2}}.$$

Hence

$$\theta V^2 f = V\theta f \quad \text{for} \quad f \in \mathscr{R}(T_{(\mu*\bar{\mu})^{*2}}).$$

Hence, if $h = T_{\mu_2}f$ for a certain $f \in B(G)$, then

$$\begin{aligned} VT_{\mu_2}h &= VT_{\mu_1}f = VT_{\mu_1}\theta(\theta^{-1}f) = V\theta T_{(\mu*\bar{\mu})^{*2}}\theta^{-1}f \\ &= \theta V^2 T_{(\mu*\bar{\mu})^{*2}}\theta^{-1}f = \theta T_{\mu^{*2}}T_{\mu*\bar{\mu}}\theta^{-1}f \\ &= \theta T_{\mu^{*2}}\theta^{-1}T_{\mu_2}f = \theta T_{\mu^{*2}}\theta^{-1}h. \end{aligned} \tag{3.7}$$

Thus, if we show that the formula

$$V'T_{\mu_2}h = \theta T_{\mu^{*2}}\theta^{-1}h \quad \text{for} \quad h \in B(G)$$

defines a linear mapping $V'$ on a subspace $\mathscr{R}(T_{\mu_2})$ of $L_2(\mu_1)$ into $L_2(\bar{\mu})$, then (3.7) shows that $V'$ is an extension of $V$. This extension will be denoted in the sequel by the same symbol $V$.

More generally, we will show that an extension of $V$ to $\bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$ can be defined by the following formula:

$$VT_{\mu_n}h = \theta^{n-1}T_{\mu^{*2^{n-1}}}\theta^{-n+1}h \quad \text{for} \quad h \in B(G). \tag{3.8}$$

Let $h \in B(G)$. From (3.4) and (3.5) we have

$$\begin{aligned} T_{\bar{\mu}}\big((\theta T_{\mu^{*2}}\theta^{-1}h)^2\big) &= \theta T_{\bar{\mu}^{*3}*\mu}\big((T_{\mu^{*2}}\theta^{-1}h)^2\big) \\ &= \theta T_{\bar{\mu}^{*2}*\mu*\mu_1}\big((T_{\mu_1*\mu}\theta^{-1}h)^2\big) \\ &= T_{(\mu*\bar{\mu})^{*2}}\big((T_{\mu*\bar{\mu}}\theta^{-1}h)^2\big) = T_{\mu_1}\big((T_{\mu_2}h)^2\big). \end{aligned}$$

By induction,

$$T_{\bar{\mu}}\big((\theta^{n-1}T_{\mu^{*2^{n-1}}}\theta^{-n+1}h)^2\big) = T_{\mu_1}\big((T_{\mu_n}h)^2\big), \tag{3.9}$$

for every $h \in B(G)$ and $n = 1, 2, \ldots$ Moreover, if

$$h = T_{\mu_n}f_0 = T_{\mu_{n+1}}f_1 = \ldots = T_{\mu_{n+m}}f_m$$

for some $f_0, \ldots, f_m \in B(G)$ and some positive integers $n, m$, then

$$T_{\mu_{n+k-1}}f_{k-1} = T_{\mu_{n+k}}T_{\mu_{n+k}}f_{k-1} = T_{\mu_{n+k}}f_k \quad \text{for} \quad 1 \leqslant k \leqslant m.$$

Thus

$$g_k = f_k - T_{\mu_{n+k}}f_{k-1} \in \operatorname{Ker} T_{\mu_{n+k}}. \tag{3.10}$$

From (3.9) and (3.10) we obtain

$$(3.11)\qquad \theta^{n+k-1}T_{\mu^{*2^{n+k-1}}}\theta^{-n-k+1}f_k = \theta^{n+k-1}T_{\mu^{*2^{n+k-1}}}\theta^{-n-k+1}T_{\mu_{n+k}}f_{k-1}$$

$\bar{\mu}$-a.e. Using (3.4), we obtain

$$(3.12)\qquad \theta^{-n-k+1}T_{\mu_{n+k}} = T_{(\mu*\bar{\mu})^{*2^{n+k-2}}}\,\theta^{-n-k+1}.$$

In virtue of (3.12) equality (3.11) can be written in the form

$$\begin{aligned}\theta^{n+k-1}T_{\mu^{*2^{n+k-1}}}\theta^{-n-k+1}f_k &= \theta^{n+k-1}T_{\mu^{*2^{n+k-1}}}T_{(\mu*\bar{\mu})^{*2^{n+k-2}}}\,\theta^{-n-k+1}f_{k-1}\\ &= \theta^{n+k-1}T_{(\mu^{*3}\bar{\mu})^{*2^{n+k-2}}}\theta^{-n-k+1}f_{k-1} = \theta^{n+k-2}T_{\mu^{*2^{n+k-2}}}\theta^{-n-k+2}f_{k-1}\end{aligned}$$

$\bar{\mu}$-a.e. for $1 \leqslant k \leqslant m$. Hence

$$(3.13)\qquad \theta^{n-1}T_{\mu^{*2^{n-1}}}\theta^{-n+1}\,f_0 = \theta^{n+m-1}T_{\mu^{*2^{n+m-2}}}\theta^{-n-m+1}f_m$$

$\bar{\mu}$-a.e., which means that (3.8) defines a linear mapping on the subspace $\bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$ of $L_2(\mu_1)$ into $L_2(\bar{\mu})$. Putting $n = 1$ in (3.13), we see that this mapping is an extension of the mapping $V$ defined by (3.6). Moreover, (3.9) shows that this extension is also an isometry defined on a subspace of $L_2(\mu_1)$ into $L_2(\bar{\mu})$.

Next, observe that if $h \in UC(G)$, then $T_{\mu_n}h$ tends uniformly to $h$ as $n\to\infty$, by the weak convergence of $\mu_n$ to 0. Hence $\mathscr{R}(T_{\mu_n})$ contains a dense subset of $UC(G)$. Since $G$ is a metric space and $\mu_1$ is regular, $\bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$ is dense in $L_2(\mu_1)$. Thus, we can extend $V$ to $L_2(\mu_1)$. This extension, denoted in the sequel by the same symbol, is also an isometry.

Now we will show that $V$ is multiplicative on $\bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$. Let $h_1, h_2 \in \bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$. Then

$$h_i = T_{\mu_{n'}}f_i$$

for an integer $n'$ and some $f_i \in B(G)$, $i = 1, 2$. As in the first part of the proof, we obtain

$$h_1h_2 = T_{\mu_{n'+1}}\varphi,$$

where

$$\varphi(y) = \int f_1(y+x_2)f_2(y-x_2)\mu_{n'+1}(dx_2).$$

Hence

$$V(h_1h_2) = \theta^{n'}T_{\mu^{*2^{n'}}}\theta^{-n}\varphi.$$



On the other hand, in a similar way

$$\begin{aligned}&(Vh_1)(x)(Vh_2)(x)\\ &= (\theta^{n'-1}T_{\mu^{*2^{n'-1}}}\theta^{-n'+1}f_1)(x)(\theta^{n'-1}T_{\mu^{*2^{n'-1}}}\theta^{-n'+1}f_2)(x)\\ &= \theta^{n'-1}\int\int(\theta^{-n'+1}f_1)\left(x+\frac{x_1+x_2}{2}\right)(\theta^{-n'+1}f_2)\left(x+\frac{x_1-x_2}{2}\right)\\ &\qquad\qquad (\mu*\bar{\mu})^{*2^{n'-1}}(dx_2)\,\mu^{*2^n}(dx_1)\\ &= \theta^{n'}\int\left\{\int(\theta^{-n'}f_1)(x+x_1+x_2)(\theta^{-n'}f_2)(x+x_1-x_2)(\mu*\bar{\mu})^{*2^{n'-1}}(dx_2)\right\}\mu^{*2^{n'}}(dx_1)\\ &= \theta^{n'}T_{\mu^{*2^{n'}}}\varkappa(x),\end{aligned}$$

where

$$\begin{aligned}\varkappa(y) &= \int(\theta^{-n'}f_1)(y+x_2)(\theta^{-n'}f_2)(y-x_2)(\mu*\bar{\mu})^{*2^{n'-1}}(dx_2)\\ &= \theta^{-n'}\int f_1(y+x_2)f_2(y-x_2)\mu_{n'+1}(dx_2) = \theta^{-n'}\varphi(y).\end{aligned}$$

Hence

$$V(h_1h_2) = (Vh_1)(Vh_2)$$

for $h_i = \bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$, $i = 1, 2$.

Next, we show that

$$(3.14)\qquad V|h| = |Vh| \quad \text{for} \quad h \in L_2(\mu_1).$$

This part of the proof seems to be known; it is included for the sake of completeness.

We first show (3.14) for $h \in \bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$. Let $h = \bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$. Then $|h| \leqslant c$. Let $w_n(t)$ be a sequence of polynomials convergent uniformly on $[0, c]$ to $\sqrt{t}$. Then $w_n(h^2)\to|h|$ in $L_2(\mu_1)$. Also $w_n\big((Vh)^2\big)\to|Vh|$ in $L_2(\bar{\mu})$. Since $V$ is multiplicative on the algebra $\bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$, we have

$$Vw_n(h^2) = w_n(Vh^2) = w_n\big((Vh)^2\big).$$

The desired conclusion now follows from the continuity of $V$.

Since $\bigcup_{n=1}^{\infty}\mathscr{R}(T_{\mu_n})$ is dense in $L_2(\mu_1)$, we obtain (3.14). In particular, $V$ is positive.

Finally, from Theorem 2 in [14] it follows that $V$ is pointwise induced, i.e., there is a (measure preserving) transformation $k\colon (G, \bar{\mu})\to(G, \mu_1)$ such that

$$(3.15)\qquad (Vg)(x) = g\big(k(x)\big)$$

for every $g \in L_2(\mu_1)$. Denote

$$(L_y f)(x) = f(x+y), \quad \text{for} \quad f \in B(G).$$

Then

$$(L_y VT_{\mu_1})f(x) = (L_y T_\mu f)(x) = (T_\mu L_y f)(x) = (VT_{\mu_1} L_y f)(x) = (VL_y T_{\mu_1} f)(x),$$

for $f \in C(G)$ so $L_y Vh = VL_y h$ for $h \in \mathscr{R}(T_{\mu_1}) \cap C(G)$. Hence, in virtue of (3.15) we obtain

$$(VT_{\mu_1} f)(y) = (L_y VT_{\mu_1} f)(0) = (VT_{\mu_1} L_y f)(0) = (T_{\mu_1} L_y f)(k(0)) = (L_y T_{\mu_1} f)(k(0)) = (T_{\mu_1} f)(k(0)+y) = (T_{\mu_1 * x_0} f)(y) \quad \text{for} \quad f \in C(G)$$

where $x_0 = k(0) \in G$. Since

$$VT_{\mu_1} f = T_\mu f,$$

we obtain

$$T_\mu = T_{\mu_1 * x_0}.$$

Thus

$$\mu = \mu_1 * x_0 = (\mu * \bar{\mu})_{1/2} * x_0,$$

which completes the proof.

Corollary 3.1. *Let $\mu$ be a Gaussian measure on $G$ without idempotent factors. Then there exists an element $x_0 \in G$ such that*

$$\mu = (\mu * \bar{\mu})_{1/2} * x_0,$$

*where $(\mu * \bar{\mu})_{1/2}$ is the symmetric Gaussian square root of $\mu * \bar{\mu}$.*

Proof. By Proposition 2.1 we infer that there exists an $x_1 \in G$ such that $\mu * x_1$ is concentrated on a Borel subgroup $G_0$ of $G$ which is a Borel space and which has the property that $x \to 2x$ is a Borel isomorphism of $G_0$. Clearly, $\mu * x_1$ restricted to $G_0$ is a Gaussian measure on $G_0$ without idempotent factors. By Theorem 3.1 we infer that there exists an element $x_2 \in G_0$ such that

$$\mu * x_1 = (\mu * \bar{\mu})_{1/2} * x_2.$$

Thus, we have

$$\mu = (\mu * \bar{\mu})_{1/2} * x_0,$$

where $x_0 = x_2 - x_1$, which completes the proof.

In the case of vector spaces we can prove an analogue of Corollary 3.1 without using Lemma 2.2 and under slightly different assumptions.



Corollary 3.2. *Let $G$ be a real vector space which is a Borel space. Assume that $(x, y) \to x - y$ is measurable with respect to $\mathscr{B} \times \mathscr{B}$ and $\mathscr{B}$ and that the mapping $a \to ax$ of $R$ into $G$ is continuous for every fixed $x \in G$. Let $\mu$ be a Gaussian measure on $G$. Then there exists an $x_0 \in G$ such that*

$$\mu = (\mu * \bar{\mu})_{1/2} * x_0,$$

*where $(\mu * \bar{\mu})_{1/2}$ is the symmetric Gaussian square root of $\mu * \bar{\mu}$.*

Proof. Observe that in the proof of Theorem 3.1 we have only used the Borel measurability of the mappings

$$(x, y) \to x \pm y \quad \text{and} \quad x \to 2^n x, \; n = \pm 1, \pm 2, \ldots$$

and the fact that $\mu_n$ converges weakly to zero, where $\mu_n$ is the symmetric Gaussian root of $\mu * \bar{\mu}$ of order $2^n$. However, the Borel measurability of $(x, y) \to x + y$ and $x \to 2^n$ is a consequence of the Borel measurability of $(x, y) \to x - y$; hence, by Kuratowski's theorem, $x \to x/2^n$ is also Borel measurable. Further, the fact that $\mu_n$ converges weakly to zero is a consequence of the continuity of $a \to ax$ as well as (2.5) and the application of the change of variables formula

$$\int f(x) \mu_n(dx) = \int f(x/2^n)(\mu * \bar{\mu})(dx) \to f(0),$$

for every $f \in C(G)$. This completes the proof of the corollary.

**4. The zero-one law.** Throughout this section $G$ will denote a Hausdorff abelian group, unless stated otherwise.

Definition 4.1. Let $\mu_1, \mu_2$ be two probability measures on $G$. $\mu_1, \mu_2$ are called *associated probability measures* if there exist probability measures $\nu_1, \nu_2$ such that

$$\psi(\mu_1 \times \mu_2) = \nu_1 \times \nu_2. \tag{4.1}$$

Lemma 4.1. *Assume that $x \to 2x$ is bi-measurable and one-to-one. If $\mu_1, \mu_2$ are associated and symmetric, then $\mu_1 = \mu_2$ and, by (4.1), they are Gaussian. If $\mu_1, \mu_2$ are associated and are translations of some symmetric measures, then $\mu_1, \mu_2$ are Gaussian and $\mu_1$ is a translation of $\mu_2$.*

Proof. Assume that $\mu_1, \mu_2$ are symmetric probability measures satisfying (4.1). Then, in the same way as (2.3), we obtain

$$\psi^2(\mu_1 \times \mu_2) = \psi(\nu_1 \times \nu_2). \tag{4.2}$$

Since $\nu_1 = \mu_1 * \mu_2$, $\nu_2 = \mu_1 * \bar{\mu}_2 = \mu_1 * \mu_2$, we have

$$\nu_1 = \nu_2 = \bar{\nu}_1 = \bar{\nu}_2.$$

By (4.1) we obtain

$$\mu_1(\{x;\; 2x \in A\}) = \nu_1 * \nu_2(A),$$
$$\mu_2(\{x;\; 2x \in A\}) = \nu_1 * \nu_2(A);$$

so

$$\mu_1(A) = \nu_1 * \nu_2(2A) = \nu_1 * \bar{\nu}_2(2A) = \mu_2(A).$$

Next, if $\mu_1 = \mu_1' * x_1$, $\mu_2 = \mu_2' * x_2$, where $\mu_1'$, $\mu_2'$ are symmetric, then by (4.1) and the change of variables formula it follows that

$$\psi(\mu_1' \times \mu_2') = \nu_1' \times \nu_2',$$

where $\nu_1' = \nu_1 * (-x_1 - x_2)$ and $\nu_2' = \nu_2 * (-x_1 + x_2)$. Thus $\mu_1'$, $\mu_2'$ are associated and symmetric; hence they are Gaussian and $\mu_1' = \mu_2'$, and so

$$\mu_2 = \mu_1 * (x_2 - x_1).$$

PROPOSITION 4.1. *Let $G$ be a metric group having no non-zero elements of order two. If $\mu_1$, $\mu_2$ are Gaussian measures on $G$ without idempotent factors and satisfying* (4.1), *then there exists an element $x_0 \in G$ such that*

$$\mu_1 = \mu_2 * x_0.$$

Proof. This follows immediately from Lemma 4.1 and from Corollary 3.1.

PROPOSITION 4.2. *Let $G$ be a real vector space which is a Borel space. Assume that the mapping $(x, y) \to x - y$ is $\mathscr{B} \times \mathscr{B}$ measurable and that $a \to ax$ is continuous for every fixed $x \in G$. If $\mu_1$, $\mu_2$ are two Gaussian measures satisfying* (4.1), *then $\mu_1 = \mu_2 * x_0$ for an element $x_0 \in G$.*

Proof. This is an immediate consequence of Corollary 3.2 and Lemma 4.1.

LEMMA 4.2. *Let $G$ be a countable abelian group endowed with the discrete topology. Let $\mu$ be a Gaussian measure on $G$. If $\mu(\{0\}) > 0$, then $\mu$ is the normed Haar measure concentrated on a finite subgroup $I$ of $G$. Moreover, $I$ does not contain non-zero elements of order two.*

Proof. The arguments needed to prove the first part of this lemma are standard and can be found for example, in [17], p. 101, Remark 2; for a detailed proof, see [6], Lemma 2.

We prove the last part. Let $I = \{x;\ \mu(\{x\}) > 0\}$. From the first part of the lemma, $I$ is a finite subgroup of $G$ and $\mu$ is the normed Haar measure concentrated on $I$. In particular, $\mu$ is symmetric. Since $\mu$ is Gaussian (and symmetric), its characteristic function $\hat{\mu}$ satisfies the equation

$$\hat{\mu}(\chi+\gamma)\hat{\mu}(\chi-\gamma) = \hat{\mu}(\chi)^2\hat{\mu}(\gamma)^2 \quad \text{for} \quad \chi, \gamma \in \hat{G}, \tag{4.3}$$

where $\hat{G}$ denotes the dual group of $G$. $\mu$ is an idempotent, and hence $\hat{\mu}(\gamma) = 0$ or $\hat{\mu}(\gamma) = 1$. Since

$$\hat{\mu}(\gamma) = \int \gamma(x)\mu(dx) = \int_I \gamma(x)\mu(dx),$$



$\hat{\mu}(\gamma) = 1$ iff $\gamma(I) \subseteq \{1\}$, that is, $\hat{\mu}(\gamma) = 1$ iff

$$\gamma \in A(G, I) = \{\gamma \in \hat{G};\ \gamma(x) = 1 \text{ for } x \in I\}.$$

Thus, if we denote $H = A(G, I)$, then, by the duality between annihilators (see 2.1.3 in [19]) we obtain

$$I = A(G, H).$$

By (4.3) we see that $H$ is closed under division by two:

$$2\gamma \in H \Rightarrow \gamma \in H.$$

Hence $\hat{G}/H$ does not contain any non-zero element of order 2. By the duality between subgroups and quotient groups (see 2.1 in [19]) we infer that $I = A(G, H)$ and $(\hat{G}/\hat{H})$ are isomorphically homeomorphic. Since finite groups are self-dual, $I$ is isomorphic to $\hat{G}/H$, which completes the proof.

THEOREM 4.1. *Let $G$ be a Hausdorff abelian group such that $x \to 2x$ is one-to-one and bi-measurable. Assume that every Gaussian measure on $G$ without idempotent factors is a translation of a symmetric measure. Let $\mu$ be a Gaussian measure without idempotent factors. Then for every $\mathscr{B}^\mu$-measurable subgroup $F$ of $G$ we have*

$$\mu(F) = 0 \quad \text{or} \quad \mu(F) = 1.$$

Proof. It is standard that it suffices to prove the theorem for $\sigma$-compact subgroups of G. Indeed, if $F \in \mathscr{B}^\mu$ is a subgroup of $G$ and $\mu(F) > 0$, then there exists a compact subset $K \subseteq F$ such that $\mu(K) > 0$. Then the subgroup $F_0$ generated by $K$ is $\sigma$-compact (hence Borel), $\mu(F_0) > 0$, and $F_0 \subseteq F$. Thus, if we prove that $\mu(F_0) = 1$, then also $\mu(F) = 1$. Hence, we can assume without loss of generality that $\mu(F) > 0$ and that $F$ is a $\sigma$-compact (hence Borel) subgroup of $G$.

Now, let $E$ be the subgroup of $G$ generated by the set

$$\{x \in G;\ \mu(F+x) > 0\}.$$

Since there are at most countably many cosets of $F$ having positive measure $\mu$, $E$ consists of countably many cosets of $F$ and therefore

$$E \in \mathscr{B}.$$

By the definition of $E$ we have

$$\mu(E+x) = 0 \quad \text{if} \quad x \notin E.$$

Hence

$$\begin{aligned} 0 < \mu(F) \leqslant \mu(E) \leqslant \mu(\{x;\ 2x \in E\}) &= \nu_1 * \nu_2(E) \\ &= \int \bar{\mu}(E-x)\mu^{*3}(dx) \\ &= \mu(E)\mu^{*3}(E) = \ldots = (\mu(E))^4, \end{aligned}$$

which gives $\mu(E) = 1$. Thus, $\mu$ restricted to $E$ is Gaussian. Let $\pi$ be the canonical homomorphism of $E$ into $E/F$ (with the discrete topology). Since $E/F$ is countable, $\pi$ is Borel measurable. It is easy to see that $\pi(\mu)$ is a Gaussian measure on $E/F$. By Lemma 4.2 $\pi(\mu)$ is the normed Haar measure concentrated on a finite subgroup $I$ which does not contain non-zero elements of order two. Clearly, we can assume that $E/F = I$, and hence $F$ is closed (in $E$) under division by 2:

$$\{x \in E;\ 2x \in F\} = F.$$

Assume that card $E/F > 1$. Then $E/F$ is isomorphic to the direct sum of cyclic groups

$$E/F \cong Z(k_1) \oplus \ldots \oplus Z(k_n),$$

where $Z(k_i)$ denotes the cyclic group of order $k_i$, $k_i > 1$, $i = 1, \ldots, n$. Let

$$F_1 = \pi^{-1}\big(Z(k_2) \oplus \ldots \oplus Z(k_n)\big).$$

Then $F_1$ is a Borel subgroup of $G$, $E/F_1 = Z(k_1)$, $\pi_1(\mu)$ is the Haar measure on $Z(k_1)$, where $\pi_1$ is the canonical homomorphism of $E$ into $E/F_1$, and

$$0 < \mu(F_1) < 1. \tag{4.4}$$

Thus, in order to prove our theorem it suffices to show that (4.4) leads to a contradiction. Hence we can assume that $E/F = Z(k)$, $k > 1$.

Now, since $\pi(\mu)$ is the Haar measure on $E/F$, we have

$$\mu(F+x) = \mu(F) = 1/k,$$

for every $x \in E$. Hence

$$\nu_2(F+x) = \mu * \bar{\mu}(F+x) = \int\limits_E \mu(F-x+y)\,\mu(dy) = \mu(F),$$

for every $x \in E$. Next, let $A \subseteq F$, $u, v \in E$. Since $F$ is closed under division by 2 (in $E$), we have

$$\begin{matrix} x \in F-v, \\ y \in F+u+v, \end{matrix} \quad \Leftrightarrow \quad \begin{matrix} x+y \in F+u, \\ x-y \in F-u-2v. \end{matrix}$$

Hence

$$\begin{aligned} \mu \times \mu\big(\{(x,y) \in E \times E;\ x+y \in A+u,\ x-y \in F-u-2v\}\big) \\ = \nu_1(A+u)\nu_2(F-u-2v) = \nu_1(A+u)\mu(F). \end{aligned}$$

On the other hand,

$$\begin{aligned} \mu \times \mu\big(\{(x,y) \in E \times E;\ x+y \in A+u,\ x-y \in F-u-2v\}\big) \\ = \mu \times \mu\big(\{(x,y) \in (F-v) \times (F+u+v);\ x+y \in A+u\}\big) \\ = \int\limits_{F-v} \mu(A+u-x)\,\mu(dx). \end{aligned}$$



Thus, we obtain

$$\nu_1(A+u) = 1/\mu(F) \int\limits_{F-v} \mu(A+u-x)\,\mu(dx), \tag{4.5}$$

for $A \subseteq F$, $u, v \in E$. Now, by $\mu|_B$ we will denote the restriction of $\mu$ to $B$, $B \in \mathscr{B}$. Let $u \in E \setminus F$ be such that $iu \notin F$ for $0 < i < k$. Let us denote

$$\mu_l' = k\mu|F+lu, \quad l = 0, 1, \ldots, 2k.$$

$\mu_l'$ is a probability measure concentrated on $F+lu$. Hence, if we define

$$\mu_l = (-lu) * \mu_l',$$

then $\mu_l$ is a probability measure concentrated on $F$, $l = 0, \ldots, 2k$. We will show that $\mu_0, \mu_1, \ldots, \mu_k$ are associated Gaussian measures without idempotent factors. By (4.5) we obtain

$$\begin{aligned} \mu*\mu(A+lu) &= 1/\mu(F) \int\limits_{F+nu} \mu(A+lu-x)\,\mu(dx) \\ &= k \int\limits_{F+nu} \big(\mu*(n-l)u\big)\big(A-(x-nu)\big)\,\mu(dx) \\ &= k \int\limits_{F} \big(\mu*(n-l)u\big)(A-y)\,(\mu*-nu)(dy) \\ &= \mu(F)(\mu_{l-n}*\mu_n)(A) \quad \text{for} \quad 0 \leqslant l \leqslant n \leqslant 2k. \end{aligned}$$

Hence

$$\nu_1(A+lu) = \mu*\mu(A+lu) = \mu(F)\mu_{l-n}*\mu_n(A) \tag{4.6}$$

for $0 \leqslant n \leqslant l \leqslant 2k$ and $A \in E \cap \mathscr{B}$. Analogously we obtain

$$\nu_2(A+lu) = \mu*\bar{\mu}(A+lu) = \mu(F)\mu_n*\bar{\mu}_{n-l}(A) \tag{4.7}$$

for $0 \leqslant l \leqslant n \leqslant 2k$ and $A \in F \cap \mathscr{B}$.

Now, let $A \subseteq E$, $A \in \mathscr{B}$. Then, for fixed $n$, $0 \leqslant n \leqslant k$

$$A = \bigcup_{i=n}^{n+k-1} A_i,$$

where $A_i = A \cap (F+iu)$. By (4.6) we obtain

$$\begin{aligned} \mu*\mu(A_i) &= \mu*\mu\big((A_i-iu)+iu\big) = \mu(F)(\mu_{i-n}*\mu_n)(A_i-iu) \\ &= \mu(F) \int\limits_F \mu_{i-n}(A_i-iu-x)\,\mu_n(dx) \\ &= \int\limits_F \mu(A_i-nu-x)\,\mu_n(dx) \\ &= \mu_n*(\mu*nu)(A_i) \quad \text{for} \quad n = 0, \ldots, k, \end{aligned}$$

and so

$$\mu*\mu(A) = \sum_{i=n}^{n+k-1} \mu*\mu(A_i) = \mu_n*\mu*nu(A).$$

Thus

$$(4.8)\qquad \mu*\mu = \mu_n*(\mu*nu) \quad \text{for} \quad n = 0, \ldots, k.$$

By (4.8) we infer that if $\mu_n$ has an idempotent factor $\lambda$, then $\mu*\mu$ has the same idempotent factor $\lambda$. Hence $\mu^{*3}*\bar{\mu}$ has the same idempotent factor $\lambda$. This means that

$$x*\mu^{*3}*\bar{\mu} = \mu^{*3}*\bar{\mu}$$

for every $x \in C(\lambda)$. Since, by (2.3)

$$\mu(A) = \mu^{*3}*\bar{\mu}(2A),$$

we have

$$x*\mu(A) = 2x*\mu^{*3}*\bar{\mu}(2A) = \mu^{*3}*\bar{\mu}(2A) = \mu(A) \quad \text{for} \quad x \in C(\lambda).$$

Thus, $\mu$ has the same idempotent factor $\lambda$. By the assumption, $\lambda = 0$, hence $\mu_n$ has no idempotent factors, $n = 0, \ldots, k$.

Next, observe that for $x, y \in E$ and $0 \leqslant j, l \leqslant k$

$$\begin{matrix} x \in F+ju, \\ y \in F+lu, \end{matrix} \Leftrightarrow \begin{matrix} x+y \in F+(j+l)u, \\ x-y \in F+(j-l)u. \end{matrix}$$

Hence, for $A, B \subseteq$ we have

$$\begin{aligned}\mu\times\mu\big(\{(x, y);\ x+y \in A+(j+l)u,\ x-y \in B+(j-l)u\}\big) &= \mu(F)^2\big(\mu*(-ju)\big)\times\big(\mu*(-lu)\big)\big(\psi^{-1}(A\times B)\big)\\ &= \mu(F)^2(\mu_j\times\mu_l)\big(\psi^{-1}(A\times B)\big).\end{aligned}$$

On the other hand, by (4.6) and (4.7) we obtain

$$\begin{aligned}\mu\times\mu\big(\{(x, y);\ x+y \in A+(j+l)u,\ x-y \in B+(j-l)u\}\big) &= \mu*\mu\big(A+(j+l)u\big)\,\mu*\bar{\mu}\big(B+(j-l)u\big)\\ &= \big(\mu(F)\big)^2\mu_j*\mu_l(A)\,\mu_j*\bar{\mu}_l(B).\end{aligned}$$

Thus, we have

$$\mu_j\times\mu_l\big(\psi^{-1}(A\times B)\big) = \mu_j*\mu_l(A)\,\mu_j*\bar{\mu}_l(B),$$

for $j, l = 0, 1, \ldots, k$, and $A, B \in F\cap\mathscr{B}$. Hence $\mu_j$, $\mu_l$ are associated Gaussian measures on $F$ without idempotent factors for $j, l = 0, 1, \ldots, k$. By Lemma 4.1 we infer that

$$\mu_i = x_i*\mu_0 \quad \text{for some } x_i\text{'s},\ x_i \in F,\ i = 1, 2, \ldots, k.$$

Since, by (4.6),

$$\mu_{l-n}*\mu_n = \mu_l*\mu_0 \quad \text{for} \quad 0 \leqslant n \leqslant l \leqslant k,$$



we have

$$(x_{l-n}+x_n)*\mu_0^{*2} = x_l*\mu_0^{*2}.$$

As before, we see that $\mu_0^{*2}$ has no idempotent factors, and hence

$$x_{l-n}+x_n = x_l \quad \text{for} \quad 0 \leqslant n \leqslant l \leqslant k.$$

Thus $x_l = lx_1$. Since $\mu_k = (-ku)*\mu_0$, we have

$$k(u+x_1)*\mu_0 = \mu_0.$$

Since $\mu_0$ has no idempotent factors, we obtain

$$k(u+x_1) = 0.$$

Let $u' = u+x_1$. Then $u' \in E\setminus F$, $u'$ generates a cyclic group of order $k$ and

$$\{lu';\ l = 0, 1, \ldots, k-1\}\cap F = 0.$$

Hence

$$E \cong Z(k)\oplus F.$$

Moreover,

$$\mu|F+iu = iu*\mu|F,$$

and hence

$$\mu = \lambda_Z*\mu_0,$$

where $\lambda_Z$ is the normed Haar measure concentrated on $Z = \{lu';\ l = 0, 1, \ldots, k-1\}$. This contradicts the assumption that $\mu$ has no idempotent factors and completes the proof of the theorem.

COROLLARY 4.1. *Let $G$ be an LCA group such that $x \to 2x$ is an automorphism. If $\mu$ is a Gaussian measure on $G$ without idempotent factors, then*

$$\mu(F) = 0 \quad \text{or} \quad \mu(F) = 1$$

*for every $\mathscr{B}^\mu$-measurable subgroup $F$ of $G$.*

Proof. This follows immediately from Theorem 4.1 and the fact that every Gaussian measure on such a group is a translation of a symmetric measure (see [13]).

COROLLARY 4.2. *Let $G$ be a metric abelian group having no non-zero elements of order two. Let $\mu$ be a Gaussian measure on $G$ without idempotent factors. If $F$ is a $\mathscr{B}^\mu$-measurable subgroup of $G$, then $\mu(F) = 0$ or $\mu(F) = 1$.*

Proof. This is a direct consequence of Theorem 4.1 and Corollary 3.1.

Remark 4.1. If $G$ is a complete separable locally convex space, then this corollary can be proved without using Corollary 3.1. Indeed, every Gaussian measure $\mu$ on $G$ is then of the form $x_0 * \mu_1$, where $\mu_1$ is symmetric and $x_0$ is the expectation of $\mu$ (see [8]). These facts are also true if $G$ is a separable Orlicz space (non-necessarily locally convex) (see [5]).

THEOREM 4.2. *Let $G$ be a metric abelian group. Let $\mu$ be a Gaussian measure such that $\mu * \bar{\mu}$ is embeddable into a 0-continuous semigroup of Gaussian measures. Let $F$ be a $\mathscr{B}^\mu$-measurable subgroup of $G$. Then*

$$\mu(F) = 0 \quad \text{or} \quad \mu(F) = 1.$$

Proof. As in the proof of Theorem 4.1, we can assume that $F$ is $\sigma$-compact. We can also assume that $\mu$ is symmetric (see [10]). Hence, throughout the remaining part of the proof we assume that $\mu$ is a symmetric Gaussian measure which is embedded into a 0-continuous semigroup $(\nu_t)_{t>0}$ of symmetric Gaussian measures and that $F$ is a $\sigma$-compact subgroup of $G$ such that $\mu(F) > 0$. From the first part of the proof of Theorem 4.1 we infer that

$$E = \{x \in G;\ \mu(F+x) > 0\}$$

is a $\sigma$-compact subgroup of $G$ such that $E/F = I$ is finite and has no non-zero elements of order two. Moreover, $\pi(\mu)$ is the normed Haar measure on $I$, where $\pi$ is the canonical homomorphism of $E$ into $E/F$. Since $I$ does not contain non-zero elements of order two, $F$ is closed (in $E$) under division by 2, that is

$$\{x \in E;\ 2x \in F\} = F.$$

On the other hand, $G_0 = \{x;\ 2x = 0\}$ is a closed subgroup of $G$, and hence $G/G_0$ is Hausdorff. Hence $G/G_0$ is metrizable. Let $\pi_0\colon G \to G/G_0$ be the canonical homomorphism of $G$ into $G/G_0$. Since $\pi_0$ is continuous and $E$ is $\sigma$-compact, $\pi_0(E)$ is also $\sigma$-compact (hence Borel) in $G/G_0$. Hence we infer that $\pi_0(\mu)$ is concentrated on $\pi_0(E)$. Next, if $x \in E$ and $\pi_0(x) \in \pi_0(F)$, then $2(x-f) = 0$ for an element $f \in F$. Since $x \in E$, $x-f \in E$. Since $F$ is closed (in $E$) under division by 2,

$$x - f \in F.$$

Thus

$$\{x \in E;\ \pi_0(x) \in \pi_0(F)\} = F.$$

Hence

$$\mu(F) = \mu(\{x \in E;\ \pi_0(x) \in \pi_0(F)\}) = \mu\big(\pi_0^{-1}(\pi_0(F))\big) = \pi_0(\mu)\big(\pi_0(F)\big)$$

because $\pi_0(F)$ is also $\sigma$-compact in $G/G_0$, and hence Borel measurable.

Thus, $\pi_0(\mu)$ is a Gaussian measure on $G/G_0$, $G/G_0$ is a metric group having no non-zero elements of order 2 and $\pi_0(F)$ is a Borel subgroup of $G/G_0$ with the property

$$\pi_0(\mu)\big(\pi_0(F)\big) = \mu(F). \tag{4.9}$$



Moreover, $\pi_0(\mu)$ is embedded into a 0-continuous semigroup of symmetric Gaussian measures, namely into $(\pi_0(\nu_t))_{t>0}$. By Proposition 2.2, $\pi_0(\mu)$ has no idempotent factors. By Corollary 4.2 we obtain $\pi_0(\mu)(\pi_0(F)) = 1$, which, in virtue of (4.9), completes the proof.

Now, let us recall Parthasarathy's definition of Gaussian measures on LCA groups (see [17], IV).

Let $G$ be a second countable LCA group. A probability measure $\mu$ is called *Gaussian in the sense of Parthasarathy* (*P-Gaussian*, briefly) if its characteristic function $\hat{\mu}$ is of the form

$$\hat{\mu}(\gamma) = \gamma(x_0)\exp(-\varphi(\gamma)), \quad \gamma \in \hat{G},$$

where $x_0 \in G$ and $\varphi\colon \hat{G} \to R$ is a non-negative continuous function satisfying

$$\varphi(\chi+\gamma) + \varphi(\chi-\gamma) = 2\varphi(\chi) + 2\varphi(\gamma), \quad \chi, \gamma \in \hat{G}. \tag{4.10}$$

Every $P$-Gaussian measure is Gaussian in our sense and, if it is symmetric (that is, if $x_0 = 0$), it is embeddable into a unique 0-continuous semigroup of symmetric Gaussian measures. It is well known that no $P$-Gaussian measure has any idempotent factors. Thus, from Theorem 4.2 we obtain

COROLLARY 4.3. *Let $G$ be a second countable* LCA *group. Let $\mu$ be a Gaussian measure in the sense of Parthasarathy. If $F$ is a $\mathscr{B}$-measurable subgroup of $G$, then*

$$\mu(F) = 0 \quad \text{or} \quad \mu(F) = 1.$$

Remark 4.2. It is well known that every symmetric $P$-Gaussian measure $\mu$ is concentrated on a connected subgroup of $G$. Thus, we can assume that $G$ is connected and that the support of a Gaussian measure $\mu$ is equal to $G$. Now, if $\mu$ is absolutely continuous with respect to the Haar measure, then every Borel subgroup of $G$ having positive measure $\mu$ must be open: this is an immediate consequence of (20.17) in [12]. Since $G$ is connected, every non-empty open subgroup of $G$ is equal to $G$.

However, it is well known that there are Gaussian measures which are singular with respect to the Haar measure (see [17], IV). In this situation we do not know any direct proof of Corollary 4.3.

In the end of this section we prove that a certain version of the zero-one law for Gaussian measures on measurable groups can be derived from the proof of Theorem 4.1. First of all, observe that in the definition of a Gaussian measure we have only used the measurability of the mappings

$$(x, y) \to x \pm y$$

(with respect to the appriopriate $\sigma$-fields). This fact provides a motivation for introducing the following definition:

DEFINITION 4.2. Let $G$ be an abelian group and let $\mathscr{B}$ be a $\sigma$-field of subsets of $G$. $(G, \mathscr{B})$ is called a *measurable group* (m.g.) if the mapping $(x, y) \to x - y$ is measurable with respect to $\mathscr{B}$ and the product $\sigma$-field $\mathscr{B} \times \mathscr{B}$.

The importance of this notion follows from the fact that many frequently occurring vector spaces with the usual $\sigma$-fields are of this type; e.g., the space $D[0, 1]$ with the Borel $\sigma$-field (with respect to the Skorohod topology) (see [2]) or an arbitrary vector space with the $\sigma$-field generated by a family of linear functionals.

If $(G, \mathscr{B})$ is a m.g., we can define the convolution of probability measures and we can consider Gaussian measures on $(G, \mathscr{B})$ in the sense of Definition 2.1. It is easy to see that equality (2.3) as well as Lemma 2.2 and Lemma 4.1 remain true for Gaussian measures defined on a m.g.

Now, it is easily seen that in the proof of Theorem 4.1 we have only used the fact that a Gaussian measure $\mu$ has the property

$$\mu * x = \mu \Rightarrow x = 0$$

and (roughly speaking) the possibility of using Lemma 4.2. The latter fact can be established exactly in the same way as in the first part of the proof of Theorem 4.1, if $F$ is a $\mathscr{B}$-measurable subgroup of a m.g. $(G, \mathscr{B})$. However, if $F$ is only $\mathscr{B}^\mu$-measurable, the previous arguments cannot be applied and the situation becomes more complicated. In the proof of the next theorem we show that we can again ensure the possibility of using Lemma 4.2. Some ideas used in this proof are similar to those used in the first part of Theorem 4.1 and in Lemma 1 in [6]. However, for the sake of clearity and convenience, we give a complete proof.

THEOREM 4.3. *Let $(G, \mathscr{B})$ be a* m.g. *such that $x \to 2x$ is one-to-one and bi-measurable. Assume that every Gaussian measure $\nu$ satisfying*

$$\nu * x = \nu \Rightarrow x = 0 \tag{4.11}$$

*is a translation of a symmetric measure. Let $\mu$ be a Gaussian measure satisfying* (4.11). *If $F$ is a $\mathscr{B}^\mu$-measurable subgroup of $G$, then*

$$\mu(F) = 0 \quad \text{or} \quad \mu(F) = 1.$$

Proof. Let $\mu$ be a Gaussian measure on $(G, \mathscr{B})$ satisfying (4.11). We can assume without loss of generality that $\mu$ is symmetric. Let $F$ be $\mathscr{B}^\mu$-measurable subgroup of $G$ such that $\mu(F) > 0$. Let $\mu_{1/2^n}$ be the symmetric Gaussian root of order $2^n$ of $\mu$. We show that $F \in \mathscr{B}^{\mu_{1/4}}$ and that $\mu_{1/4}(F) = 1$. Since $\mu_{1/4}^{*4} = \mu$, this will imply that $\mu(F) = 1$.

First of all, we show that for every $x \in G$, $F + x \in \mathscr{B}^{\mu_{1/2}}$ and that $\mu_{1/2}(F) > 0$. To show this, let $C, D \in \mathscr{B}$ be such that

$$C \subseteq F \subseteq D \quad \text{and} \quad \mu(D \setminus C) = 0.$$



Since $\mu = \mu_{1/2} * \mu_{1/2}$, we have

$$\int \mu_{1/2}\big((D \setminus C) - x\big)\, \mu_{1/2}(dx) = 0. \tag{4.12}$$

Let

$$F_1 = \{x;\ F + x \in \mathscr{B}^{\mu_{1/2}}\}.$$

By the symmetry of $\mu_{1/2}$ and (4.12) we infer that $F_1 \in \mathscr{B}^{\mu_{1/2}}$ and that $\mu_{1/2}(F_1) = 1$. Since $F + F_1 \supseteq F_1$, we have $F + F_1 \in \mathscr{B}^{\mu_{1/2}}$ and $\mu_{1/2}(F + F_1) = 1$. We thus have two possibilities: $F + x \subseteq F + F_1$ and then $F + x \in \mathscr{B}^{\mu_{1/2}}$ or $F + x$ is disjoint from $F + F_1$, which implies that $F + x$ is a null set, and hence $F + x \in \mathscr{B}^{\mu_{1/2}}$. We have thus proved that $F + x \in \mathscr{B}^{\mu_{1/2}}$ for every $x \in G$. Moreover, since $\mu_{1/2} * \mu_{1/2}(F) > 0$, there exists an element $x_0 \in G$ such that

$$\mu_{1/2}(F + x_0) > 0. \tag{4.13}$$

Now, let us denote

$$F_2 = \{x;\ F + x_0 - x \in \mathscr{B}^{\mu_{1/4}}\}, \qquad F_3 = \{x;\ F - x_0 + x \in \mathscr{B}^{\mu_{1/4}}\}.$$

By the arguments we have just used and by the symmetry of $\mu_{1/4}$ we infer that $F_2, F_3 \in \mathscr{B}^{\mu_{1/4}}$ and that $\mu_{1/4}(F_2) = \mu_{1/4}(F_3) = 1$. Hence, in virtue of (4.13), there exists an $x_1 \in G$ such that

$$\begin{aligned} &F + x_0 - x_1 \in \mathscr{B}^{\mu_{1/4}}, \qquad F - x_0 + x_1 \in \mathscr{B}^{\mu_{1/4}}, \\ &\mu_{1/4}(F + x_0 - x_1) = \mu_{1/4}(F - x_0 + x_1) > 0. \end{aligned} \tag{4.14}$$

Now, let $z_1, z_2 \in G$. Since $F + z_i \in \mathscr{B}^{\mu_{1/2}}$, $i = 1, 2$, there exist $C_i$, $D_i \in \mathscr{B}$ such that $C_i \subseteq F + z_i \subseteq D_i$ and $\mu_{1/2}(D_i \setminus C_i) = 0$, $i = 1, 2$. Since

$$\begin{aligned} \mu_{1/2}(D_i \setminus C_i) &= \mu_{1/4} * \mu_{1/4}(D_i \setminus C_i) \\ &= \mu_{1/4} \times \mu_{1/4}\big(\{(x, y);\ x + y \in D_i \setminus C_i\}\big) \\ &= \mu_{1/4} \times \mu_{1/4}\big(\{(x, y);\ x - y \in D_i \setminus C_i\}\big), \end{aligned} \tag{4.15}$$

we obtain

$$\{(x, y);\ x \pm y \in F + z_i\} \in (\mathscr{B} \times \mathscr{B})^{\mu_{1/4} \times \mu_{1/4}}, \qquad i = 1, 2. \tag{4.16}$$

Moreover, in virtue of (2.1)

$$\begin{aligned} 0 &= \mu_{1/4} \times \mu_{1/4}\big(\{(x, y);\ x + y \in D_1 \setminus C_1,\ x - y \in D_2\}\big) \\ &= \mu_{1/2} \times \mu_{1/2}\big((D_1 \setminus C_1) \times D_2\big) \\ &= \mu_{1/2} \times \mu_{1/2}\big(D_1 \times (D_2 \setminus C_2)\big); \end{aligned}$$

so

$$\begin{aligned} \psi(\mu_{1/4} \times \mu_{1/4})\big((F + z_1) \times (F + z_2)\big) &= \psi(\mu_{1/4} \times \mu_{1/4})(C_1 \times C_2) \\ &= \mu_{1/2} \times \mu_{1/2}(C_1 \times C_2) \\ &= \mu_{1/2} \times \mu_{1/2}\big((F + z_1) \times (F + z_2)\big). \end{aligned} \tag{4.17}$$

Next, we have

$$\{(x,y);\ x+y\in F\} = \{(x,y);\ x+y\in (F+x_0-x_1)+(F-x_0+x_1)\}$$
$$\supseteq (F+x_0-x_1)\times(F-x_0+x_1);$$

hence from (4.14), (4.15) and from the first part of (4.16) (applied to $z_1 = 0$) we obtain

$$\mu_{1/2}(F) = \mu_{1/2}(C_1) = \mu_{1/4}*\mu_{1/4}(C_1) = \mu_{1/4}\times\mu_{1/4}(\{(x,y);\ x+y\in C_1\})$$
$$= \mu_{1/4}\times\mu_{1/4}(\{(x,y);\ x+y\in F\})$$
$$\geqslant \mu_{1/4}(F+x_0-x_1)\,\mu_{1/4}(F-x_0+x_1) > 0.$$

By repeating this reasoning we infer that $F+x\in\mathscr{B}^{\mu_{1/4}}$ for every $x\in G$, and that $\mu_{1/4}(F)>0$. Analogously, $F+x\in\mathscr{B}^{\mu_{1/8}}$, for every $x\in G$, and $\mu_{1/8}(F)>0$.

Now, let $E$ be the subgroup of $G$ generated by the set

$$\{x;\ \mu_{1/8}(F+x)>0\}.$$

Then $E$ consists of countably many cosets of $F$, and hence

$$E\in\mathscr{B}^{\mu_{1/2}}\cap\mathscr{B}^{\mu_{1/4}}\cap\mathscr{B}^{\mu_{1/8}}.$$

Clearly

(4.18) $$\mu_{1/8}(E-x) = 0 \quad \text{if} \quad x\notin E.$$

It is not difficult to see that also

(4.19) $$\mu_{1/4}(E-x) = \mu_{1/2}(E-x) = 0 \quad \text{if} \quad x\notin E.$$

Moreover, it can be checked, by arguments similar to those used above, that $E\in\mathscr{B}^{\mu_{1/2}}$ implies $1/2E\in\mathscr{B}^{\mu_{1/8}}$, and then

$$\mu_{1/8}(1/2\,E) = \mu_{1/2}(E).$$

Since $E\subseteq 1/2\ E$, from (4.18), (4.19) and the above equality we obtain

$$0<\mu_{1/8}(E)\leqslant\mu_{1/8}(1/2\,E) = \mu_{1/2}(E) = \int\mu_{1/4}(E-x)\,\mu_{1/4}^{*4}(dx)$$
$$= \mu_{1/4}(E)\,\mu_{1/4}^{*3}(E) = \big(\mu_{1/4}(E)\big)^4 = \ldots = \big(\mu_{1/8}(E)\big)^8.$$

Hence

$$1 = \mu_{1/8}(E) = \mu_{1/8}(1/2\,E) = \mu_{1/2}(E) = \big(\mu_{1/4}(E)\big)^2 = \mu_{1/4}(E).$$

Hence we infer that $\mu_{1/4}$ is a Gaussian measure on a m.g. $(E, E\cap\mathscr{B})$. Moreover, $F+x\in\mathscr{B}^{\mu_{1/4}}$ for every $x\in G$, and $\mu_{1/4}(F)>0$. Let $\pi$ be the canonical homomorphism from $E$ into $E/F$. In virtue of (4.17) we see that $\pi(\mu_{1/4})$ is a Gaussian measure on $E/F$ endowed with the $\sigma$-field of all subsets. Thus, we can apply Lemma 4.2 to $\pi(\mu_{1/4})$. Finally, it remains to observe that the arguments used in the second part of the proof of Theorem 4.1 can be applied without any change. This completes the proof.



**5. Gaussian processes with values in LCA groups.** Throughout this section $G$ will denote an LCA group satisfying the second countability axiom and such that $2G = G$.

Let $T$ be a set. By $(G^T, \mathscr{B}^T)$ we will denote the Cartesian product of $T$ copies of $G$ with the product $\sigma$-field. If $T = \{1,\ldots,n\}$, we write $(G^n, \mathscr{B}^n)$; if $T$ is countable, we write $(G^\infty, \mathscr{B}^\infty)$. Now, let $X$ be a subgroup of $G^T$ with coordinatewise addition. By $\mathscr{B}(X)$ we will denote the $\sigma$-field induced on $X$ by $\mathscr{B}^T$, that is, the $\sigma$-field generated by the sets of the form

$$\{x\in X;\ \langle x(t_1),\ldots,x(t_k)\rangle\in B\},$$

where $t_1,\ldots,t_k\in T$ and $B\in\mathscr{B}^k$. Let us define, for $t_1,t_2,\ldots,t_k\in T$, the natural projection $\pi_{t_1\ldots t_k}$ from $X$ into $G^k$:

$$\pi_{t_1\ldots t_k}(x) = \langle x(t_1),\ldots,x(t_k)\rangle.$$

Let $\mu$ be a probability measure on $(X,\mathscr{B}(X))$. By $\mu_{t_1\ldots t_k}$ we will denote the probability measure on $(G^k,\mathscr{B}^k)$ induced by $\mu$ and $\pi_{t_1\ldots t_k}$. It is easy to verify that $(X,\mathscr{B}(X))$ is a m.g. and that $\mu$ is a Gaussian measure on $(X,\mathscr{B}(X))$ if and only if $\mu_{t_1\ldots t_k}$ is a Gaussian measure on $(G^k,\mathscr{B}^k)$ for every $t_1,\ldots,t_k\in T$. It can also be checked that $\mu$ satisfies the condition

(5.1) $$\mu * x = \mu \Rightarrow x = 0$$

whenever $\mu_t$ satisfies this condition on $(G,\mathscr{B})$ for every $t\in G$, that is, if $\mu_t$ has no idempotent factors. If $X = G^T$, then (5.1) is equivalent to the fact that $\mu_t$ has no idempotent factors for any $t\in T$.

Now, we are in a position to state and prove the zero-one law for Gaussian measures on $(G^\infty,\mathscr{B}^\infty)$.

THEOREM 5.1. *Let $\mu$ be a Gaussian measure on $(G^\infty,\mathscr{B}^\infty)$. Assume that $\mu_t$ has no idempotent factors for any $t\in T$. If $F$ is a $(\mathscr{B}^\infty)^\mu$-measurable subgroup of $G^\infty$, then*

$$\mu(F) = 0 \quad or \quad \mu(F) = 1.$$

Proof. By the previous remarks we infer that $\mu$ satisfies (5.1). However, $G^\infty$ with the product topology is a metrizable topologically complete group and $\mathscr{B}^\infty$ is its Borel $\sigma$-field. Thus $\mu$ is a (tight) probability measure on $G^\infty$ and therefore has no idempotent factors. In virtue of Theorem 4.2 and Corollary 2.1 it suffices to construct a sequence of symmetric Gaussian measures $\mu_n$ on $G$ such that

(5.2) $$\mu_0 = \mu*\bar\mu, \qquad \mu_n*\mu_n = \mu_{n-1}.$$

Let $h_{t_1\ldots t_k}$ be the characteristic function of $\pi_{t_1\ldots t_k}(\mu*\bar\mu) = \mu_{t_1\ldots t_k}*\bar\mu_{t_1\ldots t_k}$. Since $h_{t_1\ldots t_k}$ is a characteristic function of a Gaussian measure on $G^k$, it satisfies equation (4.3). Moreover, since $h_{t_1\ldots t_k}$ is positive, it is of the form

$$h_{t_1 \dots t_k} = \exp(-\varphi_{t_1 \dots t_k}),$$

where $\varphi_{t_1 \dots t_k}$ is a nonnegative continuous function from $\hat{G}^k$ into $R$ satisfying (4.10). Let $n$ be an integer and let

$$h^{(n)}_{t_1 \dots t_k} = \exp(-1/2n\varphi_{t_1 \dots t_k}).$$

It is easily seen that $h^{(n)}_{t_1 \dots t_n}$ defines a consistent family of symmetric Gaussian measures $\{\mu^{(n)}_{t_1 \dots t_k}\}$. By Kolmogoroff's Extension Theorem we infer that there exists a probability measure, say $\mu_n$, on $(G^\infty, \mathscr{B}^\infty)$ extending this family. It is easy to see that $\{\mu_n\}$ satisfy (5.2).

Now, we state and prove an analogue of Kallianpur's zero-one law [16]. It also contains the result of Jain [15].

THEOREM 5.2. *Let $\mu$ be a Gaussian measure on $(X, \mathscr{B}(X))$. Assume that $\mu_t$ has no idempotent factors for any $t \in T$. Let $F$ be a $\mathscr{B}(X)^\mu$-measurable subgroup of $X$. Then $\mu(F) = 0$ or $\mu(F) = 1$.*

Proof. Let $F$ be a $\mathscr{B}(X)^\mu$-measurable subgroup of $X$ such that $\mu(F) > 0$. Then there exists a sequence $\{t_i\}$, $t_i \in T$, and $B \in \mathscr{B}^\infty$ such that

$$\{x \in X;\ \langle x(t_1), \dots, x(t_n), \dots\rangle \in B\} \subseteq F$$

and

$$\mu(\{x \in X;\ \langle x(t_1), \dots, x(t_n), \dots\rangle \in B\}) > 0.$$

Since $G^\infty$ is a metrizable topologically complete space, we can assume that $B$ is compact. Let $D$ be the subgroup generated by $B$. $D$ is $\sigma$-compact (hence Borel) and

$$0 < \mu(\{x \in X;\ \langle x(t_1), \dots, x(t_n), \dots\rangle \in D\}) \leqslant \mu(F).$$

Now, let $\nu$ be a probability measure defined on $(G^\infty, \mathscr{B}^\infty)$ by the formula

$$\nu(A) = \mu(\{x \in X;\ \langle x(t_1), \dots, x(t_n), \dots\rangle \in A\}).$$

It is easy to see that $\nu$ is a Gaussian measure on $G^\infty$, without idempotent factors and such that $\nu(D) > 0$. By Theorem 5.1 we obtain $\nu(D) = 1$. Hence $\mu(F) = 1$, which completes the proof.

Putting $G = R$, we obtain Kallianpur's result (in fact, a stronger version of his theorem, without any additional assumption concerning the covariance function, etc.). Observe also that Theorem 5.2 can be obtained independently of Theorem 3.1. Indeed, in the case of $G^\infty$ (or even $G^T$, for arbitrary $T$) the conclusion of Theorem 3.1 is an easy consequence of the correspondence between (Gaussian) measures and families of (Gaussian) finite-dimensional distributions.

We now list some natural examples of groups having the form $(X, B(X))$.



EXAMPLES. 1. Let $D_G = D_G[0, 1]$ be the space of functions $f$ defined on $[0, 1]$ with values in $G$ which are right-continuous and have left-hand limits. $D_G$, endowed with the so-called Skorohod topology, is a separable topologically complete space (see [2] for $G = R$, and [3], for the general case). Moreover, it is well known that the Borel $\sigma$-field in $D_G$ is generated by projections $\pi_{t_1 \dots t_k}$, that is, equals $\mathscr{B}(D_G)$.

2. Let $C_G = C_G[0, 1]$ be the space of all continuous functions defined on $[0, 1]$ into $G$ with uniform convergence. As before, the Borel $\sigma$-field in $C_G$ is equal to $\mathscr{B}(C_G)$.

3. Obviously, $(G^T, \mathscr{B}^T)$ is also a space of the above type.

Now, let $(\Omega, \Sigma, P)$ be a probability space. A family $\xi = \{\xi(t);\ t \in T\}$ of random variables with values in $G$ will be called a *$G$-valued stochastic process*. $\xi$ is called *Gaussian* if for every $t_1, t_2, \dots, t_k;\ t_i \in T$, the distribution induced on $(G^k, \mathscr{B}^k)$ by

$$\langle \xi(t_1), \dots, \xi(t_k)\rangle$$

is Gaussian. Now, let $X$ denote, as before, a subgroup of $G^T$. If $\xi$ has the sample paths in $X$, then it induces a mapping $\tilde{\xi}\colon \Omega \to X$ taking $\omega$ into the sample path corresponding to $\omega$. It is easy to see that $\tilde{\xi}$ is measurable with respect to $\Sigma$ and $\mathscr{B}(X)$. Let $\mu_\xi$ be the distribution of $\tilde{\xi}$ on $(X, \mathscr{B}(X))$. Now, we can formulate Theorem 5.2 in terms of stochastic processes.

COROLLARY 5.1. *Let $\xi$ be a $G$-valued Gaussian stochastic process with the sample paths in $X$. Let $\mu = \mu_\xi$ be the distribution of $\tilde{\xi}$. Assume that $\mu_t$ has no idempotent factors for any $t \in T$. If $F$ is a $\mathscr{B}(X)^\mu$-measurable subgroup of $X$, then*

$$P(\{\tilde{\xi} \in F\}) = 0 \quad \text{or} \quad P(\{\tilde{\xi} \in F\}) = 1.$$

Now, let $\xi = \{\xi(t);\ t \in [0, 1]\}$ be a Wiener process with values in $G$, that is, a separable (in the sense of Doob) $G$-valued stochastic process with independent increments such that the characteristic function of $\xi(t)$ is of the form $\exp(-t\varphi)$, where $\varphi$ is a continuous non-negative function satisfying (4.10). It is known that $\xi$ has continuous paths (with probability one). Using Corollary 5.1, we obtain

COROLLARY 5.2. *Let $\xi = \{\xi(t);\ t \in [0, 1]\}$ be a Wiener process with values in $G$. Let $\mu$ be the distribution induced on $C_G$ by $\tilde{\xi}$. If $F$ is a $\mathscr{B}(C_G)^\mu$-measurable subgroup of $C_G$, then*

$$P(\{\tilde{\xi} \in F\}) = 0 \quad \text{or} \quad P(\{\tilde{\xi} \in F\}) = 1.$$

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**Added in proof.** Recently, the author has revealed that the proof of Theorem 4.2 is incorrect; namely, we cannot claim that $G/G^0$ has no elements of order 2. At this time, the author does not know whether such a theorem is true or not. However, a detailed inspection of the proof of Theorem 4.1 yields the following result:

*Let $G$ a Hausdorff abelian group. Assume that every Gaussian measure on $G$ without idempotent factors is essentially symmetric. Let $\mu$ be a Gaussian measure on $G$ without idempotent factors. Suppose that $\mu$ has a symmetric Gaussian root of order 4. Then for every $\mathscr{B}^\mu$-measurable subgroup $F$ we have $\mu(F) = 0$ or 1.*

In virtue of this modified version of Theorem 4.2, Corollary 4.3 as well as all results of Section 5 remain valid, under additional assumption that $2G = G$. Also Professor A. Tortrat has pointed out that the assumption of bi-measurability of $0$ is superfluous (except in Th. 4.3) since we may assume that $G$ is $\sigma$-compact.

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