STUDIA MATHEMATICA, T. LXIX. (1980)

Conjugate transforms and limit theorems for τ_T semigroups

by

RICHARD MOYNIHAN* (Bedford, MA)

Abstract. This paper continues the development of the theory of *conjugate transforms* in the study of τ_T semigroups of probability distribution functions. If Δ^+ is the space of probability distribution functions which are concentrated on $[0, \infty)$ and T is a *t-norm*, i.e., a suitable binary operation on $[0, 1]$, then the operation τ_T is defined for F, G in Δ^+ by

$$\tau_T(F, G)(x) = \sup_{u+v=x} T(F(u), G(v))$$

for all x. The pair (Δ^+, τ_T) is then a semigroup. For any *Archimedean* t-norm T, we defined in [6] a conjugate transform C_T on (Δ^+, τ_T) and established the basic algebraic properties of these transforms. In this paper we first establish that conjugate transforms and their inverses are 'continuous' mappings on these τ_T semigroups. We then show that conjugate transforms are a very effective tool in studying the convergence of distribution functions and their τ_T products in τ_T semigroups. In general, these transforms are shown to have many of the same properties that the Laplace transform has on the convolution semigroup.

1. Introduction. τ_T semigroups are the most prominent of several classes of semigroups of probability distribution functions which are of central concern in the theory of probabilistic metric (PM) spaces [10], [11], [13]. An understanding of the basic properties of these semigroups is therefore a major area of study in this theory [2], [4], [5], [8], [12]. Conjugate transforms have proven to be a very valuable tool in this study. In particular, B. Schweizer and the author have shown in [7] that these transforms have application to the study of betweenness in PM spaces. Conjugate transforms are an adaptation of the "conjugate function" concept used in Convex Analysis [9] and are similar to the *maximum transform* used by R. Bellman and W. Karush [1]. Another recent applications of conjugate transforms to τ_T semigroups appears in [17].

If Δ^+ is the space of probability distribution functions which are concentrated on $[0, \infty)$, then, for any τ_T operation corresponding to

* The contents of this paper are drawn from the author's Ph. D. thesis at the University of Massachusetts. The author would like the thank his thesis advisor, Professor B. Schweizer, for his very helpful comments ~~and criticisms~~ while this research was carried out.

an Archimedean t-norm T, we defined in [6] the *T-conjugate transform* C_T on the semigroup (Δ^+, τ_T). For any $F \in \Delta^+$, $C_T F$ is a function from $[0, \infty)$ into $[0, 1]$. If $\mathscr{A}_T$ is the corresponding space of T-conjugate transforms, i.e., $\mathscr{A}_T = C_T(\Delta^+)$, then in Section 2 we show that if $\{F_n\}$ is a sequence in Δ^+ which converges weakly to F in Δ^+, written $F_n \xrightarrow{w} F$, then $C_T F_n(z) \to C_T F(z)$ for all $z > 0$. Conversely, if $\{\varphi_n\}$ is a sequence in $\mathscr{A}_T$ and, for some $\varphi \in \mathscr{A}_T$, $\varphi_n(z) \to \varphi(z)$ for all $z > 0$, then $C_T^* \varphi_n \xrightarrow{w} C_T^* \varphi$ in Δ^+, where C_T^* is the *inverse* T-conjugate transform. Thus both C_T and C_T^* are continuous under the above convergence. Moreover, in this setting if Δ_T^+ is the subspace of *T-log-concave* distribution functions, then the map $C_T\colon \Delta_T^+ \to \mathscr{A}_T$ is a homeomorphism with inverse C_T^*. The spaces Δ_T^+ and $\mathscr{A}_T$ are each shown to be compact and complete, where their respective topologies are metrically induced.

In Section 3 we apply our results to the convergence of τ_T products of distribution functions. Our key result is that, for any Archimedean t-norm T, if $\{F_n\}$ is a sequence in Δ^+, then the sequence $\{\tau_T(F_1, \ldots, F_n)\}$ has a non-trivial weak limit in Δ^+ if and only if there is a sequence of positive numbers $\{a_i\}$ such that

$$\sum_{i=1}^{\infty} a_i < \infty \quad \text{and} \quad \lim_{n\to\infty} T\big(F_1(a_1), \ldots, F_n(a_n)\big) > 0.$$

Before we present our results we state some definitions and known facts: The space of probability distribution functions under consideration is:

$$\Delta^+ = \{F\colon \boldsymbol{R} \to [0, 1] \mid F \text{ is left-continuous, non-decreasing and } F(0) = 0\}. \tag{1.1}$$

In particular ε_0 and ε_∞ in Δ^+ are defined by

$$\varepsilon_0(x) = \begin{cases} 0, & x \leqslant 0, \\ 1, & 0 < x; \end{cases} \quad \text{and} \quad \varepsilon_\infty(x) = 0 \text{ for all } x. \tag{1.2}$$

A *t-norm* is a two-place function T from $[0, 1] \times [0, 1]$ into $[0, 1]$ which is symmetric, associative (i.e., $T(a, T(b, c)) = T(T(a, b), c)$), non-decreasing in each place, and has 1 as a unit. In particular, we say a continuous t-norm T is (a) *Archimedean* if $T(a, a) < a$ for all $a \in (0, 1)$; and (b) *strict* if T is strictly increasing in each place on $(0, 1] \times (0, 1]$. Note that a strict t-norm is Archimedean. Standard examples of t-norms are Product, Minimum, and $T_m(a, b) = \max\{a + b - 1, 0\}$.

For any t-norm T, we define the operation τ_T on F, G in Δ^+ at any real x by

$$\tau_T(F, G)(x) = \sup_{u+v=x} T\big(F(u), G(v)\big). \tag{1.3}$$

If the t-norm T is left continuous as a two-place function, then τ_T is a commutative semigroup operation on Δ^+ with unit ε_0 and null element ε_∞. Moreover, τ_T is non-decreasing in each place, i.e., $\tau_T(F, H) \geqslant \tau_T(G, H)$ whenever $F \geqslant G$, where $F \geqslant G$ means $F(x) \geqslant G(x)$ for all x [11], [13]. Hence, we call each such pair (Δ^+, τ_T) a *τ_T semigroup*.

A key tool in dealing with Archimedean t-norms is the following representation theorem [3]:

The t-norm T is Archimedean if and only if there exists a continuous and increasing function h: $[0, 1] \to [0, 1]$ with $h(1) = 1$ such that T is representable in the form

$$T(x, y) = h^{[-1]}\big(h(x) \cdot h(y)\big), \tag{1.4}$$

where $h^{[-1]}$ is the pseudo-inverse of h, i.e.,

$$h^{[-1]}(x) = \begin{cases} 0, & 0 \leqslant x \leqslant h(0), \\ h^{-1}(x), & h(0) \leqslant x \leqslant 1, \end{cases} \tag{1.5}$$

where h^{-1} is the usual inverse of h on $[h(0), 1]$.

The function h in (1.4) is called a *multiplicative generator* of the Archimedean t-norm T. Note from (1.5) that $h^{[-1]}$ is a continuous and non-decreasing function on the unit interval $[0, 1]$. In addition, $h^{[-1]} = h^{-1}$ if and only if $h(0) = 0$. These facts yield the result that, if h is a multiplicative generator of the Archimedean t-norm T, then T is strict if and only if $h(0) = 0$, i.e., if and only if $h^{[-1]} = h^{-1}$.

For studying convergence we will consider Δ^+ as endowed with the topology of weak convergence of distribution functions. To be precise, if $\{F_n\}$ and F are in Δ^+, then we say the sequence $\{F_n\}$ *converges weakly* to F, written $F_n \xrightarrow{w} F$, if $F_n(x) \to F(x)$ at each continuity point x of the limit function F. This topology is metrizable via the *modified Lévy metric* $\mathscr{L}$ which is defined for any F, G in Δ^+ by:

$$\mathscr{L}(F, G) = \inf\{\delta \mid F(x) \leqslant G(x+\delta) + \delta \text{ and } G(x) \leqslant F(x+\delta) + \delta \text{ for } 0 < x < 1/\delta\}.$$

In [14] D. A. Sibley established the following:

(1) $\mathscr{L}$ is a metric on Δ^+.

(2) For any sequence in Δ^+ and F in Δ^+, $\mathscr{L}(F_n, F) \to 0$ as $n \to \infty$ if and only if $F_n \xrightarrow{w} F$.

(3) The metric space $(\Delta^+, \mathscr{L})$ is compact—hence complete.

In addition from [11] we have that if T is a continuous t-norm, then τ_T is continuous (as a two-place function) on the metric space $(\Delta^+, \mathscr{L})$.

Finally we list those properties of conjugate transforms, established in [6], which are necessary to the sequel. *Below, for any Archimedean*

t*-norm* T, *the function* h *will be a fixed multiplicative generator of* T *and* $h^{[-1]}$ *the pseudo-inverse of* h. In particular, for $T = $ Product (i.e., $T(a, b) = a \cdot b$) h will be the identity function.

DEFINITION 1.1. For any Archimedean t-norm T, the T*-conjugate transform for the semigroup* (Δ^+, τ_T) is the mapping C_T defined for $F \in \Delta^+$ via:

$$C_T F(z) = \sup_{x \geqslant 0} e^{-xz} hF(x), \quad \text{for all } z \geqslant 0, \tag{1.6}$$

where $hF \in \Delta^+$ is given by

$$hF(x) = \begin{cases} 0, & x \leqslant 0, \\ h(F(x)), & 0 < x. \end{cases} \tag{1.7}$$

Note that if C denotes the Product-conjugate transform, then, for any $F \in \Delta^+$,

$$CF(x) = \sup_{x \geqslant 0} e^{-xz} F(x), \tag{1.8}$$

so that, for any Archimedean t-norm T, we have $C_T F = C(hF)$.

T-conjugate transforms are completely characterized by the following:

THEOREM 1.1. *Let*

$$\mathscr{A} = \{\varphi\colon [0, \infty) \to (0, 1] \mid \varphi \text{ is non-increasing, positive, continuous and log-convex}\} \cup \{\theta_\infty\}, \tag{1.9}$$

where $\theta_\infty(z) = 0$ *for all* $z \geqslant 0$. *Then, for any Archimedean* t*-norm* T, *if*

$$\mathscr{A}_T = \{\varphi \in \mathscr{A} \mid \varphi(z) \geqslant h(0) \text{ for all } z \geqslant 0\}, \tag{1.10}$$

then $\mathscr{A}_T = \{C_T F \mid F \in \Delta^+\}$. *Thus* $\mathscr{A}_T \subseteq \mathscr{A}$ *and* $\mathscr{A}_T = \mathscr{A}$ *if and only if* T *is strict.*

DEFINITION 1.2. Let T be an Archimedean t-norm. Then:

(i) C_T^* is the mapping defined for any $\varphi \in \mathscr{A}_T$ via:

$$C_T^* \varphi(x) = h^{[-1]}\big(\inf_{z \geqslant 0} e^{xz} \varphi(z)\big), \quad \text{for all } x, \tag{1.11}$$

and where, in addition, $C_T^*\varphi$ is normalized so as to be left-continuous. ($C_T^*\varphi$ will have at most one discontinuity.)

(ii) $F \in \Delta^+$ is (Product)-*log-concave* if $\log F$ is concave on (b_F, ∞), where $b_F = \sup\{x \mid F(x) = 0\}$. Furthermore,

$$\Delta_P^+ = \{F \in \Delta^+ \mid F \text{ is log-concave}\}.$$

(iii) $F \in \Delta^+$ is T*-log-concave* if hF is log-concave, where $hF \in \Delta^+$ is given by (1.7). Also,

$$\Delta_T^+ = \{F \in \Delta^+ \mid F \text{ is } T\text{-log-concave}\}.$$

(iv) For any $F \in \Delta^+$, the *log-concave envelope* of F, denoted $\overline{F}$, is defined as follows: $\overline{F}(x) = 0$ for $x \leqslant b_F$ and on (b_F, ∞) the graph of $\log(\overline{F})$ is the upper boundary of the concave hull of the graph of $\log F$.

(v) For any $F \in \Delta^+$, the T*-log-concave envelope* of F is the function F_T in Δ^+ given by $F_T = h^{[-1]}(\overline{hF})$, where $\overline{hF}$ is the log-concave envelope of hF. (Note that $\overline{F} = F_T$ for $T = $ Product.)

Remark. Even though multiplicative generators are not unique, we showed in [6] that the results obtained using T-conjugate transforms do not depend on the choice of a multiplicative generator h in (1.6), so long as it remains fixed. In particular, the concepts of T-log-concavity and T-log-concave envelopes are independent of the choice of a multiplicative generator and depend on the t-norm T alone.

We then have:

THEOREM 1.2. *Let T be an Archimedean t-norm. Then, for any* $F, G \in \Delta^+$,

(P1) $C_T(\tau_T(F, G))(z) = \max\{h(0), C_T F(z) \cdot C_T G(z)\}$ *for all* $z \geqslant 0$. (*Thus, if* T *is strict,* $C_T(\tau_T(F, G)) = C_T F \cdot C_T G$.)

(P2) $C_T\colon \Delta_T^+ \to \mathscr{A}_T$ *is one-to-one, onto, with inverse* C_T^*.

(P3) $F_T \in \Delta_T^+$, $F_T \geqslant F$, *and if* $F \in \Delta_T^+$, *then* $F_T = F$.

(P4) $C_T F = C_T(F_T)$, *whence* $C_T^* C_T F = F_T$.

(P5) *If* $F \geqslant G$, *then* $C_T F \geqslant C_T G$.

(P6) $C_T F(0) = \lim_{x \to \infty} hF(x)$ *and* $\lim_{z \to \infty} C_T F(z) = hF(0^+)$.

2. Continuity of conjugate transforms and general convergence. Our first task is to define a useful topology on the space of conjugate transforms $\mathscr{A}$ via:

DEFINITION 2.1. Let $\mathscr{L}^*$ be the mapping from $\mathscr{A} \times \mathscr{A}$ into $[0, 1]$, which for any $\varphi, \theta \in \mathscr{A}$ is given by

$$\mathscr{L}^*(\varphi, \theta) = \inf\{\delta \mid |\varphi(z) - \theta(z)| < \delta \text{ for all } \delta < z < 1/\delta\}. \tag{2.1}$$

THEOREM 2.1. $\mathscr{L}^*$ *is a metric on* $\mathscr{A}$.

Proof. Clearly, since all conjugate transforms are continuous at 0, we have $\mathscr{L}^*(\varphi, \theta) = 0$ if and only if $\varphi = \theta$.

It is also clear that $\mathscr{L}^*$ is symmetric.

To establish the triangle inequality let $\varphi_1, \varphi_2, \varphi_3 \in \mathscr{A}$ with $\mathscr{L}^*(\varphi_1, \varphi_2) = r$, and $\mathscr{L}^*(\varphi_2, \varphi_3) = s$. Assume, without loss of generality, that $r \leqslant s$. Then, for any $\varepsilon > 0$, if $s + \varepsilon < z < 1/(s+\varepsilon)$, then $r + \varepsilon < z < 1/(r+\varepsilon)$, so that, by (2.1),

$$|\varphi_1(z) - \varphi_3(z)| \leqslant |\varphi_1(z) - \varphi_2(z)| + |\varphi_2(z) - \varphi_3(z)| < s + \varepsilon + r + \varepsilon. \tag{2.2}$$

In particular then, (2.2) holds for any z with $r + s + 2\varepsilon < z < 1/(r+s+2\varepsilon)$. Thus $\mathscr{L}^*(\varphi_1, \varphi_3) \leqslant r + s + 2\varepsilon$, whence, since $\varepsilon > 0$ was arbitrary, the proof is complete.

By convergence in $\mathscr{A}$ we will mean convergence with respect to the metric $\mathscr{L}^*$. This type of convergence is characterized in:

THEOREM 2.2. *Let $\{\varphi_n\}$ be a sequence in $\mathscr{A}$ and let $\varphi \in \mathscr{A}$. Then $\mathscr{L}^*(\varphi_n, \varphi) \to 0$ if and only if $\varphi_n(z) \to \varphi(z)$ for all $z > 0$.*

Proof. Assume $\varphi_n(z) \to \varphi(z)$ for all $z > 0$. Let $\varepsilon > 0$ be given. Since φ is continuous, non-increasing, and bounded on $[0, \infty)$, it is uniformly continuous. Thus there is a $\delta > 0$ so that

$$|\varphi(z_1) - \varphi(z_2)| < \varepsilon/2 \quad \text{if} \quad |z_1 - z_2| < \delta. \tag{2.3}$$

Choose a finite set of points $y_1, y_2, \ldots, y_n$ so that $\varepsilon = y_1 < y_2 < \ldots < y_n = 1/\varepsilon$ and

$$|y_i - y_{i+1}| < \delta \quad \text{for} \quad i = 1, 2, \ldots, n-1. \tag{2.4}$$

For each $i = 1, 2, \ldots, n$ there is an integer N_i so that

$$|\varphi_n(y_i) - \varphi(y_i)| < \varepsilon/2 \quad \text{for} \quad n \geqslant N_i. \tag{2.5}$$

Let $N = \max\{N_i |\ i = 1, 2, \ldots, n\}$. Then, for any z with $\varepsilon < z < 1/\varepsilon$, there is an integer k, $1 \leqslant k \leqslant n-1$, such that $y_k \leqslant z < y_{k+1}$. Thus if $n \geqslant N$, then since φ_n, φ are non-increasing, we have using (2.3), (2.4), and (2.5) that

$$\begin{aligned}
&|\varphi_n(z) - \varphi(z)| \\
&\quad \leqslant \max\{|\varphi_n(y_k) - \varphi(z)|, |\varphi_n(y_{k+1}) - \varphi(z)|\} \\
&\quad \leqslant \max\{|\varphi_n(y_k) - \varphi(y_k)| + |\varphi(y_k) - \varphi(z)|, |\varphi_n(y_{k+1}) - \varphi(y_{k+1})| + \\
&\qquad\qquad + |\varphi(y_{k+1}) - \varphi(z)|\} < \varepsilon.
\end{aligned}$$

Consequently, $\mathscr{L}^*(\varphi_n, \varphi) \leqslant \varepsilon$ for $n \geqslant N$, whence $\mathscr{L}^*(\varphi_n, \varphi) \to 0$. Conversely, assume $\mathscr{L}^*(\varphi_n, \varphi) \to 0$ and let $z > 0$. For any $\varepsilon > 0$ there is an integer $k > 0$ so that

$$1/k < \min\{\varepsilon, z\} \quad \text{and} \quad k > z. \tag{2.6}$$

Also, by assumption, for some integer M we have $\mathscr{L}^*(\varphi_n, \varphi) < 1/k$ for $n \geqslant M$. But then, by (2.1) and (2.6), it follows that

$$|\varphi_n(z) - \varphi(z)| < 1/k < \varepsilon \quad \text{for} \quad n \geqslant M,$$

whence $\varphi_n(z) \to \varphi(z)$, completing the proof.

THEOREM 2.3. *Let $\mathscr{L}$ be the modified Lévy metric on Δ^+ and let $\mathscr{L}^*$ be the metric on $\mathscr{A}$ given by (2.1). Then the Prod-conjugate transform C: $(\Delta^+, \mathscr{L}) \to (\mathscr{A}, \mathscr{L}^*)$ is continuous.*

Proof. Assume that for $\{G_n\}$ and G in Δ^+, we have $\mathscr{L}(G_n, G) \to 0$. Choose $z > 0$ and let $\varepsilon > 0$ with $\varepsilon < z$ be given. Now, for fixed z, e^{-xz} is decreasing and bounded, and hence uniformly continuous, on $[0, \infty)$.

Thus there is a $\gamma > 0$ so that

$$|e^{-xz} - e^{-yz}| < \varepsilon/2 \quad \text{if} \quad |x - y| < \gamma. \tag{2.7}$$

Furthermore, there is an $M > 0$ such that for $x > M$,

$$e^{-xz} < e^{-Mz} < \varepsilon. \tag{2.8}$$

Next, for any δ with $0 < \delta < \min\{\varepsilon/2, \gamma, 1/M\}$, there is an $N > 0$ such that $n \geqslant N$ implies $\mathscr{L}(G_n, G) < \delta$. Therefore, if $n \geqslant N$ and $0 \leqslant x \leqslant M$, then, by (2.7) and the definition of $\mathscr{L}$, we have

$$\begin{aligned}
e^{-xz}G_n(x) &\leqslant e^{-xz}\big(G(x+\delta) + \delta\big) \leqslant e^{-xz}G(x+\delta) + \delta \\
&\leqslant (e^{-(x+\delta)z} + \varepsilon/2)G(x+\delta) + \delta \\
&\leqslant e^{-(x+\delta)z}G(x+\delta) + \varepsilon \\
&\leqslant CG(z) + \varepsilon,
\end{aligned}$$

which, combined with (2.8), yields, for $n \geqslant N$, that

$$CG_n(z) = \sup_{x \geqslant 0} e^{-xz}G_n(x) \leqslant CG(z) + \varepsilon. \tag{2.9}$$

Similarly, interchanging G_n and G in the above argument, we obtain, for $n \geqslant N$, that $CG(z) \leqslant CG_n(z) + \varepsilon$, which with (2.9) implies, for $n \geqslant N$, that

$$|CG(z) - CG_n(z)| < \varepsilon. \tag{2.10}$$

Since ε with $0 < \varepsilon < z$ was arbitrary, (2.10) implies that $CG_n(z) \to CG(z)$ for any $z > 0$. In other words, $\mathscr{L}^*(CG_n, CG) \to 0$, whence C is continuous, completing the proof.

Note that by definition, for any Archimedean t-norm T, we have $\mathscr{A}_T \subseteq \mathscr{A}$. Thus $\mathscr{L}^*$ is a metric on $\mathscr{A}_T$. Using Theorem 2.3, this yields:

COROLLARY 2.1. *Let T be an Archimedean t-norm. Then the T-conjugate transform C_T: $(\Delta^+, \mathscr{L}) \to (\mathscr{A}_T, \mathscr{L}^*)$ is continuous.*

Proof. By (1.6) and (1.8) we have, for any $F \in \Delta^+$, that $C_T F = C(hF)$, where hF is given by (1.7). Define the map h^*: $(\Delta^+, \mathscr{L}) \to (\Delta^+, \mathscr{L})$ by $h^*(F) = hF$. Since h is continuous and increasing, it is clear that if $\{F_n\}$ is a sequence in Δ^+ such that $F_n \xrightarrow{w} F$, then $hF_n \xrightarrow{w} hF$. In other words, h^* is continuous. But C_T can be factored as

$$(\Delta^+, \mathscr{L}) \xrightarrow{h^*} (\Delta^+, \mathscr{L}) \xrightarrow{C} (\mathscr{A}, \mathscr{L}^*).$$

Since C is continuous and $C_T(\Delta^+) = \mathscr{A}_T$ by Theorem 1.5, the desired result follows.

COROLLARY 2.2. *Let T be an Archimedean t-norm and let $\{G_n\}$ be a sequence in Δ^+. If $G_n \xrightarrow{w} G$, then $C_T G_n(z) \to C_T G(z)$ for all $z > 0$.*

Remarks. (1) The results of R. A. Wijsman [15] can be used to deduce that the prod-conjugate transform C is continuous on the subspace Δ_P^+ of all log-concave distribution functions. However, since using this fact explicitly would not lead to any significant condensation of our presentation, we have proven our results (e.g., Theorem 2.3) in straightforward and self-contained arguments.

(2) The converse of the preceding corollary is not true. For, from (P4), for any T, the T-conjugate transform C_T is not one-to-one on Δ^+. Thus there exist $G_1, G_2 \in \Delta^+$ with $G_1 \neq G_2$ so that $C_T G_1 = C_T G_2$. Letting $\{H_n\}$ be the sequence defined by letting $H_n = G_1$ if n is odd and $H_n = G_2$ if n is even, we have that $C_T H_n(z) \to C_T G_1(z)$ for all $z > 0$, but the sequence $\{H_n\}$ does not converge weakly in Δ^+.

(3) It is not true that $G_n \xrightarrow{w} G$ implies $C_T G_n(0) \to C_T G(0)$. For example, for each integer $n > 0$, let ε_n be defined by $\varepsilon_n(x) = \varepsilon_0(x-n)$. Then $\varepsilon_n \xrightarrow{w} \varepsilon_\infty$. But by (P6), for all n,

$$C\varepsilon_n(0) = \lim_{x\to\infty} \varepsilon_n(x) = 1 \neq 0 = C\varepsilon_\infty(0).$$

We can obtain a partial converse to Corollary 2.2 by restricting our attention to T-log-concave functions. Using the fact that $(\Delta^+, \mathscr{L})$ is compact, hence complete, our first step in this direction is:

LEMMA 2.1. *The space Δ_P^+ of all log-concave distribution functions is a closed, and hence compact, subset of $(\Delta^+, \mathscr{L})$.*

Proof. Let $\{F_n\}$ be a sequence in Δ_P^+ and suppose $F_n \xrightarrow{w} F$ so that $F \in \Delta^+$. Let $b_F = \sup\{x \mid F(x) = 0\}$. Let $x_0, x_1, x_2 > b_F$ be continuity points of F so that for some constants $p, q \geqslant 0$ with $p+q = 1$ we have $x_0 = px_1 + qx_2$. Then since log is continuous for positive reals and the $\{F_n\}$ are log-concave, we have

$$\log F(x_0) = \lim_{n\to\infty} \log F_n(x_0) \geqslant \lim_{n\to\infty} \big(p\log F_n(x_1) + q\log F_n(x_2)\big) = p\log F(x_1) + q\log F(x_2). \tag{2.11}$$

Next let $x_0, x_1, x_2 \geqslant b_F$ be arbitrary with $x_1 < x_0 < x_2$, so that if

$$p = (x_0 - x_1)/(x_2 - x_1) \quad \text{and} \quad q = 1-p, \tag{2.12}$$

then $x_0 = px_2 + qx_1$ with $p, q > 0$. Now $\mathscr{C}(F)$, the set of continuity points of F, is dense on the real line. Thus we may choose sequences $\{x_0(n)\}$, $\{x_1(n)\}$, $\{x_2(n)\}$ in $\mathscr{C}(F) \cap (b_F, \infty)$ so that $x_i(n) \nearrow x_i$ for $i = 0, 1, 2$ (i.e., $x_i(n)$ converges to x_i from the left for $i = 0, 1, 2$). For each integer n let $p_n = \big(x_0(n) - x_1(n)\big) / \big(x_2(n) - x_1(n)\big)$ and $q_n = (1-p_n)$, so that $x_0(n) = p_n x_2(n) + q_n x_1(n)$ for all n. Clearly the sequences $\{x_i(n)\}$ can be chosen so that $p_n, q_n > 0$ for all n, i.e. $x_1(n) < x_0(n) < x_2(n)$ for all n. Note by (2.12) that we then have $p_n \to p$ and $q_n \to q$. Thus, using the left continuity of F, (2.11), and the continuity of log we have

$$\log F(x_0) = \lim_{n\to\infty} \log F\big(x_0(n)\big) \geqslant \lim_{n\to\infty} \Big(p_n \log F\big(x_2(n)\big) + q_n \log F\big(x_1(n)\big)\Big) = p\log F(x_2) + q\log F(x_1),$$

whence F is log-concave, completing the proof.

LEMMA 2.2. *For any Archimedean t-norm T, the space Δ_T^+ of all T-log-concave distribution functions is a closed, and hence compact, subset of $(\Delta^+, \mathscr{L})$.*

Proof. Let $\{F_n\}$ be a sequence in Δ_T^+ and suppose $F_n \xrightarrow{w} F$, where $F \in \Delta^+$. Now h, the multiplicative generator of T, is a continuous, increasing function, whence if hF_n is given by (1.7), then clearly $hF_n \xrightarrow{w} hF$. But, by the definition of T-log-concavity, $\{hF_n\}$ is a sequence in Δ_P^+. Thus Lemma 2.1 implies that $hF \in \Delta_P^+$ or, equivalently that F is T-log-concave, completing the proof.

We can now prove the following key result:

THEOREM 2.4. *For any Archimedean t-norm T, the map*

$$C_T\colon (\Delta_T^+, \mathscr{L}) \to (\mathscr{A}_T, \mathscr{L}^*) \tag{2.13}$$

is a homeomorphism.

Proof. By (P2) the map (2.13) is one-to-one and onto. In addition, it easily follows from Corollary 2.1 that the map (2.13) is continuous. But then by [16], Thm 17.14, a continuous, one-to-one, onto map from a compact space to a Hausdorff space is a homeomorphism and $(\mathscr{A}_T, \mathscr{L}^*)$, being a metric space, is Hausdorff. This completes the proof.

By (P2) we then also have:

COROLLARY 2.3. *For any Archimedean t-norm T, the map $C_T^*\colon (\mathscr{A}_T, \mathscr{L}^*) \to (\Delta_T^+, \mathscr{L})$ is a homeomorphism.*

Another property that now follows easily is:

COROLLARY 2.4. *For any Archimedean t-norm T, the operation of forming T-log-concave envelopes, i.e., the mapping $F \to F_T$, is continuous. In particular if $F_n \xrightarrow{w} F$ in Δ^+, then $(F_n)_T \xrightarrow{w} F_T$.*

Proof. By Corollaries 2.1 and 2.3 we have that the map $C_T^* C_T$ is continuous on $(\Delta^+, \mathscr{L})$. Since, by (P4), for any $F \in \Delta^+$ we have $C_T^* C_T F = F_T$, the result follows.

The following immediate consequence of Lemma 2.2 and Theorem 2.4 is useful for studying the convergence of T-conjugate transforms.

COROLLARY 2.5. *For any Archimedean t-norm T, the metric space $(\mathscr{A}_T, \mathscr{L}^*)$ is compact, and, hence, complete.*

Two further properties which easily follow from the compactness and completeness of $(\mathscr{A}_T, \mathscr{L}^*)$ are:

COROLLARY 2.6. *Let T be an Archimedean t-norm and let $\{\varphi_n\}$ be a sequence in $\mathscr{A}_T$ such that the sequence $\{\varphi_n(z)\}$ converges for all $z > 0$. Then*

(1) *The function φ defined by $\varphi(z) = \lim_{n\to\infty}\varphi_n(z)$ for $z > 0$ and $\varphi(0) = \lim_{z\to 0+}\varphi(z)$ belongs to $\mathscr{A}_T$.*

(2) *If, furthermore, the sequence $\{\varphi_n(0)\}$ also converges and if $\theta(z) = \lim_{n\to\infty}\varphi_n(z)$ for all $z \geqslant 0$, then $\theta \in \mathscr{A}_T$ if and only if θ is continuous at 0.*

Proof. The proof is an easy exercise using the fact that conjugate transforms are continuous functions on $[0, \infty)$.

We can now characterize the relation between convergence of distribution functions and convergence of the corresponding conjugate transforms.

THEOREM 2.5. *Let T be an Archimedean t-norm. Then for any sequence $\{\varphi_n\}$ in $\mathscr{A}_T$, $\varphi_n(z) \to \varphi(z)$ for all $z > 0$ if and only if $C_T^*\varphi_n \xrightarrow{w} C_T^*\varphi$.*

Proof. The theorem is just a restatement of Corollary 2.3.

THEOREM 2.6. *Let T be an Archimedean t-norm and let $\{F_n\}$ be a sequence in Δ^+. Then, for $F \in \Delta^+$, $(F_n)_T \xrightarrow{w} F_T$ if and only if $C_T F_n(z) \to C_T F(z)$ for all $z > 0$.*

Proof. The result follows from Theorem 2.5 on letting $\varphi_n = C_T F_n$ for all n and $\varphi = C_T F$ and applying property (P4).

Two important special cases are convergence in Δ^+ to ε_0 and to ε_∞. As in the theory of infinite products, convergence to ε_∞ corresponds to divergence. Here we have:

THEOREM 2.7. *Let T be an Archimedean t-norm and let $\{F_n\}$ be a sequence in Δ^+. Then*

(i) *$F_n \xrightarrow{w} \varepsilon_0$ if and only if $C_T F_n(z) \to 1$ for all $z > 0$.*

(ii) *$F_n \xrightarrow{w} \varepsilon_\infty$ if and only if $C_T F_n(z) \to h(0)$ for all $z > 0$.*

Proof. If $F_n \xrightarrow{w} \varepsilon_0$, then, by Corollary 2.2, $C_T F_n(z) \to C_T\varepsilon_0(z) = 1$ for all $z > 0$. Conversely, suppose $C_T F_n(z) \to 1$ for all $z > 0$ and let $y > 0$. Then

$$C_T F_n(1) = \max\{\sup_{x\geqslant y} e^{-x} hF_n(x), \sup_{0\leqslant x\leqslant y} e^{-x} hF_n(x)\} \tag{2.14}$$
$$\leqslant \max\{e^{-y}, hF_n(y)\}.$$

Since $e^{-y} < 1$ and $C_T F_n(1) \to 1$, (2.14) implies that $hF_n(y) \to 1$, or, since h is continuous and increasing with $h(1) = 1$, that $F_n(y) \to 1$, establishing (i).

Similarly, if $F_n \xrightarrow{w} \varepsilon_\infty$, then, $C_T F_n(z) \to C_T\varepsilon_\infty(z) = h(0)$ for all $z > 0$.

Conversely, let $\theta_T(z) = h(0)$ for all $z \geqslant 0$ so that, by (1.10), $\theta_T \in \mathscr{A}_T$ and assume that $C_T F_n(z) \to h(0) = \theta_T(z)$ for all $z > 0$. Now it is easily shown that $\theta_T = C_T\varepsilon_\infty$, whence Theorem 2.6 implies that $(F_n)_T \xrightarrow{w} (\varepsilon_\infty)_T = \varepsilon_\infty$. But, by (P3), $(F_n)_T \geqslant F_n$ for all n, whence, necessarily, $F_n \xrightarrow{w} \varepsilon_\infty$, yielding (ii) and completing the proof.

The next results yield a much simpler test for convergence to ε_0 and to ε_∞ in Δ^+.

LEMMA 2.3. *Let T be an Archimedean t-norm and let $\{\varphi_n\}$ be a sequence in $\mathscr{A}_T$. Then*

(i) *If $\varphi_n(y) \to 1$ for some $y > 0$, then $\varphi_n(z) \to 1$ for all $z \geqslant 0$.*

(ii) *If $\varphi_n(y) \to 0$ for some $y > 0$, then $\varphi_n(z) \to 0$ for all $z > 0$.*

Proof. If $\varphi_n(y) \to 1$ for some $y > 0$, then, by the non-increasing character and boundedness of the $\{\varphi_n\}$, we have that $\varphi_n(z) \to 1$ for all $0 \leqslant z \leqslant y$. Also, using the log-convexity of the $\{\varphi_n\}$, we have, for any $\alpha > 1$, that

$$\log\varphi_n(y) \leqslant (1-1/\alpha)\log\varphi_n(0) + 1/\alpha\log\varphi_n(\alpha y)$$

or, since $\log\varphi_n(0) \leqslant 0$,

$$0 \geqslant \log\varphi_n(\alpha y) \geqslant \alpha\log\varphi_n(y) \to 0.$$

Thus $\log\varphi_n(\alpha y) \to 0$ or $\varphi_n(\alpha y) \to 1$ for any $\alpha > 1$, which establishes (i).

Similarly, if $\{\varphi_n\}$ is a sequence in $\mathscr{A}_T$ such that $\varphi_n(y) \to 0$ for some $y > 0$, then, since each φ_n is non-increasing and non-negative, we have $\varphi_n(z) \to 0$ for all $z \geqslant y$. Also, for any α with $0 < \alpha \leqslant 1$, we have, using the log-convexity of each φ_n and the fact that $\log\varphi_n(0) \leqslant 0$ for all n, that

$$\log\varphi_n(\alpha y) \leqslant \alpha\log\varphi_n(y) + (1-\alpha)\log\varphi_n(0) \leqslant \alpha\log\varphi_n(y). \tag{2.15}$$

But $\log\varphi_n(y) \to -\infty$, whence (2.15) implies that $\log\varphi_n(\alpha y) \to -\infty$ or $\varphi_n(\alpha y) \to 0$ for all $0 < \alpha \leqslant 1$. Therefore $\varphi_n(z) \to 0$ for all $0 < z \leqslant y$, yielding (ii) and completing the proof.

Note that if T is Archimedean but not strict, then, by Corollary 1.1, $h(0) > 0$. Thus, for any $\varphi \in \mathscr{A}_T$, $\varphi(z) \geqslant h(0) > 0$ for all $z \geqslant 0$. Hence, for any sequence $\{\varphi_n\}$ in $\mathscr{A}_T$, it is impossible for $\varphi_n(z) \to 0$ for any $z \geqslant 0$. Thus, part (ii) of Lemma 2.3 is only of use when T is strict.

Combining Lemma 2.3 with Theorem 2.7 yields:

THEOREM 2.8. *Let T be an Archimedean t-norm and let $\{F_n\}$ be a sequence in Δ^+. Then*

(i) *If, for some $y > 0$, $C_T F_n(y) \to 1$, then $F_n \xrightarrow{w} \varepsilon_0$ in Δ^+.*

(ii) *If, for some $y > 0$, $C_T F_n(y) \to 0$, then $F_n \xrightarrow{w} \varepsilon_\infty$ in Δ^+.*

We conclude this section by pointing out that the space of T-conjugate transforms, $\mathscr{A}_T$, can be endowed with the structure of a topological semigroup. For any Archimedean t-norm T, define the operation p_T, for $\varphi, \theta \in \mathscr{A}_T$, by

$$(2.16)\qquad p_T(\varphi, \theta)(z) = \max\{h(0), \varphi(z)\theta(z)\} \quad \text{for all } z \geqslant 0.$$

We then have:

THEOREM 2.9. *For any Archimedean t-norm T, $(\mathscr{A}_T, p_T, \mathscr{L}^*)$ is a commutative topological semigroup, with zero θ_T, where $\theta_T(z) = h(0)$ for all $z \geqslant 0$, and unit θ_0, where $\theta_0(z) = 1$ for all $z \geqslant 0$.*

Proof. To show $\mathscr{A}_T$ is closed under the operation p_T we first note by (P2) that, for any $\varphi, \theta \in \mathscr{A}_T$, there exist $F, G \in \Delta_T^+$ so that $C_T F = \varphi$ and $C_T G = \theta$. But then, by (2.16) and (P1),

$$(2.17)\qquad p_T(\varphi, \theta) = \max\{h(0), C_T F \cdot C_T G\} = C_T\big(\tau_T(F, G)\big) \in \mathscr{A}_T.$$

The remainder of the proof is easily verified.

If T is an Archimedean t-norm which is not strict, then $(\Delta_T^+, \tau_T, \mathscr{L})$ is not a topological semigroup because, as shown in [6], Δ_T^+ is not closed under τ_T. But by [6] if T is strict, then Δ_T^+ is closed under τ_T so that $(\Delta_T^+, \tau_T, \mathscr{L})$ is a sub-topological semigroup of $(\Delta^+, \tau_T, \mathscr{L})$. Also, from (2.17), for any $F, G \in \Delta_T^+$, we have $C_T\big(\tau_T(F, G)\big) = p_T(C_T F, C_T G)$, i.e., C_T preserves the operations τ_T and p_T. Combined with Theorem 2.9 these facts yield:

THEOREM 2.10. *For any strict t-norm T, $(\Delta_T^+, \tau_T, \mathscr{L})$ is topologically isomorphic to $(\mathscr{A}_T, p_T, \mathscr{L}^*)$, via C_T.*

3. Convergence of τ_T Products. To introduce this topic we begin with:

DEFINITION 3.1. For any t-norm and any sequence $\{F_i\}$ in Δ^+, let the sequence $\{\tau_T(F_1, F_2, \ldots, F_n)\}$ be defined recursively by $\tau_T(F_1) = F_1$ and for $n \geqslant 2$

$$\tau_T(F_1, \ldots, F_n) = \tau_T\big(\tau_T(F_1, \ldots, F_{n-1}), F_n\big).$$

Since τ_T is an associative operation on Δ^+, $\tau_T(F_1, \ldots, F_n)$ is well-defined.

Note, since each τ_T operation is non-decreasing, that for any $n \geqslant 1$,

$$\begin{aligned} \tau_T(F_1, \ldots, F_n) &= \tau_T\big(\tau_T(F_1, \ldots, F_n), \varepsilon_0\big) \\ &\geqslant \tau_T\big(\tau_T(F_1, \ldots, F_n), F_{n+1}\big) = \tau_T(F_1, \ldots, F_{n+1}). \end{aligned}$$

Thus $\{\tau_T(F_1, \ldots, F_n)\}$ is a non-increasing sequence in Δ^+ and, hence, has a weak limit in Δ^+. Our primary interest is in determining when this limit is non-trivial, i.e., $\neq \varepsilon_\infty$.

The following extension of (P1) is useful in this respect.

THEOREM 3.1. *Let T be Archimedean and let $F_i \in \Delta^+$ for $i = 1, 2, \ldots, n$. Then, for any $z \geqslant 0$,*

$$C_T\big(\tau_T(F_1, \ldots, F_n)\big)(z) = \max\Big\{h(0), \prod_{i=1}^{n} C_T F_i(z)\Big\}.$$

Proof. The result is easily established using (P1) by induction on n.

For our main result we will need some more notation.

DEFINITION 3.2. For any t-norm T and any sequence of points $\{x_n\}$ in $[0, 1]$, define, recursively; $\overset{1}{\underset{i=1}{T}} x_i = x_1$, $\overset{2}{\underset{i=1}{T}} x_i = T(x_1, x_2)$, and $\overset{n}{\underset{i=1}{T}} x_i = T\big(\overset{n-1}{\underset{i=1}{T}} x_i, x_n\big)$.

Since T is associative, we may also denote $\overset{n}{\underset{i=1}{T}} x_i$ by $T(x_1, \ldots, x_n)$. Also let $\overset{\infty}{\underset{i=1}{T}} x_i = \lim_{n\to\infty} \overset{n}{\underset{i=1}{T}} x_i$. Note that the sequence $\{\overset{n}{\underset{i=1}{T}} x_i\}$ is non-negative and non-increasing, so that $\overset{\infty}{\underset{i=1}{T}} x_i$ is well-defined.

LEMMA 3.1. *Let T be an Archimedean t-norm and let $\{x_n\}$ be any sequence in $[0, 1]$. Then*

(1) *For any positive integer n, $\overset{n}{\underset{i=1}{T}} x_i = h^{[-1]}\big(\prod_{i=1}^{n} h(x_i)\big)$; and*

(2) *$\overset{\infty}{\underset{i=1}{T}} x_i > 0$ if and only if $\prod_{i=1}^{\infty} h(x_i) > h(0)$.*

Proof. (i) For $n = 1$, using (1.5), we have $\overset{1}{\underset{i=1}{T}} x_i = x_1 = h^{[-1]}\big(h(x_1)\big)$. Also, for $n = 2$, $\overset{2}{\underset{i=1}{T}} x_i = T(x_1, x_2) = h^{[-1]}\big(h(x_1)\cdot h(x_2)\big)$ by (1.4).

(ii) In [6] we showed, for any $a, b \in [0, 1]$, that

$$h^{[-1]}\big(hh^{[-1]}(a)\cdot b\big) = h^{[-1]}(a \cdot b).$$

Thus assume (1) holds for $n = 1, 2, \ldots, k$. Then we have that

$$\begin{aligned} \overset{k+1}{\underset{i=1}{T}} x_i &= T\big(\overset{k}{\underset{i=1}{T}} x_i, x_{k+1}\big) = h^{[-1]}\Big(h\big(\overset{k}{\underset{i=1}{T}} x_i\big)\cdot h(x_{k+1})\Big) \\ &= h^{[-1]}\Big(hh^{[-1]}\Big(\prod_{i=1}^{k} h(x_i)\Big)\cdot h(x_{k+1})\Big) = h^{[-1]}\Big(\Big(\prod_{i=1}^{k} h(x_i)\Big)\cdot h(x_{k+1})\Big) \\ &= h^{[-1]}\Big(\prod_{i=1}^{k+1} h(x_i)\Big), \end{aligned}$$

completing the induction and the proof of (1).

Next, by (1) and the continuity of $h^{[-1]}$ we have $\mathop{T}_{i=1}^{\infty} x_i = h^{[-1]}\big(\prod_{i=1}^{\infty} h(x_i)\big)$, from which (2) then follows since, by (1.5), $h^{[-1]}(x) > 0$ if and only if $x > h(0)$.

THEOREM 3.2. *Let T be an Archimedean t-norm and let $\{F_n\}$ be a sequence in Δ^+. Let $G_n = \tau_T(F_1, \ldots, F_n)$, for $n = 1, 2, \ldots$ Then $\{G_n\}$ does not converge weakly to ε_∞, i.e., has a non-trivial limit, if and only if there is a sequence of positive numbers $\{a_i\}$ such that*

$$\sum_{i=1}^{\infty} a_i < \infty \quad \text{and} \quad \mathop{T}_{i=1}^{\infty} F_i(a_i) > 0.$$

Proof. Assumes that $\{G_n\}$ does not converge to ε_∞. Then by Theorem 2.7 there is a $w > 0$ such that $\{C_T G_n(w)\}$ does not converge to $h(0)$. But by the remark after Definition 3.1 and (P5), $\{C_T G_n(w)\}$ is a non-increasing sequence of real numbers, each of which is greater than or equal to $h(0)$, hence the limit of the sequence exists and, from the above remarks, necessarily,

$$\lim_{n\to\infty} C_T G_n(w) > h(0),$$

whence, in particular, $C_T G_n(w) > h(0)$ for all n.

Thus, using Theorem 3.1, we have

$$\begin{aligned} h(0) < \lim_{n\to\infty} C_T G_n(w) &= \lim_{n\to\infty} C_T\big(\tau_T(F_1, \ldots, F_n)\big)(w) \\ &= \lim_{n\to\infty} \max\Big\{h(0), \prod_{i=1}^{n} C_T F_i(w)\Big\} = \lim_{n\to\infty} \prod_{i=1}^{n} C_T F_i(w) \\ &= \prod_{i=1}^{\infty} C_T F_i(w) = \prod_{i=1}^{\infty} \sup_{x\geq 0} e^{-xw} hF_i(x). \end{aligned}$$

Next, select a sequence $\{p_i\}$ with $0 < p_i < 1$ for all i so that

$$\prod_{i=1}^{\infty} p_i > h(0)\Big/\Big(\prod_{i=1}^{\infty} \sup_{x\geq 0} e^{-xw} hF_i(x)\Big). \tag{3.1}$$

Now, for each integer i, there is an $a_i > 0$ such that

$$e^{-a_i w} hF_i(a_i) \geq p_i\big(\sup_{x\geq 0} e^{-xw} hF_i(x)\big).$$

For these a_i, using (3.1), we have

$$\begin{aligned} \prod_{i=1}^{\infty} e^{-a_i w} hF_i(a_i) &\geq \prod_{i=1}^{\infty} p_i\big(\sup_{x\geq 0} e^{-xw} hF_i(x)\big) \\ &= \Big(\prod_{i=1}^{\infty} p_i\Big)\Big(\prod_{i=1}^{\infty} \sup_{x\geq 0} e^{-xw} hF_i(x)\Big) > h(0). \end{aligned} \tag{3.2}$$

It follows from (3.2) that $\prod_{i=1}^{\infty} e^{-a_i w} = \exp\big[-\big(\sum_{i=1}^{\infty} a_i\big) w\big] > 0$, whence $\sum_{i=1}^{\infty} a_i < \infty$. Moreover, since $\prod_{i=1}^{\infty} e^{-a_i w} \leq 1$, (3.2) also implies that $\prod_{i=1}^{\infty} hF_i(a_i) > h(0)$ which combined with Lemma 3.1 yields that $\mathop{T}_{i=1}^{\infty} F_i(a_i) > 0$.

Conversely, assume there exists a sequence of positive numbers $\{a_i\}$ satisfying $\sum_{i=1}^{\infty} a_i < \infty$ and $\mathop{T}_{i=1}^{\infty} F_i(a_i) > 0$. We will show by induction that for each positive integer n,

$$G_n(x) \geq \mathop{T}_{k=1}^{n} F_k(a_k) \quad \text{for all } x > \sum_{k=1}^{n} a_k. \tag{3.3}$$

(i) For $n = 1$, (3.3) just states that $F_1(x) \geq F_1(a_1)$ for all $x > a_1$, which is clearly true since F_1 is non-decreasing.

(ii) Suppose (3.3) holds for $k = 1, \ldots, n$. For any $x > \sum_{k=1}^{n+1} a_k$, choose points s, t with $s+t = x$ so that $s > \sum_{k=1}^{n} a_k$ and $t > a_{n+1}$. Then, by (1.3) and the induction hypotheses,

$$\begin{aligned} G_{n+1}(x) = \tau_T(G_n, F_{n+1})(x) &\geq T\big(G_n(s), F_{n+1}(t)\big) \\ &\geq T\big(\mathop{T}_{k=1}^{n} F_k(a_k), F_{n+1}(a_{n+1})\big) = \mathop{T}_{k=1}^{n+1} F_k(a_k), \end{aligned}$$

completing the induction.

In particular if $x > \sum_{k=1}^{\infty} a_k$, then by (3.3), $G_n(x) \geq \mathop{T}_{k=1}^{n} F_k(a_k) \geq \mathop{T}_{k=1}^{\infty} F_k(a_k) > 0$ for all n, whence $\{G_n\}$ does not converge weakly to ε_∞. This completes the proof.

Note that the second half of the above proof holds for any t-norm T that induces an associative operation τ_T on Δ^+. Thus we have:

COROLLARY 3.1. *Let T be any left-continuous t-norm and let $\{F_n\}$ be a sequence in Δ^+. Suppose there exists a sequence of positive numbers, $\{a_i\}$, such that*

$$\sum_{i=1}^{\infty} a_i < \infty \quad \text{and} \quad \mathop{T}_{i=1}^{\infty} F_i(a_i) > 0.$$

Then the sequence $\{\tau_T(F_1, \ldots, F_n)\}$ has a non-trivial weak limit, i.e., it does not converge to ε_∞.

Remark. The condition $\mathop{T}_{i=1}^{\infty} F_i(a_i) > 0$ takes various forms, depending on the t-norm T. For example, for T = product the condition is $\prod_{i=1}^{\infty} F_i(a_i)$

> 0; for $T = \mathrm{Min}$ it is $\inf_i \{F_i(a_i)\} > 0$; and for $T = T_m$, where $T_m(a, b) = \max\{a+b-1, 0\}$, it is $\sum_{i=1}^{\infty} (1-F_i(a_i)) < 1$.

We close with two consequences of the preceding results. First:

COROLLARY 3.2. *Let T be Archimedean and let $\{F_n\}$ be a sequence in Δ^+. If for some number $M < 1$ there exists a sequence of non-negative numbers $\{a_i\}$ such that*

$$\sum_{i=1}^{\infty} a_i = \infty \quad \text{and} \quad F_k(a_k) \leqslant M \text{ for all } k, \tag{3.4}$$

then

$$\tau_T(F_1, \ldots, F_n) \xrightarrow{w} \varepsilon_\infty.$$

Proof. Let $\{a_i\}$ satisfy (3.4) and let $\{b_i\}$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} b_i < \infty$. Then clearly by (3.4) the set $J = \{k|\ b_k \leqslant a_k\}$ is infinite. Hence

$$\prod_{k=1}^{\infty} hF_k(b_k) \leqslant \prod_{k\in J} hF_k(b_k) \leqslant \prod_{k\in J} hF_k(a_k) \leqslant \prod_{k\in J} h(M) = 0$$

since $h(M) < 1$. Therefore, by Lemma 3.1, $\operatorname*{T}_{k=1}^{\infty} F_k(b_k) = 0$, whence by Theorem 3.2 $\tau_T(F_1, \ldots, F_n) \xrightarrow{w} \varepsilon_\infty$, completing the proof.

Remark. Corollary 3.2 gives a sufficient, but not necessary, condition for $\tau_T(F_1, \ldots, F_n) \xrightarrow{w} \varepsilon_\infty$. Consider the following example. Let

$$F_i(x) = \begin{cases} 0, & x \leqslant 0, \\ 1-1/(i+1), & 0 < x; \end{cases} \qquad i = 1, 2, \ldots; \tag{3.5}$$

and let $T = \text{Product}$. Then by (3.5), for any $M < 1$ and any sequence of non-negative numbers $\{a_i\}$ such that $\sum_{i=1}^{\infty} a_i = \infty$, there are only a finite number of indices i with $a_i > 0$ such that $F_i(a_i) \leqslant M$. Thus the second half of (3.4) is not satisfied. Yet for any sequence of positive numbers $\{b_i\}$ we have that $\prod_{i=1}^{\infty} F_i(b_i) = \prod_{i=1}^{\infty} (1-1/(i+1)) = 0$ since $\sum_{i=1}^{\infty} 1/(i+1) = \infty$. Hence by Theorem 3.2, $\tau_{\mathrm{Prod}}(F_1, \ldots, F_n) \xrightarrow{w} \varepsilon_\infty$. Note, since $F_i \neq \varepsilon_\infty$ for any i, that $\tau_{\mathrm{Prod}}(F_1, \ldots, F_n) \neq \varepsilon_\infty$ for any integer n because Prod is a strict t-norm.

COROLLARY 3.3. *Let T be an Archimedean t-norm and let $\{F_n\}$ be a sequence in Δ^+. Then either $\tau_T(F_1, \ldots, F_n) \xrightarrow{w} \varepsilon_\infty$ or $F_n \xrightarrow{w} \varepsilon_0$, and possibly both.*

Proof. Suppose $\{\tau_T(F_1, \ldots, F_n)\}$ does not converge weakly to ε_∞. Let $y > 0$ be fixed and $\varepsilon > 0$ arbitrary. For each integer i, let

$$a_i = \begin{cases} 0, & \text{if } \quad F_i(y) > 1-\varepsilon, \\ y, & \text{if } \quad F_i(y) \leqslant 1-\varepsilon. \end{cases}$$

By Corollary 3.2 we must have $\sum_{i=1}^{\infty} a_i < \infty$, i.e., the set $J = \{i|\ F_i(y) \leqslant 1-\varepsilon\}$ must be finite. But then, for some integer N, we must have that $i \geqslant N$ implies $F_i(y) > 1-\varepsilon$ so that $F_i(y) \to 1$ and hence $F_n \xrightarrow{w} \varepsilon_0$.

The "possibly both" part of the Corollary is demonstrated by the $\{F_i\}$ defined in (3.5), where we have $F_i \xrightarrow{w} \varepsilon_0$, even though $\tau_T(F_1, \ldots, F_n) \xrightarrow{w} \varepsilon_\infty$. Thus our proof is complete.

Remark. If T is a continuous non-Archimedean t-norm, then $T(c, c) = c$ for some $c \in (0, 1)$. Then, letting $F_i = F$ for $i = 1, 2, \ldots$, where

$$F(x) = \begin{cases} 0, & x \leqslant 0, \\ c, & 0 < x; \end{cases}$$

it can easily be shown that $\tau_T(F, F) = F$. Hence $\tau_T(F_1, \ldots, F_n) = F$ for all n so that $\tau_T(F_1, \ldots, F_n) \overset{w}{\not\to} \varepsilon_\infty$ and $F_n \overset{w}{\not\to} \varepsilon_0$, i.e., Corollary 3.3 does not hold for non-Archimedean t-norms.

References

[1] R. Bellman and W. Karush, *On a new functional transform in analysis: the maximum transform*, Bull. Amer. Math. Soc. 67 (1961), pp. 501–503.

[2] M. J. Frank, *Associativity in a class of operations on spaces of distribution functions*, Aequationes Math. 12 (1975), pp. 121–144.

[3] C. H. Ling, *Representation of associative functions*, Publ. Math. Debrecen 12 (1965), pp. 189–212.

[4] R. Moynihan, *On the class of τ_T semigroups of probability distribution functions*, Aequationes Math. 12 (1975), pp. 249–261.

[5] — *On τ_T semigroups of probability distribution functions II*, ibid. 17 (1978), pp. 19–40.

[6] — *Conjugate transforms for τ_T semigroups of probability distribution functions*, J. Math. Anal. Appl. 74 (1980), pp. 15–30.

[7] R. Moynihan and B. Schweizer, *Betweenness relations in probabilistic metric spaces*, Pacific J. Math. 81 (1979), pp. 175–196.

[8] R. Moynihan, B. Schweizer and A. Sklar, *Inequalities among operations on probability distribution functions*, in: *General inequalities* I, *Proceeding of the First International Conference on General Inequalities*, Oberwolfach, 1976, ed. by E. F. Beckenbach, International Series of Numerical Mathematics, Vol. 41 (1978), pp. 133–149, Birkhauser Verlag, Basel.

[9] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.

[10] B. Schweizer, *Probabilistic metric spaces — the first 25 years*, The New York Statistician 19 (1967), pp. 3–6.

[11] — *Multiplications on the space of probability distribution functions*, Aequationes Math. 12 (1975), pp. 156–183.

[12] B. Schweizer and A. Sklar, *Operations on distribution functions not derivable from operations on random variables*, Studia Math. 52 (1974), pp. 43–52.

[13] A. N. Serstnev, *The notion of a random normed space*, Dokl. Acad. Nauk SSSR 149 (1963), pp. 280–283 [translated in Soviet Math. Dokl. 4 (1963), pp. 388–391].

[14] D. A. Sibley, *A metric for weak convergence of distribution functions*, Rocky Mountain J. Math. 1 (1971), pp. 427–430.

[15] R. A. Wijsman, *Convergence of sequences of convex sets, cones and functions II*, Trans. Amer. Math. Soc. 123 (1966), pp. 32–45.

[16] S. Willard, *General topology*, Addison-Wesley, Reading, Mass. 1970.

[17] R. Moynihan, *Infinite τ_T products of distribution functions*, J. Austral. Math. Soc. (Series A) 26 (1978), pp. 227–240.

ANALYSIS DEPARTMENT
THE MITRE CORPORATION
Bedford, Massachusetts

Received April 7, 1977 (1289)

STUDIA MATHEMATICA, T. LXIX. (1980)

Weighted inequalities for vector-valued maximal functions and singular integrals

by

KENNETH F. ANDERSEN (Edmonton, Alta) and RUSSEL T. JOHN (Holmdel, N. J.)

Abstract. Weighted weak and strong type norm inequalities are derived for a vector-valued analogue of the Hardy–Littlewood maximal function operator and these in turn are used to obtain weighted inequalities for the classical Marcinkiewicz integral and a wide class of singular integral operators defined on $\boldsymbol{R}^n$.

§ 1. Introduction. The *Hardy-Littlewood maximal function* $f^*(x)$ is defined for locally integrable functions f on $\boldsymbol{R}^n$ by

$$f^*(x) = \sup \frac{1}{|Q|} \int_Q |f(y)|\, dy \qquad (x \in \boldsymbol{R}^n),$$

the supremum being taken over all cubes Q of Lebesgue measure $|Q|$, centered at x with sides parallel to the co-ordinate axis. The operator $M\colon f \to f^*$ and its variants have been widely studied, in particular, the well-known inequalities

$$\int_{\boldsymbol{R}^n} |f^*(x)|^p\, dx \leqslant C_p \int_{\boldsymbol{R}^n} |f(x)|^p\, dx \qquad (1 < p < \infty), \tag{1.1}$$

$$|\{x \in \boldsymbol{R}^n\colon f^*(x) > a\}| \leqslant Ca^{-1} \int_{\boldsymbol{R}^n} |f(x)|\, dx \qquad (\forall a > 0), \tag{1.2}$$

$$\operatorname*{ess\,sup}_{x \in \boldsymbol{R}^n} f^*(x) \leqslant \operatorname*{ess\,sup}_{x \in \boldsymbol{R}^n} |f(x)| \tag{1.3}$$

have been generalized and extended in various directions.

Let $\omega(x)$ be non-negative, locally integrable on $\boldsymbol{R}^n$ and for measurable $E \subset \boldsymbol{R}^n$ put $\omega(E) = \int_E \omega(x)\, dx$. We say that $\omega \in A_p$, $(1 \leqslant p < \infty)$ if there is a constant K such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x)\, dx\right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)}\, dx\right)^{p-1} \leqslant K \tag{1.4}$$