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Weighted inequalities for vector-valued maximal functions and singular integrals

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Abstract. Weighted weak and strong type norm inequalities are derived for a vector-valued analogue of the Hardy–Littlewood maximal function operator and these in turn are used to obtain weighted inequalities for the classical Marcinkiewicz integral and a wide class of singular integral operators defined on $\boldsymbol{R}^n$.

§ 1. Introduction. The *Hardy-Littlewood maximal function* $f^*(x)$ is defined for locally integrable functions f on $\boldsymbol{R}^n$ by

$$f^*(x) = \sup \frac{1}{|Q|} \int_Q |f(y)|\, dy \qquad (x \in \boldsymbol{R}^n),$$

the supremum being taken over all cubes Q of Lebesgue measure $|Q|$, centered at x with sides parallel to the co-ordinate axis. The operator $M\colon f \to f^*$ and its variants have been widely studied, in particular, the well-known inequalities

$$\int_{\boldsymbol{R}^n} |f^*(x)|^p\, dx \leqslant C_p \int_{\boldsymbol{R}^n} |f(x)|^p\, dx \qquad (1 < p < \infty), \tag{1.1}$$

$$|\{x \in \boldsymbol{R}^n\colon f^*(x) > a\}| \leqslant C a^{-1} \int_{\boldsymbol{R}^n} |f(x)|\, dx \qquad (\forall a > 0), \tag{1.2}$$

$$\operatorname*{ess\,sup}_{x\in\boldsymbol{R}^n} f^*(x) \leqslant \operatorname*{ess\,sup}_{x\in\boldsymbol{R}^n} |f(x)| \tag{1.3}$$

have been generalized and extended in various directions.

Let $\omega(x)$ be non-negative, locally integrable on $\boldsymbol{R}^n$ and for measurable $E \subset \boldsymbol{R}^n$ put $\omega(E) = \int_E \omega(x)\, dx$. We say that $\omega \in A_p$, $(1 \leqslant p < \infty)$ if there is a constant K such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x)\, dx\right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)}\, dx\right)^{p-1} \leqslant K \tag{1.4}$$

for all cubes $Q \subset \boldsymbol{R}^n$. In (1.4), products of the form $0\cdot\infty$ are taken to be zero, and for $p=1$ the second factor on the left is understood to be $\operatorname{ess\,sup}_{x\in Q} \omega(x)^{-1}$. In [13] B. Muckenhoupt proved that $\omega\in A_p$ is both necessary and sufficient in order that the following weighted analogues of (1.1) and (1.2) should hold:

$$(1.5)\qquad \int_{\boldsymbol{R}^n} |f^*(x)|^p\omega(x)\,dx \leqslant C_p\int_{\boldsymbol{R}^n}|f(x)|^p\omega(x)\,dx \qquad (1<p<\infty),$$

$$(1.6)\qquad \omega\big(\{x\in\boldsymbol{R}^n:\ f^*(x)>a\}\big)\leqslant C_p a^{-p}\int_{\boldsymbol{R}^n}|f(x)|^p\omega(x)\,dx \qquad (1\leqslant p<\infty).$$

These results were used later by R. Hunt, B. Muckenhoupt and R. Wheeden [9] and by R. Coifman and C. Fefferman [5] to obtain weighted norm inequalities for Hilbert transforms in $\boldsymbol{R}^1$ and singular integrals in $\boldsymbol{R}^n$, respectively.

In a different direction C. Fefferman and E. Stein [7] obtained vector-valued analogues of (1.1)–(1.3) and applied these to obtain certain estimates for the Marcinkiewicz integral. If $f=\{f_k\}_1^\infty$, is a sequence of locally integrable functions on $\boldsymbol{R}^n$, $f^*=\{f_k^*\}_1^\infty$, and $|f(x)|_r=\big(\sum_{k=1}^\infty |f_k(x)|^r\big)^{1/r}$, $1<r<\infty$, their main result may be stated as follows:

$$(1.7)\qquad \int_{\boldsymbol{R}^n}|f^*(x)|_r^p\,dx\leqslant C_{r,p}\int_{\boldsymbol{R}^n}|f(x)|_r^p\,dx \qquad (1<p<\infty),$$

$$(1.8)\qquad |\{x\in\boldsymbol{R}^n:\ |f^*(x)|_r>a\}|\leqslant C_{r,p}a^{-p}\int_{\boldsymbol{R}^n}|f(x)|_r^p\,dx \qquad (1\leqslant p<\infty).$$

(1.9) If $|E|<\infty$ and $|f(x)|_r$ is bounded and supported on E, then $|f^*(x)|_r^r\in \exp L(E)$.

The purpose of this paper is to obtain the weighted analogues of (1.7)–(1.9), the corresponding weighted estimates for the Marcinkiewicz integral and weighted norm inequalities for a wide class of vector-valued singular integral operators. Section 3 is devoted to the statement and proof of results for the vector-valued maximal function operator; the application to the Marcinkiewicz integral is given in Section 4 while Section 5 contains the results for vector-valued singular integrals. Section 2 contains the statement of two interpolation lemmas as well as several facts about the A_p condition which are required in the sequel.

As usual $C_p,\ldots$ will denote an absolute constant, not necessarily the same at each occurrence, depending only on n, ω and the parameters indicated by subscripts.

§ 2. Preliminaries. In addition to the A_p condition already defined for $1\leqslant p<\infty$ we shall require also the A_∞ condition, namely $\omega\in A_\infty$

if there are positive constants K, δ such that

$$\frac{\omega(E)}{\omega(Q)}\leqslant K\left(\frac{|E|}{|Q|}\right)^{\delta}$$

for every cube Q and measurable $E\subset Q$.

The following properties, used frequently in the sequel, are stated here for easy reference. For proofs, see [13], [5], [9] and [14].

$$(2.1)\qquad \omega(x)\in A_p \Leftrightarrow \omega(x)^{1-p'}\in A_{p'},\qquad 1<p<\infty,\ 1/p+1/p'=1,$$

$$\omega(x)\in A_p\Rightarrow \omega(x)\in A_q\ \forall q>p,$$

$$(2.2)\qquad \omega(x)\in A_p,\ (p>1)\Rightarrow \omega(x)\in A_q \text{ for some } q<p,$$

$$\omega(x)\in A_\infty\Leftrightarrow \omega(x)\in A_p \text{ for some } p\geqslant 1,$$

(2.3) $\omega(x)\in A_p\Rightarrow \omega(x)>0$ a.e. and $\omega(x)^q$ is locally integrable for some $q>1$,

(2.4) $\omega(x)\in A_p\Rightarrow \omega(\bar{Q})\leqslant C\omega(Q)$ for all cubes Q where $\bar{Q}$ is the cube concentric with Q and diameter $(\bar{Q})=2$ diameter (Q).

The following interpolation results will be required. The first is essentially the Marcinkiewicz interpolation Theorem, see [1]. The second is a vector-valued analogue of the Riesz convexity Theorem; for linear operators it is proved in [2] while the extension to the sublinear case may be patterned along the line of proof given in [4] for the scalar-valued case.

Let S denote the linear space of sequences $f=\{f_k\}$ of the form: $f_k(x)$ is a simple function on $\boldsymbol{R}^n$ and $f_k(x)\equiv 0$ for all sufficiently large k. S is dense in $L^p_\omega(l^r)$, $1\leqslant p,\ r<\infty$, see [2].

LEMMA 2.1. *Let $\omega(x)\geqslant 0$ be locally integrable on $\boldsymbol{R}^n$, $1<r<\infty$, $1\leqslant p_i\leqslant q_i<\infty$ and suppose T is a sublinear operator defined on S satisfying*

$$\omega\big(\{x\in\boldsymbol{R}^n:\ |Tf(x)|_r>a\}\big)\leqslant M_i^{q_i}a^{-q_i}\Big(\int_{\boldsymbol{R}^n}|f(x)|_r^{p_i}\omega(x)\,dx\Big)^{q_i/p_i}$$

for $i=0,1$ and $f\in S$. Then T extends uniquely to a sublinear operator on $L^p_\omega(l^r)$ and there is a constant M_θ such that

$$\Big(\int_{\boldsymbol{R}^n}|Tf(x)|_r^q\omega(x)\,dx\Big)^{1/q}\leqslant M_\theta\Big(\int_{\boldsymbol{R}^n}|f(x)|_r^p\omega(x)\,dx\Big)^{1/p}$$

where

$$(1/p,1/q)=(1-\theta)(1/p_0,1/q_0)+\theta(1/p_1,1/q_1),\qquad 0<\theta<1.$$

LEMMA 2.2. *Let $\omega(x)\geqslant 0$ be locally integrable on $\boldsymbol{R}^n$, $1<r_i,\ s_i<\infty$, $1\leqslant p_i,\ q_i<\infty$ and suppose T is a sublinear operator defined on S satisfying*

$$\Big(\int_{\boldsymbol{R}^n}|Tf(x)|_{s_i}^{q_i}\omega(x)\,dx\Big)^{1/q_i}\leqslant M_i\Big(\int_{\boldsymbol{R}^n}|f(x)|_{r_i}^{p_i}\omega(x)\,dx\Big)^{1/p_i}$$

for $i = 0, 1$ and $f \in S$. Then T extends uniquely to a sublinear operator on $L^p_\omega(l^r)$ such that

$$\Big(\int_{\mathbf{R}^n} |Tf(x)|_s^q\,\omega(x)\,dx\Big)^{1/q} \leqslant M_0^{1-\theta} M_1^{\theta} \Big(\int_{\mathbf{R}^n} |f(x)|_r^p\,\omega(x)\,dx\Big)^{1/p}$$

where

$$(1/p, 1/q, 1/s, 1/r) = (1-\theta)(1/p_0, 1/q_0, 1/s_0, 1/r_0) + \theta(1/p_1, 1/q_1, 1/s_1, 1/r_1), \quad 0 \leqslant \theta \leqslant 1.$$

§ 3. Maximal function inequalities. In this section we prove the following theorem.

THEOREM 3.1. *Let $1 < r < \infty$ and suppose $E \subset \mathbf{R}^n$ with $|E| < \infty$.*

(a) *If $1 \leqslant p < \infty$, there is a constant $C_{r,p}$ such that*

$$(3.1) \qquad \omega(\{x \in \mathbf{R}^n:\ |f^*(x)|_r > \alpha\}) \leqslant C_{q,p}\alpha^{-p} \int_{\mathbf{R}^n} |f(x)|_r^p\,\omega(x)\,dx$$

if and only if $\omega \in A_p$.

(b) *If $1 < p < \infty$, there is a constant $C_{r,p}$ such that*

$$(3.2) \qquad \int_{\mathbf{R}^n} |f^*(x)|_r^p\,\omega(x)\,dx \leqslant C_{r,p} \int_{\mathbf{R}^n} |f(x)|_r^p\,\omega(x)\,dx$$

if and only if $\omega \in A_p$.

(c) *If $\omega(x)^q$ is integrable on E for some $q > 1$, in particular if $\omega \in A_\infty$ and E is a cube, then $|f^*(x)|_r^r \in \exp L_\omega(E)$ whenever $|f(x)|_r$ is bounded and supported on E.*

H. Heinig [8] has recently obtained (a) in the case $p = 1$ and has also given some results for p in the range $0 < p < 1$. The weights considered there satisfy the condition $\omega^*(x) \leqslant K\omega(x)$ a.e. which is readily seen to be equivalent to the A_1 condition.

For the proof of Theorem 3.1 we require the following duality result of C. Fefferman and E. Stein [7], Lemma 1.

LEMMA 1. *Let f, φ be non-negative real-valued functions on $\mathbf{R}^n$ and suppose $r > 1$. There exists a constant C_r independent of f, φ such that*

$$(3.3) \qquad \int_{\mathbf{R}^n} |f^*(x)|^r \varphi(x)\,dx \leqslant C_r \int_{\mathbf{R}^n} |f(x)|^r \varphi^*(x)\,dx.$$

Proof of Theorem 3.1. We shall prove (a) and (b) first, then (c).

Concerning the necessity of $\omega \in A_p$, there is nothing to show since $\omega \in A_p$ is already necessary in the scalar-valued case, $f = \{f_k\}$, $f_k(x) = 0$, $k = 2, 3, \ldots$ according to Muckenhoupt's results (1.5) and (1.6).

The sufficiency of $\omega \in A_p$ for (a) and (b) will be achieved as follows. We first obtain (3.2) for $p = r$ as an easy consequence of (1.5); then (3.1) is derived for $r > p$ and (3.2) is then obtained for $r > p$ by an appeal to Lemma 2.1. A duality argument then yields (3.2) for $r < p$ and r sufficiently small; finally (3.2) follows for all $r < p$ by an application of Lemma 2.2. Since (3.2) always implies (3.1), the proof of (a) and (b) will be complete.

Observe first that (3.2) for the case $r = p$ is an easy consequence of (1.5) since

$$(3.4) \qquad \int_{\mathbf{R}^n} |f^*(x)|_r^r\,\omega(x)\,dx = \sum \int_{\mathbf{R}^n} |f_k^*(x)|^r\,\omega(x)\,dx \leqslant C_r \sum \int_{\mathbf{R}^n} |f_k(x)|^r\,\omega(x)\,dx = C_r \int_{\mathbf{R}^n} |f(x)|_r^r\,\omega(x)\,dx.$$

Now suppose $r > p$, $\omega \in A_p$ and $\alpha > 0$. As usual, we can assume without loss of generality that $f \in S$, for then the general case follows by a standard limiting argument. The Calderón–Zygmund decomposition [15], pp. 17–18, yields a sequence of non-overlapping cubes $\{Q_j\}$ such that

$$(3.5) \qquad |f(x)|_r \leqslant \alpha, \quad x \notin \Omega = \bigcup_1^\infty Q_j,$$

$$(3.6) \qquad \alpha < \frac{1}{|Q_j|} \int_{Q_j} |f(x)|_r\,dx \leqslant 2^n\alpha, \quad j = 1, 2, \ldots$$

Let $f = f' + f''$ where $f' = \{f_k'\}$, $f_k'(x) = f_k(x)\chi_{\mathbf{R}^n - \Omega}(x)$. Minkowski's inequality shows that

$$|f^*(x)|_r \leqslant |f'^*(x)|_r + |f''^*(x)|_r$$

so that (3.1) will follow if we show that

$$(3.7) \qquad \omega(\{x \in \mathbf{R}^n:\ |f'^*(x)|_r > \alpha\}) \leqslant C_{r,p}\alpha^{-p} \int_{\mathbf{R}^n} |f(x)|_r^p\,\omega(x)\,dx$$

and

$$(3.8) \qquad \omega(\{x \in \mathbf{R}^n:\ |f''^*(x)|_r > \alpha\}) \leqslant C_{r,p}\alpha^{-p} \int_{\mathbf{R}^n} |f(x)|_r^p\,\omega(x)\,dx.$$

Since $\omega \in A_r$ by (2.2), (3.4) holds with f' in place of f, and hence the Chebyshev inequality yields

$$\omega(\{x \in \mathbf{R}^n:\ |f'^*(x)|_r > \alpha\}) \leqslant C_r\alpha^{-r} \int_{\mathbf{R}^n} |f'(x)|_r^r\,\omega(x)\,dx$$

and since, by (3.5), $|f'(x)|_r^r \leqslant \alpha^{r-p}|f'(x)|_r^p$, we obtain (3.7). To prove (3.8), define $\bar f = \{\bar f_k\}$ by

$$\bar f_k(x) = \begin{cases} \dfrac{1}{|Q_j|}\displaystyle\int_{Q_j} |f_k(y)|\,dy, & x \in Q_j,\ j = 1, 2, \ldots, \\ 0 & \text{otherwise} \end{cases}$$

and let $\bar Q_j$ denote the cube with the same center as Q_j but with diameter $(\bar Q_j) = (2n)\,\text{diameter}\,(\bar Q_j)$. It is shown in [7] that $f_k''^*(x) \leqslant C\bar f_k^*(x)$ for

$x \notin \bar{\Omega} = \bigcup_1^\infty \bar{Q}_j$. Thus (3.8) will follow if we show

$$\omega(\bar{\Omega}) \leqslant C\alpha^{-p} \int_{\mathbf{R}^n} |f(x)|_r^p \, \omega(x)\, dx \tag{3.9}$$

and

$$\omega\big(\{x \in \mathbf{R}^n\colon |\bar{f}^*(x)|_r > \alpha\}\big) \leqslant C_r \alpha^{-p} \int_{\mathbf{R}^n} |f(x)|_r^p \, \omega(x)\, dx. \tag{3.10}$$

If $p > 1$, Hölder's inequality applied to (3.6) shows

$$\omega(Q_j) = \int_{Q_j} \omega(x)\, dx \leqslant \alpha^{-p} \frac{1}{|Q_j|^p} \Big(\int_{Q_j} |f(x)|_r\, dx\Big)^p \int_{Q_j} \omega(x)\, dx$$
$$\leqslant \alpha^{-p} \Big(\int_{Q_j} |f(x)|_r^p \omega(x)\, dx\Big) \Big(\frac{1}{|Q_j|} \int_{Q_j} \omega(x)^{-1/(p-1)}\, dx\Big)^{p-1} \Big(\frac{1}{|Q_j|} \int_{Q_j} \omega(x)\, dx\Big)$$

and since $\omega \in A_p$ we obtain

$$\omega(Q_j) \leqslant C\alpha^{-p} \int_{Q_j} |f(x)|_r^p \, \omega(x)\, dx. \tag{3.11}$$

A similar argument shows that (3.11) holds also if $p = 1$. Using (2.4) we see that $\omega(\bar{Q}_j) \leqslant C\,\omega(Q_j)$ and hence (3.9) follows from (3.11) upon summing over j. Now Minkowski's inequality shows, by virtue of (3.6), that $|\bar{f}(x)|_r \leqslant 2^n \alpha$, and since $|\bar{f}(x)|_r$ is supported in Ω we obtain in a manner similar to the proof of (3.7)

$$\omega\big(\{x \in \mathbf{R}\colon |\bar{f}^*(x)|_r > \alpha\}\big) \leqslant C_r \alpha^{-r} \int_{\mathbf{R}^n} |\bar{f}(x)|_r^r \, \omega(x)\, dx \leqslant C_r \int_{\Omega} \omega(x)\, dx$$

which together with (3.11) yields (3.10) as required. This completes the proof of (3.1) in the case $r \geqslant p$. If $r > p > 1$, (2.2) shows that for $\omega \in A_p$, (3.1) holds with p replaced by p_1 and p_2 where $p_1 < p < p_2 < r$, hence Lemma 2.1 yields (3.2) for $r > p > 1$.

Suppose now that $p > r$ and $\omega \in A_p$. By (2.2) there is an r_0, $1 < r_0 < p$ such that $\omega \in A_q$, $q \geqslant p/r_0$. In particular, (2.1) and (2.3) yield $\omega(x) > 0$ a.e. and $\omega(x)^{1-q'} \in A_{q'}$ so that by (1.5), if $\varphi \geqslant 0$ belongs to the unit ball of $L_\omega^{q'}(\mathbf{R}^n)$, then

$$\int_{\mathbf{R}^n} |(\varphi\omega)^*(x)|^{q'} \omega(x)^{1-q'}\, dx \leqslant C_q \int_{\mathbf{R}^n} |\varphi(x)|^{q'} \omega(x)\, dx = C_q,$$

and hence Lemma 3.1 and Hölder's inequality shows

$$\int_{\mathbf{R}^n} |f^*(x)|_r^r \varphi(x)\, \omega(x)\, dx \leqslant C_r \int_{\mathbf{R}^n} |f(x)|_r^r [(\varphi\omega)^*(x)/\omega(x)]\, \omega(x)\, dx \tag{3.12}$$
$$\leqslant C_{r,q} \Big(\int_{\mathbf{R}^n} |f(x)|_r^{rq} \omega(x)\, dx\Big)^{1/q}.$$

Taking the supremum in (3.12) over such φ then yields (3.2) for $1 < r \leqslant r_0$ upon taking $q = p/r$, and this together with the case $p = r$ proved in (3.4) yields (3.2) for $r_0 < r < p$ by an application of Lemma 2.2. The proof of (b) and with it also (a), is complete.

Finally we prove (c). Observe first that the hypothesis on ω and Hölder's inequality shows

$$\int_E e^{\delta|\psi(x)|} \omega(x)\, dx \leqslant \Big(\int_E e^{\delta q'|\psi(x)|}\, dx\Big)^{1/q'} \Big(\int_E \omega(x)^q\, dx\Big)^{1/q}$$

so that $\|\psi\|_{\exp L_\omega} \leqslant C\|\psi\|_{\exp L}$. Hence

$$\|\varphi\omega\|_{L\ln L} = \sup\Big\{\int_E |\varphi(x)\psi(x)|\, \omega(x)\, dx\colon \|\psi\|_{\exp L} \leqslant 1\Big\} \tag{3.13}$$
$$\leqslant \|\varphi\|_{L_\omega \ln L} \sup\{\|\psi\|_{\exp L_\omega}\colon \|\psi\|_{\exp L} \leqslant 1\} \leqslant C\|\varphi\|_{L_\omega \ln L}.$$

Now if $\varphi(x) \geqslant 0$, $|f(x)|_r$ are supported on E and $|f(x)|_r$ is bounded, Lemma 3.1 yields

$$\int_E |f^*(x)|_r^r \varphi(x)\, \omega(x)\, dx \leqslant C_r \int_E |f(x)|_r^r (\varphi\omega)^*(x)\, dx$$
$$\leqslant C_r \big(\operatorname*{ess\,sup}_{x \in E} |f(x)|_r^r\big) \int_E (\varphi\omega)^*(x)\, dx,$$

and by a well-known result ([15], [17]) this last integral is bounded by $C\|\varphi\omega\|_{L\ln L}$, and hence from (3.13)

$$\int_E |f^*(x)|_r^r \varphi(x)\, \omega(x)\, dx \leqslant C_r \big(\operatorname{ess\,sup} |f(x)|_r^r\big) \|\varphi\|_{L_\omega \ln L}. \tag{3.14}$$

Taking the supremum in (3.14) over φ in the unit ball of $L_\omega \ln L$ we obtain

$$\||f^*(x)|_r^r\|_{\exp L_\omega} \leqslant C_r \big(\operatorname{ess\,sup} |f(x)|_r^r\big)$$

which gives (c). The Theorem is proved.

§ 4. Application to Marcinkiewicz integrals. Following C. Fefferman and E. Stein [7] we can apply Theorem 3.1 to obtain new results for the Marcinkiewicz integral H'_λ corresponding to a disjoint collection of cubes $\{Q_j\}$. If d_j is the diameter of Q_j and y_j its center, then H'_λ is equivalent to S_λ:

$$S_\lambda(x) = \sum_{j=1}^\infty \frac{d_j^{n+\lambda}}{|x - y_j|^{n+\lambda} + d_j^{n+\lambda}},$$

see [16], §§ 4, 5. On the otherhand, if $\lambda \geqslant n(r-1)$ and $f = \{f_j\}$ with f_j the characteristic function of Q_j, then $S_\lambda(x)$ is bounded by a multiple of $|f^*(x)|_r^r$. Thus we have the following theorem.

THEOREM 4.1. *Let $1 < r < \infty$, $\lambda \geqslant n(r-1)$ and suppose E is a set of finite measure with $\bigcup_1^\infty Q_j \subset E$.*

(a) *If $q \geqslant 1/r$ and $\omega \in A_{rq}$, there is a constant $C_{q,r,\lambda}$ such that for all $\alpha > 0$*

$$\omega(\{x \in \mathbf{R}^n:\ S_\lambda(x) > \alpha\}) \leqslant C_{q,r,\lambda}\alpha^{-q}\omega\big(\bigcup_1^\infty Q_j\big).$$

(b) *If $q > 1/r$ and $\omega \in A_{rq}$, there is a constant $C_{q,r,\lambda}$ such that*

$$\int_{\mathbf{R}^n} |S_\lambda(x)|^q \omega(x)\,dx \leqslant C_{q,r,\lambda}\,\omega\big(\bigcup_1^\infty Q_j\big).$$

(c) *If $\omega(x)^q$ is integrable on E for some $q > 1$, in particular if $\omega \in A_\infty$ and E is a cube, then $S_\lambda \in \exp L_\omega(E)$.*

For related results, see A. P. Calderón [3], M. Kaneko and S. Yano [11].

§5. Singular integrals. Let $K(x)$ be a convolution kernel satisfying the conditions

$$|K(x)| \leqslant B|x|^{-n}, \qquad |\hat{K}(x)| \leqslant B \tag{5.1}$$

and

$$|K(x-y) - K(x)| \leqslant \theta(|y|/|x|)\,|x|^{-n} \quad \text{for} \quad |x| \geqslant 2|y| \tag{5.2}$$

where B is a constant and $\theta(t)$ is non-decreasing for $t > 0$, $\theta(2t) \leqslant C\theta(t)$ and satisfies the Dini condition

$$\int_0^1 \frac{\theta(t)}{t}\,dt < \infty. \tag{5.3}$$

In particular, we may take $K(x) = \Omega(x)/|x|^n$ a Calderón–Zygmund kernel, i.e. $\Omega(x)$ homogeneous of degree zero, of mean value zero on the unit sphere in $\mathbf{R}^n$ and satisfying the Dini condition

$$\int_0^1 (\delta(t)/t)\,dt < \infty$$

where

$$\delta(t) = \sup\{|\Omega(x) - \Omega(y)|:\ |x| = |y| = 1,\ |x-y| \leqslant t\}.$$

Define T and T^* by

$$Tf(x) = \text{P.V.}\int_{\mathbf{R}^n} K(x-y)f(y)\,dy,$$

$$T^*f(x) = \sup_{Q_x}\Big\{\Big|\int_{\mathbf{R}^n - Q_x} K(x-y)f(y)\,dy\Big|:\ Q_x \text{ a cube centered at } x\Big\}.$$

M. Kaneko and S. Yano [11] have shown that $\omega \in A_p$ implies the inequalities

$$\int_{\mathbf{R}^n} |T^*f(x)|^p \omega(x)\,dx \leqslant C_p \int_{\mathbf{R}^n} |f(x)|^p \omega(x)\,dx \qquad (1 < p < \infty) \tag{5.4}$$

and

$$\omega\big(\{x \in \mathbf{R}^n:\ T^*f(x) > \alpha\}\big) \leqslant C\alpha^{-1}\int_{\mathbf{R}^n} |f(x)|\,\omega(x)\,dx \qquad (p = 1). \tag{5.5}$$

Earlier, R. Coifman and C. Fefferman [5] proved (5.4) for a more restricted class of kernels, namely when $\theta(t) = Bt$ in (5.2). Of course (5.4) and (5.5) yield the corresponding inequalities for T in place of T^*. Unweighted vector analogues of (5.4) for T have been given by A. Cordoba and C. Fefferman [6], see also A. Benedek, A. Calderón and R. Panzone [1], J. Marcinkiewicz and A. Zygmund [12].

Suppose now that $\{K_k(x)\}$ is a sequence of convolution kernels satisfying (5.1)–(5.3) with a uniform constant B and fixed θ independent of k. If $f = \{f_k\}$, let $Tf = \{T_k f_k\}$, $T^*f = \{T_k^* f_k\}$ where of course T_k and T_k^* are the operators defined above corresponding to the kernel K_k. We shall prove the following vector-valued analogues of (5.4) and (5.5):

THEOREM 5.1. *Let $1 < r < \infty$ and suppose $\omega \in A_1$. There exists a constant C_r such that for all $\alpha > 0$*

$$\omega\big(\{x \in \mathbf{R}^n:\ |T^*f(x)|_r > \alpha\}\big) \leqslant C_r\alpha^{-1}\int_{\mathbf{R}^n} |f(x)|_r\,\omega(x)\,dx. \tag{5.6}$$

THEOREM 5.2. *Let $1 < r < \infty$, $1 < p < \infty$ and suppose $\omega \in A_p$. There is a constant $C_{r,p}$ such that*

$$\int_{\mathbf{R}^n} |T^*f(x)|_r^p\,\omega(x)\,dx \leqslant C_{r,p}\int_{\mathbf{R}^n} |f(x)|_r^p\,\omega(x)\,dx. \tag{5.7}$$

Again (5.6) and (5.7) imply the corresponding results for T in place of T^*, however, it is possible to give an alternate proof of those results in the special case $\theta(t) = Bt$ by following the line of proof used for Theorem 3.1. In the course of such a proof, the Calderón–Zygmund decomposition is used to write $f = g + b$ (as in the proof of Theorem 5.1 below); the contribution of Tg is handled in the same way that f'^* was while that of Tb is estimated as in [15], p. 32 by the Marcinkiewicz integral and an appeal to Theorem 4.1. The required duality relation is provided in A. Cordoba and C. Fefferman [6].

The proof of Theorem 5.2 requires the following result which is a consequence of Theorem 5.1.

LEMMA 5.1. *Let $1 < r < \infty$ and suppose $\omega \in A_\infty$. There are constants C_r, $\delta > 0$ such that*

(5.8) $$\omega(\{x \in \boldsymbol{R}^n:\ |T^* f(x)|_r > 2\alpha,\ |f(\cdot)|_r^*(x) \leqslant \gamma\alpha,\ |f^*(x)|_r \leqslant \gamma\alpha\}) \leqslant C_r \gamma^\delta \omega(\{x \in \boldsymbol{R}^n:\ |T^* f(x)|_r > \alpha\})$$

for all $\alpha > 0$, $\gamma > 0$.

Proof of Theorem 5.1. Suppose $f \in S$ and $\alpha > 0$. As in the proof of Theorem 3.1 the Calderón–Zygmund decomposition yields a sequence of disjoint cubes $\{Q_j\}$ and we write $f = g + b$ where $g = \{g_k\}$,

$$g_k(x) = \begin{cases} f_k(x) & \text{if} \quad x \notin \Omega = \bigcup_1^\infty Q_j, \\ \dfrac{1}{|Q_j|} \int\limits_{Q_j} f_k(y)\,dy & \text{if} \quad x \in Q_j,\ j = 1, 2, \ldots \end{cases}$$

Since $\omega \in A_1$, we have

$$\int\limits_{Q_j} |g(x)|_r \omega(x)\,dx \leqslant \frac{1}{|Q_j|} \int\limits_{Q_j} |f(y)|_r dy \int\limits_{Q_j} \omega(x)\,dx$$
$$\leqslant \left(\frac{1}{|Q_j|} \int\limits_{Q_j} |f(y)|_r \omega(y)\,dy\right)\left(\operatorname*{ess\,sup}_{x \in Q_j} \frac{1}{\omega(x)}\right) \int\limits_{Q_j} \omega(x)\,dx$$
$$\leqslant K \int\limits_{Q_j} |f(y)|_r \omega(y)\,dy$$

so that

(5.9) $$\int\limits_{\boldsymbol{R}^n} |g(x)|_r \omega(x)\,dx \leqslant C \int\limits_{\boldsymbol{R}^n} |f(x)|_r \omega(x)\,dx.$$

Thus, it suffices to prove

(5.10) $$\omega(\{x \in \boldsymbol{R}^n:\ |T^* g(x)|_r > \alpha\}) \leqslant C_r \alpha^{-1} \int\limits_{\boldsymbol{R}^n} |g(x)|_r \omega(x)\,dx$$

and

(5.11) $$\omega(\{x \in \boldsymbol{R}^n:\ |T^* b(x)|_r > \alpha\}) \leqslant C_r \alpha^{-1} \int\limits_{\boldsymbol{R}^n} |f(x)|_r \omega(x)\,dx.$$

Since $\omega \in A_r$ by (2.2), the Chebyshev inequality, (5.4) and the fact that $|g(x)|_r \leqslant 2^n \alpha$ yields (5.10) immediately. Let Q_j^* be the cube concentric with Q_j with diameter $3\sqrt{n}$ times as large. The same proof as that given in [15], pp. 43–44, shows that for $x \notin \Omega^* = \bigcup_1^\infty Q_j^*$

$$T_k^* b_k(x) \leqslant \sum_{j=1}^\infty \int\limits_{Q_j} |K_k(x-y) - K_k(x-y_j)|\,|b_k(y)|\,dy + C b_k^*(x)$$

where y_j is the center of Q_j. Now for such x and $y \in Q_j$ we have $|x-y| \geqslant 2|y-y_j|$ so that the hypothesis (5.2) followed by Minkowski's inequality shows that

(5.12) $$|T^* b(x)|_r \leqslant \sum_{j=1}^\infty \int\limits_{Q_j} \theta\left(\frac{|y-y_j|}{|x-y|}\right) \frac{|b(y)|_r}{|x-y|^n}\,dy + C|b^*(x)|_r$$

for $x \notin \Omega^*$. Denoting the sum of terms on the right of (5.12) by $\Sigma(x)$, is suffices to show

(5.13) $$\omega(\{x \in R^n - \Omega^*:\ \Sigma(x) > \alpha\}) \leqslant C_r \alpha^{-1} \int\limits_{\boldsymbol{R}^n} |b(x)|_r \omega(x)\,dx$$

in view of Theorem 3.1 and the estimate, as in (3.9),

$$\omega(\Omega^*) \leqslant C\omega(\Omega) \leqslant C\alpha^{-1} \int\limits_{\boldsymbol{R}^n} |f(x)|_r \omega(x)\,dx.$$

Now if $d_j =$ diameter (Q_j), then

$$\int\limits_{\boldsymbol{R}^n - \Omega^*} \Sigma(x)\omega(x)\,dx \leqslant \sum_{j=1}^\infty \int\limits_{Q_j} |b(y)|_r dy \int\limits_{\boldsymbol{R}^n - Q_j^*} \theta\left(\frac{|y-y_j|}{|x-y|}\right) \frac{\omega(x)}{|x-y|^n}\,dx$$
$$\leqslant \sum_{j=1}^\infty \int\limits_{Q_j} |b(y)|_r dy \int\limits_{|x-y| > d_j} \theta\left(\frac{d_j}{|x-y|}\right) \frac{\omega(x)}{|x-y|^n}\,dx$$

and Theorem 2 of [15], pp. 62–63, shows that the inner integral is bounded by $C\omega^*(y)$, and since $\omega \in A_1$ implies $\omega^*(y) \leqslant C\omega(y)$ a.e. we have

$$\int\limits_{\boldsymbol{R}^n - \Omega^*} \Sigma(x)\omega(x)\,dx \leqslant C \int\limits_{\Omega} |b(y)|_r \omega(y)\,dy$$

which yields, by the Chebyshev inequality, (5.13) as required. The proof of Theorem 5.1 is complete.

Proof of Lemma 5.1. We follow as closely as possible the proofs given in [5] and [11] for the scalar valued case. By the Whitney Lemma [15], p. 16, the open set $\Omega = \{x \in \boldsymbol{R}^n:\ |T^* f(x)|_r > \alpha\}$ is the union of non-overlapping cubes $|Q_j|$ with the property that the distance from Q_j to $\boldsymbol{R}^n - \Omega$ is comparable to $d_j =$ diameter (Q_j). Thus there are points $x_j \in \boldsymbol{R}^n - \Omega$ such that the distance from x_j to Q_j is less than $4d_j$. Let $\bar{Q}_j$ be the cube concentric with Q_j but of diameter say, $(21\sqrt{n})d_j$. Note that $Q_j \subset \bar{Q}_j$.

The main step in the proof is the inequality

(5.14) $$|\{x \in Q_j:\ |T^* f(x)|_r > 2\alpha,\ |f(\cdot)|^*(x) \leqslant \gamma\alpha,\ |f^*(x)|_r \leqslant \gamma\alpha\}| \leqslant C_r \gamma |Q_j|$$

for then (5.8) follows by applying the definition of $\omega \in A_\infty$ and summing over j.

To prove (5.14) we may assume that there are points ξ_j and η_j in Q_j such that $|f(\cdot)|_r^*(\xi_j) \leqslant \gamma\alpha$, $|f^*(\eta_j)|_r \leqslant \gamma\alpha$ and also that γ is small, otherwise the inequality is trivial. Write $f = u+v$ where $u = \{u_k\}$, $u_k(x) = f_k(x)\chi_{\bar{Q}_j}(x)$. Since $\xi_j \in Q_j \subset \bar{Q}_j$ we have

$$\frac{1}{|\bar{Q}_j|}\int_{\mathbf{R}^n} |u(x)|_r dx = \frac{1}{|\bar{Q}_j|}\int_{\bar{Q}_j} |f(x)|_r dx \leqslant |f(\cdot)|_r^*(\xi_j) \leqslant \gamma\alpha$$

and since the weight function $\omega(x) \equiv 1$ satisfies A_1, Theorem 5.1 shows that

$$(5.15) \qquad |\{x \in \mathbf{R}^n\colon\ |T^* u(x)|_r > \alpha/2\}| \leqslant C_r \gamma |Q_j|.$$

Now it is shown in [11], pp. 579–580, that for $x \in Q_j$ we have

$$T_k^* v_k(x) \leqslant T_k^* f_k(x_j) + C f_k^*(\eta_j)$$

and hence also

$$|T^* v(x)|_r \leqslant |T^* f(x_j)|_r + C|f^*(\eta_j)|_r \leqslant \alpha + C\gamma\alpha$$

since $x_j \notin \Omega$. Hence (5.15) yields

$$|\{x \in Q_j\colon\ |T^* f(x)|_r > \alpha/2 + \alpha + C\gamma\alpha\}| \leqslant C_r \gamma |Q_j|$$

which implies (5.14) for small γ as required. The lemma is proved.

Proof of Theorem 5.2. It is sufficient to prove the inequality for $f = \{f_k\}$ with $f_k = 0$ for all sufficiently large k, say $k > N$, for then the general case follows by the monotone convergence theorem. Since $\omega \in A_p$ implies $\omega \in A_\infty$, Lemma 5.1 shows that

$$\omega\left(\{x \in \mathbf{R}^n\colon\ |T^* f(x)|_r > \alpha\}\right) \leqslant C_r \gamma^\delta \omega\left(\{x \in \mathbf{R}^n\colon\ |T^* f(x)|_r > \alpha/2\}\right) + \\ + \omega\left(\{x \in \mathbf{R}^n\colon\ |f(\cdot)|_r^*(x) > \gamma\alpha\}\right) + \omega\left(\{x \in \mathbf{R}^n\colon\ |f^*(x)|_r > \gamma\alpha\}\right).$$

Multiplying this by $p\alpha^{p-1}$ and integrating over $\alpha \in (0, \infty)$ yields

$$\int_{\mathbf{R}^n} |T^* f(x)|_r^p \omega(x)\,dx \leqslant C_{r,p}\gamma^\delta \int_{\mathbf{R}^n} |T^* f(x)|_r^p \omega(x)\,dx + \\ + C_{\gamma,p}\int_{\mathbf{R}^n} \big||f(\cdot)|_r^*(x)\big|^p \omega(x)\,dx + C_{\gamma,p}\int_{\mathbf{R}^n} |f^*(x)|_r^p \omega(x)\,dx.$$

By our assumption on f and (5.4) we see that

$$\int_{\mathbf{R}^n} |T^* f(x)|_r^p \omega(x)\,dx \leqslant C_p \Big(\sum_{k=1}^{N}\Big(\int_{\mathbf{R}^n} |f_k(x)|^p \omega(x)\,dx\Big)^{1/p}\Big)^p < \infty,$$

so that upon choosing γ such that $C_{r,p}\gamma^\sigma \leqslant 1/2$ we obtain the desired inequality from (1.5) and Theorem 3.1. The proof is complete.

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