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On inverse-closed algebras of infinitely differentiable functions

by

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Abstract. In this paper we are concerned with algebras E_M of infinitely differentiable functions with growth restrictions on their derivatives. These are similar to the classical Denjoy–Carleman classes. The main aim of the paper is to give a characterization of those E_M which are inverse closed. This question had been considered by Rudin for the Denjoy–Carleman classes and a similar result has been obtained by him for the non-quasianalytic case. The proof goes through a description of the character spectrum and a characterization of local m-convexity. Also, a problem considered by Ehrenpreis about local images is partially solved.

1. Introduction and background.

NOTATIONS. The letter I stands for a closed interval of the real line and $I_L = [-L, L]$. $E(I)$ is the vector space of all $\boldsymbol{C}$-valued, C^∞-functions defined in I; we write E for $E(\boldsymbol{R})$. $\|g\|_I$ means $\sup\{|g(x)|: x \in I\}$ and when $I = I_L$ we write $\|g\|_L$ instead of $\|g\|_I$. H will denote the vector space of all entire functions with its usual topology. We suppose that $M = (M_n)$ is a sequence of positive real numbers so that $M_n = \exp g(n)$ where $g: [0, \infty] \to \boldsymbol{R}^+$ is convex, $g(0) = 0$ and $t^{-1}g(t)$ tends to ∞ as t tends to ∞ ([5]).

1.1. DEFINITION. For an interval I, $E_M(I)$ is defined as

$$E_M(I) = \{f \in E(I) \mid \forall \varepsilon > 0\ \exists C(\varepsilon) > 0 \text{ s.t. } \|f^{(n)}\|_I \leqslant C(\varepsilon)\varepsilon^n M_n\}$$

and E_M as the projective limit of these: $E_M = \varprojlim E_M(I)$.

In order that these spaces be closed under derivation we will assume as well that there exist constants $A, B > 0$ such that

$$M_{n+1} \leqslant AB^n M_n \quad \forall n. \tag{1.1}$$

The topology of $E_M(I)$ is defined by the norms

$$P_{I,\varepsilon}(f) = \sup_n \frac{\|f^{(n)}\|_I}{\varepsilon^n M_n}, \quad \varepsilon > 0$$

(or, what is the same, by the norms $q_{I,\varepsilon}(f) = \sum_n (\|f^{(n)}\|_I/\varepsilon^n M_n))$. E_M is given the projective topology, i.e. the one defined by the system of norms $(P_{I,\varepsilon})_{I,\varepsilon}$. When $I = I_L$, we denote $P_{I,\varepsilon}$ by $P_{L,\varepsilon}$.

It is clear that $E_M(I)$ and E_M are Fréchet spaces; also (1.1) implies for $\varepsilon \leqslant B$

$$P_{I,\varepsilon}(f') \leqslant AP_{I,B^{-1}\varepsilon}(f),$$

i.e. derivation is a continuous operation in $E_M(I)$, E_M.

All definitions are motivated by the fact that when $M_n = n!$, the space E_M (and also all the $E_M(I)$) is the space H of all entire functions. We remind, though we will not use it explicitly, that E_M is not quasianalytic, i.e. contains a function with compact support, if and only if $\sum (M_n/M_{n+1}) < \infty$ ([1], [3]).

Convexity of g implies $M_k M_{n-k} \leqslant M_n$ for $0 \leqslant k \leqslant n$. Then, if $f, g \in E_M(I)$, the inequalities

$$|(fg)^{(n)}(x)| \leqslant \sum_{k=0}^{n} \binom{n}{k} |f^{(k)}(x)|\,|g^{(n-k)}(x)|$$
$$\leqslant P_{I,\varepsilon}(f)P_{I,\varepsilon}(g)\varepsilon^n \sum_{k=0}^{n} \binom{n}{k} M_k M_{n-k} \leqslant P_{I,\varepsilon}(f)P_{I,\varepsilon}(g)(2\varepsilon)^n M_n$$

prove that

$$P_{I,2\varepsilon}(fg) \leqslant P_{I,\varepsilon}(f)P_{I,\varepsilon}(g)$$

and so $E_M(I)$, E_M are *Fréchet algebras* under pointwise multiplication. In this context, the results we prove about the algebra E_M are the following:

THEOREM A. $\operatorname{Spec} E_M = \mathbf{R}$, $\operatorname{Spec} E_M(I) = I$ *if and only if the sequence* $A_n = (M_n/n!)^{1/n}$ *is not bounded above. Otherwise,* $\operatorname{Spec} E_M(I) = \operatorname{Spec} E_M = \mathbf{C}$.

THEOREM B. E_M, $E_M(I)$ *are locally m-convex algebras if and only if the sequence* (A_n) *is almost increasing, i.e. there exists* $K > 0$ *such that* $A_n \leqslant KA_m$ *for* $n \leqslant m$.

THEOREM C. *The algebra* E_M *is inverse closed, that is,* $f \in E_M$ *and* $f(x) \neq 0\ \forall x$ *imply* $f^{-1} \in E_M$ *if and only if the sequence* (A_n) *is not bounded and almost increasing.*

Now some background. Let E'_M stand for the dual space to E_M; the following definitions are then standard:

1.2. DEFINITION. For $z \in C$ and $T \in E'_M$, let

$$\hat{T}(z) = T(\exp ixz).$$

The function $z \mapsto \hat{T}(z)$ is called the *Fourier transform* of T.

The function $e_z(x) = \exp(ixz)$ belongs to E_M so that the definition of $\hat{T}$ makes sense. As ω aproaches z, $(\omega - z)^{-1}(e_\omega - e_z)$ aproaches in E_M the function $x \mapsto ixe_z(x)$; this is to say that $\hat{T}$ is an entire function and $\hat{T}'(z) = T(ixe_z(x))$.

1.3. DEFINITION. For $z \in C$, let

$$\lambda_M(z) = \sup_n \frac{|z|^n}{M_n}.$$

1.4. DEFINITION. Let $H(M)$ denote the vector space of all entire functions F such that

$$|F(z)| \leqslant A\lambda_M\left(\frac{|z|}{\varepsilon}\right)\exp(L|\operatorname{Im} z|)$$

for some $A, \varepsilon, L > 0$. The family $H(M, L, \varepsilon)$ of those F satisfying this inequality with fixed ε, L is a Banach space. $H(M)$ is the union of these Banach spaces and may therefore be given a topology as the inductive limit of these spaces.

Theorem 2.8 of [5] says in our case the following

1.5. THEOREM. *The Fourier transform* $T \mapsto \hat{T}$ *is a topological isomorphism between the strong dual of* E_M *and* $H(M)$.

Looking carefully at the proof of Theorem 2.8 of [5] one finds that the same result is true for the space $E_M(I)$ (compare with Theorem 13.13 of [2]). If $H(M, L) = \varinjlim_{\varepsilon} H(M, L, \varepsilon)$, we have

1.6. THEOREM. *The Fourier transform* $T \mapsto \hat{T}$ *is a topological isomorphism between the strong dual of* $E_M(I_L)$ *and* $H(M, L)$.

Remark. Observe that the fact that $T \mapsto \hat{T}$ is one to one implies in particular that the e_z form a total set.

2. The problem of equivalent classes. It is clear that if there exist constants $A, B > 0$ such that

$$M_n \leqslant AB^n N_n, \tag{2.1}$$

then $E_M(I) \subset E_N(I)$ and $E_M \subset E_N$. We are going to prove that (2.1) is also a necessary condition for the relation $E_M \subset E_N$ to hold.

2.1. LEMMA. *The relation* (2.1) *is equivalent to*

$$\lambda_N(t) \leqslant A\lambda_M(Bt), \quad t > 0. \tag{2.2}$$

Proof. That (2.1) implies (2.2) is trivial. For the converse, we remind ([1]) that the sequence M can be reobtained from λ_M by means of the formula

$$M_n = \sup \frac{t^n}{\lambda_M(t)}. \tag{2.3}$$ ■

2.2. THEOREM. $E_M(I) \subset E_N(I)$ *and* $E_M \subset E_N$ *if and only if* (2.1)–(2.2) *hold.*

Proof. If $E_M \subset E_N$, the inclusion map $E_M \to E_N$ is continuous, by the closed graph theorem because convergence in the spaces E_M imply punctual convergence. In particular, given the norm $P_{I,1}$ of E_M there exist $A > 0$ and J, B so that

$$P_{I,1}(f) < AP_{J,B}(f), \quad f \in E_M. \tag{2.4}$$

If we write (2.4) for $f = e_t$, where $e_t(x) = \exp(ixt)$, $t > 0$, we find (2.2). ■

2.3. COROLLARY. *$E_M = E_N$ if and only if $(M_n/N_n)^{1/n}$ remains bounded by positive numbers a, b:*

$$a < \left(\frac{M_n}{N_n}\right)^{1/n} < b. \ ■$$

3. Proof of Theorem A. We will use Theorems 1.5 and 1.6 to find the character spectrum of $E_M(I)$, E_M. First of all, we must express, in terms of $\hat{T}$, that T is a character. As the e'_z form a total set, T is a character iff $T(e_z e'_z) = T(e_z)T(e'_z)$, i.e. iff $\hat{T}(z+z') = \hat{T}(z)\hat{T}(z')$. Now, an entire function F satisfying $F(z+z') = F(z)F(z')$ is of the form $F(z) = \exp(i\omega z)$ for some $\omega \in \boldsymbol{C}$.

Thus, to find $\operatorname{Spec} E_M$, we have to look for ω such that the function $e_\omega(z) = \exp(iz\omega)$ belongs to $H(M)$, i.e.,

$$|\exp(iz\omega)| \leqslant A\lambda_M\left(\frac{|z|}{\varepsilon}\right)\exp(L|\operatorname{Im} z|), \quad z \in \boldsymbol{C}, \tag{3.1}$$

for some A, ε, L. Now, (3.1) with $z = t$ and $\omega = a - bi$, give

$$\exp(bt) \leqslant A\lambda_M\left(\frac{|t|}{\varepsilon}\right), \quad t \in \boldsymbol{R},$$

or, what is the same

$$\exp(|b|t) \leqslant A\lambda_M\left(\frac{t}{\varepsilon}\right), \quad t > 0. \tag{3.2}$$

Also, (3.1) may be obtained from (3.2) for

$$|\exp(iwz)| = |\exp(-a\operatorname{Im} z)\exp(bz)|$$
$$\leqslant \exp|a||\operatorname{Im} z|\exp|b||z| \leqslant A\lambda_M\left(\frac{|z|}{\varepsilon}\right)\exp|a||\operatorname{Im} z|.$$

So, (3.1) and (3.2) are equivalent, and thus, for $\omega = a - bi$, $e_\omega \in H(M)$ if and only if (3.2) holds for some A, ε. But, if b satisfies (3.2) then any other satisfies it. Therefore, Spec E_M is $\boldsymbol{R}$ or $\boldsymbol{C}$ and $\operatorname{Spec} E_M = \boldsymbol{C}$ if and only if (3.2) holds for some A, ε and $b = 1$. If $N_n = n!$, then

$$\lambda_N(t) = \sup_n \frac{t^n}{n!} \leqslant e^t = \sum \frac{t^n}{n!} = \sum \frac{1}{2^n}\frac{(2t)^n}{n!} \leqslant 2\lambda_N(2t).$$

Hence (3.2) is equivalent to

$$\lambda_N(t) \leqslant A\lambda_M\left(\frac{t}{\varepsilon}\right), \quad t > 0,$$

which in turn is equivalent, by Lemma 2.1 to

$$M_n \leqslant A\varepsilon^{-n}n! \tag{3.3}$$

and to the inclusion $E_M \subset H$. Thus we have proved the following

3.1. THEOREM. *$\operatorname{Spec} E_M$ is $\boldsymbol{R}$ or $\boldsymbol{C}$. $\operatorname{Spec} E_M = \boldsymbol{C}$ if and only if $A_n = (M_n/n!)^{1/n}$ is bounded above, or equivalently, $E_M \subset H$.* ■

Since E_M contains x, it is clear that the Gelfand topology in $\operatorname{Spec} E_M$ is the usual one. In case $\operatorname{Spec} E_M = \boldsymbol{C}$, it is also clear how $\omega \in \boldsymbol{C}$ acts a character: every $f \in E_M$ extends uniquely to an entire function, its Gelfand transform, which we continue to denote by f, and $\omega(f) = f(\omega)$.

We turn now to the problem of finding $\operatorname{Spec} E_M(I_L)$. The discussion is similar. We look for ω such that (3.1) holds with L fixed; if $\omega \notin \boldsymbol{R}$ satisfies it, we obtain (3.3) as before, $E_M(I) \subset E_N(I) = H$ $(N_n = n!)$ and $\operatorname{Spec} E_M(I) = \boldsymbol{C}$. If $\omega = a \in \boldsymbol{R}$ satisfies it and $|a| > L$, we write it for $z = -it$ and find

$$\exp(at) \leqslant A\lambda_M(|t|/\varepsilon)\exp(L|t|), \quad t \in \boldsymbol{R},$$

or

$$\exp(|a|t) \leqslant A\lambda_M(t/\varepsilon)\exp(Lt), \quad t > 0,$$

or

$$\exp\big((|a| - L)t\big) \leqslant A\lambda_M(t/\varepsilon)$$

and we continue as before. Hence we have

3.2 THEOREM. *$\operatorname{Spec} E_M(I)$ is I or $\boldsymbol{C}$. $\operatorname{Spec} E_M(I) = \boldsymbol{C}$ if and only if $A_n = (M_n/n!)^{1/n}$ is bounded above, or equivalently, $E_M(I) \subset H$.* ■

Collecting 3.1 and 3.2, we have

3.3. THEOREM. *If A_n is bounded above, then $E_M(I)$ and E_M are included in H and $\operatorname{Spec} E_M(I) = \operatorname{Spec} E_M = \boldsymbol{C}$. Otherwise,*

$$\operatorname{Spec} E_M(I) = I \quad \textit{and} \quad \operatorname{Spec} E_M = \boldsymbol{R}. \ ■$$

Ehrenpreis [2] considers the problem of characterizing when the restriction map

$$r\colon E_M \mapsto E_M(I)$$

is onto. Here we are able to give a partial result (see also [1], [2]):

3.4. PROPOSITION. *If A_n is not bounded and*

$$\sum \frac{M_n}{M_{n-1}} = \infty \tag{3.4}$$

(i.e. E_M is quasianalytic but not analytic), then r is not onto.

Proof. Note that (3.4) says that $E_M, E_M(I)$ are quasianalytic, and so r is one to one. The fact that A_n is not bounded yields $\operatorname{Spec} E_M(I) = I$ and $\operatorname{Spec} E_M = \boldsymbol{R}$. If r were onto, it would be a topological isomorphism and we would have $\operatorname{Spec} E_M(I) = \boldsymbol{R}$, which is contradictory. ■

4. Proof of Theorem B. We are going to give a characterization of those sequences $M = (M_n)$ such that $E_M(I)$, E_M are locally m-convex algebras ([3], [6]).

4.1. THEOREM. *The following statements are equivalent:*

(a) *$E_M(I)$ is locally m-convex, $\forall I$.*

(b) *E_M is locally m-convex.*

(c) *There exists constants $A, B, K > 0$ such that*

$$\lambda_M(mt) \leqslant AB^m \lambda_M(Kt)^m, \quad t > 0, \ m \in N. \tag{4.1}$$

(d) *The sequence $B_n = M_n^{1/n}/n$ is almost increasing.*

(e) *The sequence A_n is almost increasing.*

(f) *If $f \in E_M(I)$ and Φ is entire, $\Phi \circ f \in E_M(I)$.*

Proof. (a) $\Rightarrow$ (b) is clear because $E_M = \varprojlim E_M(I)$.

(b) $\Rightarrow$ (c): Let (q_n) be a system of seminorms defining the topology of E_M and such that $q_n(fg) \leqslant q_n(f) q_n(g)$ $\forall f, g \in E_M$. Given $P_{L,\varepsilon}$, there exist $n, A > 0$ such that

$$P_{L,\varepsilon}(f) \leqslant A q_n(f), \quad f \in E_M.$$

For that n, there exist $B, P_{R,\delta}$ such that

$$q_n(f) \leqslant B P_{R,\delta}(f), \quad f \in E_M.$$

Then

$$P_{L,\varepsilon}(f_1 \dots f_m)$$
$$\leqslant A q_n(f_1 \dots f_m) \leqslant A q_n(f_1) \dots q_n(f_m) \leqslant AB^m P_{R,\delta}(f_1) \dots P_{R,\delta}(f_m).$$

In particular,

$$P_{L,\varepsilon}(f^m) \leqslant AB^m P_{R,\delta}(f)^m.$$

For $f = e_t$ and $\varepsilon = 1$, this gives

$$\lambda_M(mt) \leqslant AB^m \lambda_M(t/\delta)^m$$

which is (c).

(c) $\Rightarrow$ (d): By (2.3) we have

$$B_n = \frac{1}{n} \sup_{t>0} \frac{1}{\lambda_M(t)^{1/n}}, \quad n \in N. \tag{4.2}$$

Fix m and take $n > m$; suppose first that $m|n$, i.e., $n = ms$. Since $t = st/s$, we have, by (4.1) assuming $B > 1$, and using $n \geqslant s$,

$$\lambda_M(t) \leqslant AB^s \lambda_M\left(K\frac{t}{s}\right)^s \leqslant AB^n \lambda_M\left(K\frac{t}{s}\right)^s;$$

taking nth roots we find

$$\lambda_M(t)^{1/n} \leqslant A^{1/n} B \lambda_M\left(K\frac{t}{s}\right)^{1/m} \leqslant CB \lambda_M\left(K\frac{t}{s}\right)^{1/m}.$$

Now

$$B_n = \frac{1}{ms} \sup_{t>0} \frac{t}{\lambda_M(t)^{1/n}} \geqslant \frac{1}{ms} \sup_{t>0} \frac{t}{CB\lambda_M(Kt/s)^{1/m}}$$
$$= \frac{1}{mCBK} \sup_{t>0} \frac{Kt/s}{\lambda_M(Kt/s)^{1/m}} = \frac{1}{CBK} B_m.$$

Hence, $B_m \leqslant CBKB_n$ if $m|n$. In the general case, we put $sm < n < (s+1)m$; formula (4.2) shows that $nB_n = M_n^{1/n}$ is increasing. Then

$$B_n \geqslant B_{ms} \frac{ms}{n} \geqslant \frac{1}{CBK} B_m \frac{ms}{m(s+1)} \geqslant \frac{B_m}{2CBK}$$

and $B_m \leqslant 2CBKB_n$ for $m < n$.

(d) $\Rightarrow$ (e): It is sufficient to observe that $B_n/A_n = (n!)^{1/n}/n$ has finite limit different from zero (by Stirling's formula) and so it remains bounded above and below by positive numbers.

(e) $\Rightarrow$ (f): We start from the formula of Faa di Bruno ([1]) about the derivatives of a composition.

$$(\Phi \circ f)^{(n)}(x) = \sum_{\nu} k_\nu \Phi^{(\mu)}(f(x)) f^{(1)}(x)^{\nu_1} \dots f^{(\nu)}(x)^{\nu_r}. \tag{4.3}$$

Here, ν runs over all the r-tuples $\nu = (\nu_1, \dots, \nu_r)$, ν_i, $r \in N$, such that $\nu_1 + 2\nu_2 + \dots + r\nu_r = n$ and μ is defined as $\nu_1 + \dots + \nu_r$. k_ν are constants depending just on ν.

Suppose $f \in E_M(I)$ and $\Phi \in H$. Fix $\varepsilon > 0$. We have

$$|f^{(n)}(x)| \leqslant P_{I,\varepsilon}(f) \varepsilon^n M_n, \quad x \in I, \ n \in N. \tag{4.4}$$

Since $f(I)$ is compact, to every $\delta > 0$ there corresponds $C(\delta) > 0$ such that

$$\left|\Phi^{(\mu)}(f(x))\right| \leqslant C(\delta) \delta^{(\mu)} \mu!, \quad x \in I. \tag{4.5}$$

Putting (4.4) and (4.5) into (4.3) and changing M_n by $A_n^n n!$,

$$|(\Phi \circ f)^{(n)}(x)| \leqslant \sum_{\nu} k_\nu C(\delta) \delta^\mu \mu! P_{I,\varepsilon}^{\nu_1}(f) \varepsilon^{\nu_1} M^{\nu_1} \dots P_{I,\varepsilon}^{\nu_r}(f) \varepsilon^{r\nu_r} M_r^{\nu_r}$$
$$= C(\delta) \varepsilon^n \sum_{\nu} k_\nu (\delta P_{I,\varepsilon}(f))^\mu \mu! (A_1^1 1!)^{\nu_1} \dots (A_r^r r!)^{\nu_r}.$$

Now we use the hypothesis $A_m \leqslant KA_n$ for $m \leqslant n$ and choose $\delta = P_{I,\varepsilon}(f)^{-1}$ obtaining

$$|(\Phi\circ f)^{(n)}(x)| \leqslant C(\varepsilon K)^n A_n^n \sum_\nu k_\nu \mu!\,(1!)^{\nu_1}\ldots(r!)^{\nu_r}, \qquad x\in I,\ n\in N.$$

We are going to see how

$$\sum_\nu k_\nu \mu!\,(1!)^{\nu_1}\ldots(r!)^{\nu_r} \tag{4.6}$$

increases with n ([1]). Specializing (4.3) to $f(x) = x/(1-x)$, $\Phi = f$ and $x = 0$, we obtain that (4.6) equals $2^{n-1}n!$. Then

$$|(\Phi\circ f)^{(n)}(x)| \leqslant C(\varepsilon K)^n \frac{M_n}{n!}\,2^{n-1}n! = \frac{C}{2}(2\varepsilon K)^n M_n$$

for $x\in I$, $n\in M$, i.e., $\Phi\circ f\in E_M(I)$.

(f) $\Rightarrow$ (a): By Theorem 13.8 of Żelazko [6], a Fréchet algebra A is locally m-convex if and only if for every $a\in A$ and every entire function $\Phi(z) = \sum_{n\geqslant 0} c_n z^n$, the series $\sum_{n\geqslant 0} c_n a^n$ converges in A to an element of A, say $\Phi(a)$. In our case, given $f\in E_M(I)$, the mapping

$$H\to E_M(I)$$
$$\Phi\mapsto\Phi\circ f$$

is linear and has a closed graph, for if $\Phi_n\to\Phi$ in H and $\Phi\circ f\to g$ in $E_M(I)$, then $g = \Phi\circ f$. Hence it is continuous and convergence of $\sum_{n\geqslant 0} c_n z^n$ towards Φ is mapped into convergence of $\sum_{n\geqslant 0} c_n f^n$ towards $\Phi\circ f$.

Thus the proof of Theorem 4.1 is completed. ■

5. Proof of Theorem C. If E_M is locally m-convex and Spec $E_M = \boldsymbol{R}$, by the general theory of locally m-convex algebras ([3], [6]), E_M is inverse closed. Here we will prove that the converse is also true.

5.1. Proposition. *If E_M is inverse closed, then* Spec$E_M = \boldsymbol{R}$.

Proof. The proof is standard: take $\chi\in\operatorname{Spec}E_M$ and define $z_0 = \chi(x)$. If z_0 were not real, we would have $x-z_0$ invertible whereas $\chi(x-z_0) = 0$, which is contradictory. So z_0 is real. We claim that $\chi(f) = f(z_0)$. We put

$$f(x)-f(z_0) = (x-z_0)\int_0^1 f'\big(z_0+t(x-z_0)\big)\,dt.$$

The function $g(x) = \int_0^1 f'(z_0+tx-tz_0)\,dt$ belongs to E_M because

$$g^{(n)}(x) = \int_0^1 f^{(n+1)}(z_0+tx-tz_0)\,t^n\,dt,$$

$$\|g^{(n)}\|_L \leqslant \|f^{(n+1)}\|_{L+|z} = \|(f')^{(n)}\|_{L+|z_0|} \quad\text{and}\quad f'\in E_M.$$

Applying χ to $f-f(z_0) = (x-z_0)\cdot g$, one finds $\chi(f) = f(z_0)$, which establishes the claim. ■

5.2. Proposition. *If E_M is inverse closed, then E_M is locally m-convex.*

Proof. We will prove that (4.1) holds for some $A, B, K > 0$. We consider the subalgebra BE_M of E_M consisting of the bounded functions of E_M ([3]). We endow BE_M with the topology defined by the $P_{I,\varepsilon}$ and the single norm $\|f\|_B = \sup\{(f(x)),\ x\in\boldsymbol{R}\}$. It is routine to check that BE_M is a Fréchet algebra. Now, the fact that E_M is inverse closed means that the invertible functions of BE_M are exactly the ones bounded below. But, if f is bounded below, i.e., $|f(x)|\geqslant m>0$, $x\in\boldsymbol{R}$, and $\|f-g\|_B < m/2$, then $|g(x)|\geqslant m/2 > 0$ and g is invertible. Thus the set of invertible elements of BE_M is open and, following Theorem 13.17 of Żelazko [6], BE_M is m-convex. Following the same argument as in (b) $\Rightarrow$ (c) of Theorem 4.1, we conclude that for each L, ε there exist a seminorm q (which is one of the $P_{R,\delta}$ or $\|\ \|_B$) such that

$$P_{L,\varepsilon}(f^m) \leqslant AB^m q(f)^m. \tag{5.1}$$

But for $f = e_t$, $q(f) = 1 \leqslant P_{R,\delta}(e_t)$ and so, when applying (5.1) to $f = e^t$ we way suppose that $q = P_{R,\delta}$. Therefore

$$P_{L,\varepsilon}(e_t^m) \leqslant AB^m P_{R,\delta}(e_t)^m$$

which is (4.1). ■

5.3. Theorem. *E_M is inverse closed if and only if* Spec$E_M = \boldsymbol{R}$ *and E_M is locally m-convex, or what is the same, if and only if A_n is almost increasing and not bounded above.* ■

Remark. The same is true for $E_M(I)$.

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Invariant measures on the shift space

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Abstract. In this paper we investigate invariant measures on the space of sequences from a finite set S. Let p be an invariant measure on $X = \prod_{-\infty}^{+\infty} S$ and let p_n be the joint distributions of p for $n = 1, 2, \ldots$ If p runs over all invariant measures on X, then the points p_n form a polygon K_n. We decsribe the set of all extremal points of K_n and we give a decomposition of Bernoulli measures by extremal points of K_n. Next, we study a class $\mathscr{M}_0$ of those measures which may be described by extremal points used in a decomposition of the Bernoulli measures. Further, we construct a complete system of invariants of the dynamical systems induced by the measures belonging to $\mathscr{M}_0$.

1. Notations and definitions. Let $S = \{0, 1, \ldots, s-1\}$, $s \geqslant 2$, be a finite alphabet and let $X = \prod_{-\infty}^{+\infty} S$. If $x = \{\ldots, x_{-1}, x_0, x_1, \ldots\}$ is a point of X, then we define $T(x)_i = x_{i+1}$, $i = 0, \pm 1, \pm 2, \ldots$, that is, T shifts every sequence. Let $\mathscr{B}$ be a σ-field of borelian subsets of X. A Borel probability measure p on $\mathscr{B}$ is called *T-invariant* (or shortly *invariant*) if $p(T^{-1}A) = p(A)$, for any $A \in \mathscr{B}$. For $n \geqslant 1$ we put $X_n = \prod_{0}^{n-1} S$. An element $B = (i_0 i_1 \ldots i_{n-1})$ of X_n will be called a *block*. We shall identify B with the cylinder $\{x \in X;\ x_0 = i_0, x_1 = i_1, \ldots, x_{n-1} = i_{n-1}\}$. Let us denote by $M(X)$ the set of all T-invariant measures on $\mathscr{B}$. For a given $p \in M(X)$ we define a measure p_n on X_n as $p_n(B) = p(B)$, $B \in X_n$, $n \geqslant 1$. The measure p_n may be considered as a point of the space R^{s^n} in the sense that the coordinates of p_n are indexed by the blocks $B \in X_n$, and the Bth coordinate of p_n is equal to $p_n(B)$. Fix $n \geqslant 1$ and denote by K_n the set of all vectors of the form $\langle p_n(B)\rangle_{B \in X_n}$, where p runs over all invariant measures on X. It is well known that the set K_n may be described by the following conditions:

(a) $$\sum_{B \in X_n} p_n(B) = 1,$$

(b) $$\sum_{i=0}^{s-1} p_n(Ci) = \sum_{i=0}^{s-1} p_n(iC), \quad \text{for every } C \in X_{n-1},$$

(c) $$p_n(B) \geqslant 0, \quad B \in X_n.$$