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Invariant measures on the shift space

by

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Abstract. In this paper we investigate invariant measures on the space of sequences from a finite set S. Let p be an invariant measure on $X=\prod_{-\infty}^{+\infty} S$ and let p_n be the joint distributions of p for $n = 1, 2, \ldots$ If p runs over all invariant measures on X, then the points p_n form a polygon K_n. We decsribe the set of all extremal points of K_n and we give a decomposition of Bernoulli measures by extremal points of K_n. Next, we study a class $\mathscr{M}_0$ of those measures which may be described by extremal points used in a decomposition of the Bernoulli measures. Further, we construct a complete system of invariants of the dynamical systems induced by the measures belonging to $\mathscr{M}_0$.

1. Notations and definitions. Let $S=\{0,1,\ldots,s-1\}$, $s\geqslant 2$, be a finite alphabet and let $X=\prod_{-\infty}^{+\infty} S$. If $x=\{\ldots,x_{-1},x_0,x_1,\ldots\}$ is a point of X, then we define $T(x)_i=x_{i+1}$, $i=0,\pm 1,\pm 2,\ldots$, that is, T shifts every sequence. Let $\mathscr{B}$ be a σ-field of borelian subsets of X. A Borel probability measure p on $\mathscr{B}$ is called *T-invariant* (or shortly *invariant*) if $p(T^{-1}A)=p(A)$, for any $A\in\mathscr{B}$. For $n\geqslant 1$ we put $X_n=\prod_{0}^{n-1} S$. An element $B=(i_0 i_1\ldots i_{n-1})$ of X_n will be called a *block*. We shall identify B with the cylinder $\{x\in X;\ x_0=i_0, x_1=i_1,\ldots,x_{n-1}=i_{n-1}\}$. Let us denote by $M(X)$ the set of all T-invariant measures on $\mathscr{B}$. For a given $p\in M(X)$ we define a measure p_n on X_n as $p_n(B)=p(B)$, $B\in X_n$, $n\geqslant 1$. The measure p_n may be considered as a point of the space R^{s^n} in the sense that the coordinates of p_n are indexed by the blocks $B\in X_n$, and the Bth coordinate of p_n is equal to $p_n(B)$. Fix $n\geqslant 1$ and denote by K_n the set of all vectors of the form $\langle p_n(B)\rangle_{B\in X_n}$, where p runs over all invariant measures on X. It is well known that the set K_n may be described by the following conditions:

(a) $$\sum_{B\in X_n} p_n(B)=1,$$

(b) $$\sum_{i=0}^{s-1} p_n(Ci)=\sum_{i=0}^{s-1} p_n(iC),\quad \text{for every } C\in X_{n-1},$$

(c) $$p_n(B)\geqslant 0,\quad B\in X_n.$$

Further, if the measures p_n, $n = 1, 2, \ldots$, are appointed by the invariant measure p, then the conditions of consistency are satisfied, i.e.

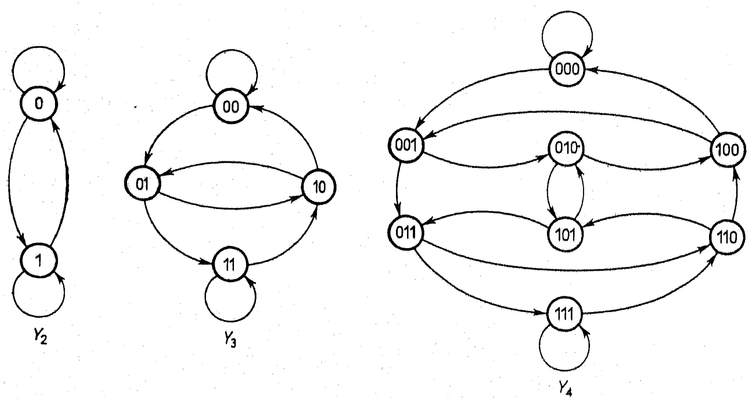
(d) $$\sum_{i=0}^{s-1} p_{n+1}(Bi) = p_n(B), \qquad B \in X_n,\ n = 1, 2, \ldots$$

Condition (d) may be regarded as a definition of a mapping f_n from K_{n+1} onto K_n. We remark that the sets K_n, $n \geq 1$, are polygons in R^{s^n} and it is easy to check that $\dim K_n = s^{n-1}(s-1)$. We obtain a sequence of the polygons K_n and the functions f_n,

$$K_1 \overset{f_1}{\leftarrow} K_2 \overset{f_2}{\leftarrow} K_3 \overset{f_3}{\leftarrow} \ldots.$$

In view of the above remarks the set $M(X)$ may be identified with $\underleftarrow{\operatorname{Lim}} K_n$. If $\bar{p} \in K_n$, $\bar{q} \in K_{n+1}$ and $\bar{p} = f_n(\bar{q})$, then we shall say that the vector $\bar{q}$ is an *extension of* $\bar{p}$.

2. Extremal points of K_n. Now, we shall describe the set of all extremal points of K_n. In order to do this we use a graph Y_n, $n = 2, 3, \ldots$ If $n = 1$, then K_1 may be identified with the simplex $T_s = \{(x_0, x_1, \ldots, x_{s-1});\ \sum x_i = 1,\ x_i \geq 0\}$, the extremal points of which are $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. The vertices of Y_n form the blocks $C \in X_{n-1}$ and two blocks $C_1 = (i_0 \ldots i_{n-2})$ and $C_2 = (j_0 \ldots j_{n-2})$ are joined by an oriented edge (write $(C_1, C_2) \in \mathscr{A}_n$) iff $(i_1 \ldots i_{n-2}) = (j_0 \ldots j_{n-3})$. This means that the end of C_1 agrees with the beginnig of C_2. In the case of $n = 2$ each two block-symbols are joined by edges. For example, if $S = \{1, 0\}$ then Y_2, Y_3, Y_4 have the following form:



Observe that the edges of Y_n may be identified with the blocks of length n in the following sense: each edge (C_1, C_2) determines a block $B = (i_0, i_1, \ldots, i_{n-2}, j_{n-2})$.

Let $\gamma = \{B_1, B_2, \ldots, B_l\}$, $B_i \in X_n$, $i = 1, 2, \ldots, l$, $1 \leq l \leq s^{n-1}$, be a closed path in Y_n not having any loop. Define a vector $\bar{p}_\gamma = \langle p_\gamma(B) \rangle_{B \in X_n}$ as follows:

$$p_\gamma(B) = \begin{cases} 1/l, & B \in \gamma, \\ 0, & B \notin \gamma. \end{cases}$$

It is easy to see that $\bar{p}_\gamma \in K_n$. Now we can prove

THEOREM 1. *A vector $\bar{p} \in K_n$ is an extremal point of K_n iff $\bar{p} = \bar{p}_\gamma$, where γ is a closed path in Y_n which does not contain any loop.*

Proof. *Sufficiency.* Suppose that $\gamma = \{B_1, \ldots, B_l\}$, $1 \leq l \leq s^{n-1}$, is a closed path without loops. Let $B_i = (b_0^i, b_1^i, \ldots, b_{n-1}^i)$, $C_i = (b_0^i, \ldots, b_{n-2}^i)$, $i = 1, 2, \ldots, l$. The blocks $C_1, C_2, \ldots, C_l$ are the vertices of γ and they are pairwise distinct since γ does not contain any loop. Further, the condition that $B_1, B_2, \ldots, B_l, B_1$ are the successive edges of γ implies $B_i = b_0^i C_{i+1}$, $i = 1, 2, \ldots, l-1$, and $B_l = b_0^l C_1$. Assume $\bar{p}_\gamma = t \cdot \bar{p} + (1-t) \cdot \bar{q}$, where $0 < t < 1$ and $\bar{p}, \bar{q} \in K_n$. Then $p(B) > 0$ implies $B \in \gamma$. Hence $p(C_i j) > 0$ implies $j = b_{n-1}^i$ for $i = 1, 2, \ldots, l$ and $p(jC_i) > 0$ implies $j = b_0^{i-1}$, $i = 2, \ldots, l$, and $j = b_0^l$ for $i = 1$. In this way we obtain

$$p(B_1) = p(b_0^1 C_2) = \sum_{j=0}^{s-1} p(jC_2) = \sum_{j=0}^{s-1} p(C_2 j) = p(C_2 b_{n-1}^2) = p(B_2).$$

Similarly we can establish $p(B_2) = p(B_3) = \ldots = p(B_l)$. Therefore the condition $\sum_{i=1}^{l} p(B_i) = 1$ implies $p(B_i) = 1/l$, $i = 1, 2, \ldots, l$, i.e. $\bar{p} = \bar{P}_\gamma$. In the same manner we obtain $\bar{p}_\gamma = \bar{q}$, so that $\bar{p}_\gamma$ is an extremal point of K_n.

Necessity. The polygon K_n is described by conditions (a), (b), (c). It is easy to remark that the order of the system of equations (a), (b) is equal to s^{n-1}. Take an extremal point $\bar{p} \in K_n$. It is well known that r ($r = s^n - s^{n-1}$) of the s^n coordinates of $\bar{p}$ are equal to zero and the remaining s^{n-1} coordinates satisfy a regular subsystem of (a), (b). So $p(B)$, $B \in X_n$, are rational numbers, say $p(B) = r(B)/N$, where $r(B)$ are non-negative integers with $\sum_{B \in X_n} r(B) = N$. In order to find a closed path γ for which $\bar{p} = \bar{p}_\gamma$ we remark that condition (b) implies the following properties:

(1) *for any $B \in X_n$ with $p(B) > 0$ there exists a $\bar{B} \in X_n$ such that $p(\bar{B}) > 0$ and $(B, \bar{B}) \in \mathscr{A}_{n+1}$.*

Let $B_0 \in X_n$ be a block such that $p(B_0) = \min\{p(B);\ p(B) > 0\}$. Starting with B_0 and using (1), we may find a finite sequence of blocks $B_0, B_1, \ldots$ of X_n such that $(B_i, B_{i+1}) \in \mathscr{A}_{n+1}$. Denote by C_i the block of length $n-1$ which forms the beginning of B_{i+1} and the end of B_i. It is clear that the sequence of blocks $C_0, C_1, \ldots$ contains pairwise different blocks $C_{s+1}, C_{s+2}, \ldots C_m$, where $s < m$ and $(C_m, C_{s+1}) \in \mathscr{A}_n$. Then the blocks $B_{s+1}, \ldots, B_m$ form a closed path of Y_n without loops. Moreover, we have $p(B) > 0$ if $B \in \gamma$.

Let $m_0 \geqslant 1$ be the length of γ. Assume $\overline{p}_\gamma \neq \overline{p}$. Then $m_0 < N$ and therefore the vector $\overline{q} = \left(\overline{p} - \frac{m_0}{N}\overline{p}_\gamma\right)\cdot\frac{N}{N-m_0}$ is an element of K_n. Hence $\overline{p} = \frac{N-m_0}{N}\cdot\overline{q} + \frac{m_0}{N}\overline{p}_\gamma$, which means that $\overline{p}$ is not an extremal point of K_n. This leads us to a contradiction, so that the theorem is proved.

THEOREM 2. *If $\overline{p}$ is an extremal point of K_n, $n = 1, 2, \ldots$, then there exists exactly one $\overline{q} \in K_{n+1}$ such that $\overline{p} = f_n(\overline{q})$. Moreover, $\overline{q}$ is an extremal point of K_{n+1}.*

Proof. First we assume $n = 1$. In this case the set of all extremal points of K_1 is identical with the set

$$\{\overline{p}_0, \overline{p}_1, \ldots, \overline{p}_{s-1}\}, \quad \text{where} \quad p_i(j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j \in S.$$

It is easy to check that the vectors

$$\overline{q}_0, \ldots, \overline{q}_{s-1} \in K_2, \quad \overline{q}_i(B) = \begin{cases} 1, & B = (ii) \\ 0, & B \neq (ii) \end{cases},$$

are the only vectors of K_2 such that $f_1(\overline{q}_i) = \overline{p}_i$.

Now, let $n \geqslant 2$ and take a path $\gamma = \{B_1, \ldots, B_l\}$ of Y_n not having loops. Then the vector $\overline{q} \in K_{n+1}$ is an extension of $\overline{p}_\gamma$ iff the following conditions are satisfied [2]

$$\text{(2)} \quad \begin{aligned} \sum_{i=0}^{s-1} q(iCj) &= p_\gamma(Cj), \quad j = 0, 1, \ldots, s-1, \\ \sum_{j=0}^{s-1} q(iCj) &= p_\gamma(iC), \quad i = 0, 1, \ldots, s-1, \end{aligned}$$

where C is any block of the length $n-1$. Therefore, in order to solve the systems of equations (2) it suffices to find matrices $Q(C) = \langle q(iCj)\rangle$, $i, j = 0, 1, \ldots, s-1$, satisfying (2) for every $C \in X_{n-1}$.

The following cases are possible:

(i) $p_\gamma(iC) = 0$, $p_\gamma(Cj) = 0$ *for every* $i, j \in S$;

(ii) there exists exactly one $i_0, j_0 \in S$ with $p_\gamma(i_0C) = p_\gamma(Cj_0) = 1/l$ and $p_\gamma(iC) = 0 = p_\gamma(Cj)$ for the remaining $i, j \in S$.

So the only solution of the system of equations (2) is a vector $\overline{q} \in K_{n+1}$ defined as follows: $q(iCj) = 0$, $i, j \in S$, if (i) holds and $q(i_0Cj_0) = 1/l$, $q(iCj) = 0$, $(i, j) \neq (i_0, j_0)$ if (ii) holds. It is easy to check that the vector $\overline{q}$ is an extremal point of K_{n+1} appointed by the vertices $B_1, B_2, \ldots, B_l$. This completes the proof of the theorem.

Remark 1. Let $S = \{0, 1\}$ and let l_n denote the number of all extremal points of K_n. We can immediately check that $l_1 = 2$, $l_2 = 3$, $l_3 = 6$, $l_4 = 19$. At the same time the dimension of the sets K_1, K_2, K_3, K_4 is equal to 1, 2, 4, 8, respectively. This means that the polygons K_n cannot be simplexes for $n = 3, 4$.

§ 3. Decomposition of the Bernoulli measures. In this section we present a decomposition of Bernoulli measures by extremal points of K_n, $n = 1, 2, \ldots$

In order to do this we define a relation of φ-equivalence between the elements of X_n. The relation φ is defined as follows: two blocks $B, \overline{B} \in X_n$ are *φ-equivalent* iff $\overline{B} = (i_t i_{t+1} \ldots i_n i_1 \ldots i_{t-1})$ for some $1 \leqslant t \leqslant n$, where $B = (i_1 \ldots i_n)$.

Denote by $\mathscr{B}_n$ the set of all classes of the φ-equivalence. Observe that each element of $\mathscr{B}_n$ contains at most n blocks of X_n. We shall show that if $\gamma \in \mathscr{B}_n$, then the edges of γ form a closed path of Y_n not having loops. It turns out that, for any Bernoulli measure p, the measures p_n may be described by the extremal points $\overline{p}_\gamma$, $\gamma \in \mathscr{B}_n$, $n = 1, 2, \ldots$

Suppose $\gamma = \{B_1, B_2, \ldots, B_l\}$, $l \leqslant n$. We may assume that if $B_1 = (i_1 i_2 \ldots i_n)$ then $B_2 = (i_2 i_3 \ldots i_n i_1)$, $B_3 = (i_3 \ldots i_n i_1 i_2)$, and so on. Let $B_i = C_i c_n^i$, $C_i \in X_{n-1}$, $c_n^i \in S$. In order to show that γ forms a closed path of Y_n without loops it remains to prove that the blocks $C_1, C_2, \ldots, C_l$ are pairwise distinct. First we observe that all blocks of γ have the same numbers of symbols. Now, if $C_1 = C_2$ then $c_n^1 \neq c_n^2$ because $B_1 \neq B_2$, and therefore the frequencies of the symbols in B_1 and B_2 are different. Thus $C_1 \neq C_2$, and similarly we obtain $C_i \neq C_j$ for $i \neq j$. This means that γ forms a closed path of Y_n not having any loop.

Let $n_0, n_1, \ldots, n_{s-1}$ be non-negative integers with $n_0 + \ldots + n_{s-1} = n$ and let $\mathscr{B}(n_0, n_1, \ldots, n_{s-1})$ be the set of all blocks of X_n containing the symbol 0 n_0 times, the symbol 1 n_1 times, and so on. Denote by b_γ the length of γ, $\gamma \in \mathscr{B}_n$. It is clear that

$$\sum_{\gamma \subset \mathscr{B}(n_0, \ldots, n_{s-1})} b_\gamma = \frac{n!}{n_0!\, n_1! \ldots n_{s-1}!}.$$

Take a Bernoulli measure p on X given by a probability vector $\bar{q} = (q_0, q_1, \ldots, q_{s-1})$. It is not difficult to check that for $n = 2, 3, \ldots$ we have

$$p_n = \sum_{n_0+\ldots+n_{s-1}=n} q_0^{n_0}\cdot q_1^{n_1}\ldots q_{s-1}^{n_{s-1}} \sum_{\gamma\subset\mathscr{B}(n_0,\ldots,n_{s-1})} \bar{p}_\gamma\cdot b_\gamma .$$

Remark 2. For any block $B \in X_n$ there exists a unique class $\gamma \in \mathscr{B}_n$ such that $B \in \gamma$. Accordingly the vectors $\bar{p}_\gamma$, $\gamma \in \mathscr{B}_n$, are linearly independent and therefore they form a simplex L_n in K_n. In general, the dimension of L_n is smaller than $\dim K_n$.

In the sequel we denote by $\mathscr{M}_0$ the set of all invariant measures p on X for which $p_n \in L_n$ for $n = 1, 2, \ldots$

§ 4. Description of the class $\mathscr{M}_0$. First we introduce the notation. If $B = (i_1 \ldots i_l)$, $C = (j_1 \ldots j_m)$ are two blocks, then we shall denote by BC the block $(i_1 \ldots i_l j_1 \ldots j_m)$. We start with the following

LEMMA 1. *A measure p on X belongs to $\mathscr{M}_0$ iff for any two blocks B, C the condition*

$$\sum_{i=0}^{s-1} p(BiC) = p(CB) \tag{3}$$

is satisfied.

Proof. *Necessity.* The condition $p \in \mathscr{M}_0$ implies that if $i \in S$ and B, C are two blocks then $p(BiC) = p(CBi)$ and further

$$\sum_{i=0}^{s-1} p(BiC) = \sum_{i=0}^{s-1} p(CBi) = p(CB).$$

Sufficiency. Taking B or C as empty blocks, we find that p is an invariant measure on X. In order to prove that $p \in \mathscr{M}_0$ it suffices to show the equality $p(iB) = p(Bi)$ for any symbol $i \in S$ and any block B. Using (3) we have

$$p(iB) = \sum_{j=0}^{s-1} p(Bji) = \sum_{j,k=0}^{s-1} p(ikBj) = \sum_{k=0}^{s-1} p(ikB) = p(Bi),$$

which completes the proof of the lemma.

DEFINITION 1. We say that an invariant measure p on X is symmetric if $p(B) = p(C)$ for any blocks $B, C \in \mathscr{B}(n_0, n_1, \ldots, n_{s-1})$, where $n_0, n_1, \ldots, n_{s-1}$ are non-negative integers.

THEOREM 3. *An invariant measure p on X belongs to $\mathscr{M}_0$ iff p is symmetric.*

Proof. *Sufficiency.* If p is symmetric then p is constant on each class $\gamma \in \mathscr{B}_n$ for $n = 1, 2, \ldots$ But this means that for $n = 1, 2, \ldots, p_n \in L_n$, i.e. $p \in \mathscr{M}_0$.

Necessity. Let T be the shift on X and let $k > 1$. Take blocks $B_0 \in B_{l_0}$, $B_1 \in X_{l_1}, \ldots, B_k \in X_{l_k}$, where $l_0, l_1, \ldots, l_k$ are positive integers and let $n_1 > l_0$, $n_2 > l_1 + n_1, \ldots, n_k > n_{k-1} + l_{k-1}$. Then we have

$$\sum_{A_1,A_2,\ldots,A_k} p(B_0A_1B_1A_2\ldots A_kB_k) = p\big(B_0\cap T^{-n_1}(B_1)\cap T^{-n_2}(B_2)\cap\ldots\cap T^{-n_k}(B_k)\big),$$

where $A_1 \in X_{n_1-l_0}$, $A_2 \in X_{n_2-n_1-l_1}$, $A_3 \in X_{n_3-n_2-l_2}$, and so on. Further, from the definition of $\mathscr{M}_0$ and (3) follows

$$\sum_{A_1,A_2,\ldots,A_k} p(B_0A_1B_1A_2\ldots A_kB_k) = p(B_0B_1\ldots B_k).$$

Thus for sufficiently large $n_k > n_{k-1} > \ldots > n_1$ we have

$$p(B_0B_1\ldots B_k) = p\big(B_0\cap T^{-n_1}(B_1)\cap\ldots\cap T^{-n_k}(B_k)\big). \tag{4}$$

Now, take a partition ζ of X on ergodic components with respect to T. Let $M = X/\zeta$, $(p_m\}_{m\in M}$ be conditional measures of ζ and let p_ζ be the quotient measure on M induced by p. Then we have

$$p\big(B_0\cap T^{-n_1}(B_1)\cap\ldots\cap T^{-n_k}(B_k)\big) = \int_M p_m\big(B_0\cap T^{-n_1}(B_1)\cap\ldots\cap T^{-n_k}(B_k)\big)p_\zeta(dm).$$

By the above equality and by (4) we obtain

$$p(B_0B_1\ldots B_k) = \int_M p_m\big(B_0\cap T^{-n_1}(B_1)\cap\ldots\cap T^{-n_k}(B_k)\big)p_\zeta(dm). \tag{5}$$

Now, fix $n_{k-1} > n_{k-2} > \ldots > n_1$. Applying the ergodic theorem to the dynamical systems $(X, \mathscr{B}, p_m, T)$, $m \in M$, we have

$$\lim_{n_k\to\infty}\frac{1}{n_k}\sum_{t=0}^{n_k-1} p_m\big(B_0\cap T^{-n_1}(B_1)\cap\ldots\cap T^{-n_{k-1}}(B_{k-1})\cap T^{-t}(B_k)\big) = p_m(B_k)p_m\big(B_0\cap T^{-n_1}(B_1)\cap\ldots\cap T^{-n_{k-1}}(B_{k-1})\big) \quad \text{for a.e. } m\in M. \tag{6}$$

Since $\frac{1}{n_k}\sum_{t=0}^{n_k-1} p_m\big(B_0\cap\ldots\cap T^{-t}(B_k)\big) \leqslant 1$, $k \geqslant 1$, we can integrate both sides of (6) and we obtain

$$\lim_{n_k\to\infty}\frac{1}{n_k}\sum_{t=0}^{n_k-1}\int_M p_m\big(B_0\cap\ldots\cap T^{-n_{k-1}}(B_{k-1})\cap T^{-t}(B_k)\big) = \int_M p_m(B_k)p_m\big(B_0\cap\ldots\cap T^{-n_{k-1}}(B_{k-1})\big)p_\zeta(dm). \tag{7}$$

Further, (5) and (7) imply

(8) $$p(B_kB_{k-1}\dots B_1B_0) = \int_M p_m(B_k)\cdot p_m\big(B_0\cap T^{-n_1}(B_1)\cap\dots\cap T^{-n_{k-1}}(B_{k-1})\big)p_\zeta(dm),$$

for any sufficiently large integers $n_{k-1} > n_{k-2} > \dots > n_1$. Repeating the above arguments for fixed $n_{k-2} > n_{k-3} > \dots > n_1$ and for $n_{k-1}\to\infty$, we obtain

$$p(B_kB_{k-1}\dots B_1B_0) = \int_M p_m(B_k)p_m(B_{k-1})p_m\big(B_0\cap T^{-n_1}(B_1)\cap\dots\cap T^{-n_{k-2}}(B_{k-2})\big)p_\zeta(dm).$$

Proceeding in the same manner, we have

(9) $$p(B_kB_{k-1}\dots B_1B_0) = \int_M p_m(B_k)p_m(B_{k-1})\dots(p_m)B_1p_m(B_0)p_\zeta(dm),$$

for any blocks $B_0, B_1, \dots, B_k$. Therefore, if B is a block having n_0 0's, n_1 1's, and so on, then (9) gives

(10) $$p(B) = \int_M p_0(m)\cdot p_1(m)\dots p_{s-1}(m)p_\zeta(dm),$$

where $p_i(m) = p_m(\{i\})$, $i\in S$. Thus (10) implies the theorem.

Remark 3. If $p\in\mathcal{M}_0$ and p is an ergodic measure, then p_ζ is a δ-measure concentrated at a point $m_0\in M$. Therefore p is a Bernoulli measure given by the probability vector $\langle p_0(m_0), \dots, p_{s-1}(m_0)\rangle$.

Example 1. Take a simplex T_s of the space R^{s-1} defined in the following way: $\overline{x} = (x_0, x_1, \dots, x_{s-1})\in T_s$ iff $\sum_{i=0}^{s-1} x_i = 1$ and $x_i\geqslant 0$, $i = 0, 1, \dots$ $\dots, s-1$. Assume that $\overline{p}$ is a normalized, borelian measure on T_s. Then we can define a measure p on X as follows:

(11) $$p(B) = \int_{T_s} x_0^{n_0}x_1^{n_1}\dots x_{s-1}^{n_{s-1}}\overline{p}(d\overline{x}),$$

where $B\in\mathcal{B}(n_0, n_1, \dots, n_{s-1})$. It is easy to verify that p is an invariant measure on X and $p\in\mathcal{M}_0$.

Now, using the well-known theorem of de Finetti [6] we have the following

Theorem 4. *For every $p\in\mathcal{M}_0$ there exists a unique measure $\overline{p}$ on T_s such that* (11) *holds.*

§ 5. Isomorphism theorems. Consider a probability measure $\overline{p}$ on T_s and let $p = \psi(\overline{p})$ be a measure on X defined by (11). The measure $\overline{p}$ determines a dynamical system $Z(\overline{p}) = \big(X, \mathcal{B}, \psi(\overline{p}), T\big)$. In this section we

give a necessary and sufficient condition for two dynamical systems $Z(\overline{p}_1)$ and $Z(\overline{p}_2)$ to be isomorphic. First we describe a decomposition of $Z(\overline{p})$ on ergodic components.

Let $\mathcal{M}_e$ be the set of all invariant ergodic measures on X. On $\mathcal{M}_e$ we can define a topology [1] induced by neighbourhoods of the following form: $M(\varepsilon, \mu_0, C_1, \dots, C_k) = \bigcap_{i=1}^{k}\{\mu\in\mathcal{M}_e;\ |\mu(C_i)-\mu_0(C_i)| < \varepsilon\}$, where $\mu_0\in\mathcal{M}_0$, $\varepsilon > 0$ and $C_1, C_2, \dots, C_k$ are cylinders. Denote by $\mathcal{B}_e$ the σ-field of borelian subsets of $\mathcal{M}_e$. Now, for a given probability measure ν on $\mathcal{B}_e$, we can define a T-invariant measure $\tilde{\mu}$ on X by

(12) $$\tilde{\mu}(C) = \int_{\mathcal{M}_e}\mu(C)\nu(d\mu),$$

where C is a block. It is well known [4] that any invariant measure $\tilde{\mu}$ on X has the form (12). Moreover, the correspondence $\nu\to\tilde{\mu}$ is one-to-one. The measure ν is called the *decomposition of $\tilde{\mu}$ on ergodic components*.

Now, we remark that the measure $\overline{p}$ on T_s may be identified with a decomposition of $\psi(\overline{p})$ on ergodic components. In fact, the set T_s may be identified with the set $\mathcal{M}_b$ of all Bernoulli measures on X. Moreover, the natural topology of T_s is identical with the restriction of the topology of $\mathcal{M}_e$ to $\mathcal{M}_b$($\mathcal{M}_b$ is a closed subset of $\mathcal{M}_e$). Therefore, the measure $\overline{p}$ may be regarded as a measure on $\mathcal{M}_b$ and on $\mathcal{M}_e$, too. For this measure equation (12) reduces to (11).

Having a decomposition of $\psi(\overline{p})$ on ergodic components, we can construct a complete system of invariants of $Z(\overline{p})$. To do this we consider a function H on T_s defined by

$$H(\overline{x}) = -\sum_{i=0}^{s-1} x_i\log x_i.$$

Let $I_s = \langle 0, \log s\rangle$ and let ζ be a partition of T_s on the sets $C_a = \{\overline{x}\in T_s;\ H(\overline{x}) = a\}$, $a\in I_s$. The measure $\overline{p}$ determines the quotient measure $\hat{p}$ on $I_s = T_s/\zeta$ and conditional measures $\{p_a\}$, $a\in I_s$. Let $\{m_n(a)\}$ be the type of $\overline{p}_a$, that is, let $\{m_n\}$ be a sequence of measurable functions defined on I_s such that

$$\sum_{n=1}^{\infty} m_n(a)\leqslant 1,\qquad m_{n+1}(a)\leqslant m_n(a),\qquad m_n(a)\geqslant 0\quad\text{ for }\quad n = 1, 2, \dots,$$

and for almost all $a\in I_s$ with respect to $\hat{p}$. We obtain a pair $\theta(\overline{p}) = \big(\hat{p}, \{m_n(a)\}_{a\in I_s}\big)$.

Theorem 5. *Given two probability measures $\overline{p}_1, \overline{p}_2$ on T_s, the dynamical systems $Z(\overline{p}_1)$, $Z(\overline{p}_2)$ are isomorphic iff $\theta(\overline{p}_1) = \theta(\overline{p}_2)$.*

The proof of the theorem can be obtained by using Ornstein's Isomorphism Theorem [3] and Roklin's decomposition theorem [5], which can be formulated in the following form:

THEOREM 6. *Let* $\tilde{\mu}_1, \tilde{\mu}_2$ *be two invariant measures on* X *and let* ν_1, ν_2 *be the decomposition on ergodic components of* $\tilde{\mu}_1$ *and* $\tilde{\mu}_2$, *respectively. The dynamical systems* $(X, \mathscr{B}, T, \tilde{\mu}_1)$ *and* $(X, \mathscr{B}, T, \tilde{\mu}_2)$ *are isomorphic iff there exists an invertible measure-preserving transformation* $S\colon (\mathscr{M}_e, \nu_1) \to (\mathscr{M}_e, \nu_2)$ *such that for a.e.* $(\operatorname{mod} \nu_1)$ $\mu \in \mathscr{M}_e$ *the ergodic dynamical systems* $(X, \mathscr{B}, T, \mu)$ *and* $(X, \mathscr{B}, T, S_\mu)$ *are isomorphic.*

Remark 4. If $s = 2$ then $m_1(\log 2) = 1$, $m_n(\log 2) = 0$ for $n \geqslant 2$ and for $a \in \langle 0, \log 2)$ we must have $m_n(a) = 0$ for $n = 3, 4, \ldots$ If $s > 2$ then $m_1(\log s) = 1$, $m_n(\log s) = 0$, $n = 2, 3, \ldots$, and the remaining measures may have arbitrary types.

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On certain subspaces of a nuclear power series spaces of finite type

by

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Dedicated to Professor Arf on his 70th birthday

Abstract. Let $\alpha = (\alpha_n)$ denote a stable nuclear exponent sequence of finite type. It is shown that a subspace X of $\Lambda_1(\alpha)$ with a basis is either isomorphic to a subspace of $\Lambda_\infty(\alpha)$ or X has a complemented subspace which is isomorphic to a power series space of finite type. Also applications of this result to spaces of analytic functions are discussed.

Introduction. Throughout, we let $\alpha = (\alpha_n)$ denote a nuclear exponent sequence of finite type which is assumed to be *stable* [6] (i.e. (α_{2n}/α_n) is bounded). By *subspace* we mean a closed, infinite-dimensional subspace. Recently, Dubinsky [5] characterized Köthe spaces which are isomorphic to subspaces of a power series space $\Lambda_1(\alpha)$ of finite type. In particular, the power series space $\Lambda_\infty(\alpha)$ of infinite type is isomorphic to a subspace of $\Lambda_1(\alpha)$ ([3]). The main result of this note is the following.

THEOREM 1. *A subspace* X *of* $\Lambda_1(\alpha)$ *with a basis is either isomorphic to a subspace of* $\Lambda_\infty(\alpha)$ *or* X *has a complemented subspace which is isomorphic to a power series space of finite type.*

We note that a subspace of $\Lambda_\infty(\alpha)$ cannot have a subspace isomorphic to a power series space of finite type ([20]). For the special case of $\alpha_n = n^{1/d}$, the corresponding power series space of infinite type is isomorphic to the space $O(C^d)$ of entire functions in d variables and the corresponding power series spaces of finite type is isomorphic to the space $O(\Delta^d)$ of functions analytic in the d-dimensional unit polycylinder ([15]). In the final section, we apply Theorem 1 to spaces of analytic functions and obtain the following result.

THEOREM 2. *Let* M *be a Stein manifold of dimension* d *and assume that* $O(M)$ *has a basis. Then* $O(M)$ *is either isomorphic to* $O(C^d)$ *or* $O(M)$ *has a complemented subspace isomorphic to a power series space of finite type.*

We introduce some terminology in the following section, which leads to the proof of Theorem 1. For any undefined terminology we refer to [9], [12], and [6]. This research was supported by the Scientific and Technical Research Council of Turkey.