The proof of the theorem can be obtained by using Ornstein's Isomorphism Theorem [3] and Roklin's decomposition theorem [5], which can be formulated in the following form:

THEOREM 6. *Let* $\tilde{\mu}_1, \tilde{\mu}_2$ *be two invariant measures on* X *and let* ν_1, ν_2 *be the decomposition on ergodic components of* $\tilde{\mu}_1$ *and* $\tilde{\mu}_2$, *respectively. The dynamical systems* $(X, \mathscr{B}, T, \tilde{\mu}_1)$ *and* $(X, \mathscr{B}, T, \tilde{\mu}_2)$ *are isomorphic iff there exists an invertible measure-preserving transformation* $S\colon (\mathscr{M}_e, \nu_1) \to (\mathscr{M}_e, \nu_2)$ *such that for a.e.* $(\operatorname{mod} \nu_1)$ $\mu \in \mathscr{M}_e$ *the ergodic dynamical systems* $(X, \mathscr{B}, T, \mu)$ *and* $(X, \mathscr{B}, T, S_\mu)$ *are isomorphic.*

Remark 4. If $s = 2$ then $m_1(\log 2) = 1$, $m_n(\log 2) = 0$ for $n \geq 2$ and for $a \in \langle 0, \log 2)$ we must have $m_n(a) = 0$ for $n = 3, 4, \ldots$ If $s > 2$ then $m_1(\log s) = 1$, $m_n(\log s) = 0$, $n = 2, 3, \ldots$, and the remaining measures may have arbitrary types.

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STUDIA MATHEMATICA, T. LXIX. (1980)

On certain subspaces of a nuclear power series spaces of finite type

by

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Dedicated to Professor
Arf on his 70th birthday

Abstract. Let $\alpha = (\alpha_n)$ denote a stable nuclear exponent sequence of finite type. It is shown that a subspace X of $\Lambda_1(\alpha)$ with a basis is either isomorphic to a subspace of $\Lambda_\infty(\alpha)$ or X has a complemented subspace which is isomorphic to a power series space of finite type. Also applications of this result to spaces of analytic functions are discussed.

Introduction. Throughout, we let $\alpha = (\alpha_n)$ denote a nuclear exponent sequence of finite type which is assumed to be *stable* [6] (i.e. (α_{2n}/α_n) is bounded). By *subspace* we mean a closed, infinite-dimensional subspace. Recently, Dubinsky [5] characterized Köthe spaces which are isomorphic to subspaces of a power series space $\Lambda_1(\alpha)$ of finite type. In particular, the power series space $\Lambda_\infty(\alpha)$ of infinite type is isomorphic to a subspace of $\Lambda_1(\alpha)$ ([3]). The main result of this note is the following.

THEOREM 1. *A subspace* X *of* $\Lambda_1(\alpha)$ *with a basis is either isomorphic to a subspace of* $\Lambda_\infty(\alpha)$ *or* X *has a complemented subspace which is isomorphic to a power series space of finite type.*

We note that a subspace of $\Lambda_\infty(\alpha)$ cannot have a subspace isomorphic to a power series space of finite type ([20]). For the special case of $\alpha_n = n^{1/d}$, the corresponding power series space of infinite type is isomorphic to the space $O(C^d)$ of entire functions in d variables and the corresponding power series spaces of finite type is isomorphic to the space $O(\Delta^d)$ of functions analytic in the d-dimensional unit polycylinder ([15]). In the final section, we apply Theorem 1 to spaces of analytic functions and obtain the following result.

THEOREM 2. *Let* M *be a Stein manifold of dimension* d *and assume that* $O(M)$ *has a basis. Then* $O(M)$ *is either isomorphic to* $O(C^d)$ *or* $O(M)$ *has a complemented subspace isomorphic to a power series space of finite type.*

We introduce some terminology in the following section, which leads to the proof of Theorem 1. For any undefined terminology we refer to [9], [12], and [6]. This research was supported by the Scientific and Technical Research Council of Turkey.

Proof of Theorem 1. A locally convex space X is said to have *property* (DN) if the topology of X can be defined by an increasing sequence of norms $\| \ \|_k$ such that the inequality

$$\|x\|_k^2 \leqslant \|x\|_{k-1}\|x\|_{k+1}$$

is satisfied for each $x \in X$ and $k = 1, 2, \ldots$ ([18]: 2.1, Satz). Such a space is necessarily metrizable. This condition was introduced by Vogt in [18], where it was proved that a nuclear locally convex space X is isomorphic to a subspace of the space (s) of rapidly decreasing sequences if and only if X has property (DN).

A countable family $P = \{(p_n^k)\}$ of sequences will be called a G_∞*-set* ([16], [6]) if the following are satisfied:

(i) $p_n^0 = 1$ *and* $1 \leqslant p_n^k \leqslant p_{n+1}^k$ *for every* k *and* $n \in N = \{0, 1, 2, \ldots\}$,

(ii) *for each* $k \in N$ *there is a* j *with* $(p_n^k)^2 = O(p_n^j)$.

The Köthe space $\lambda(P)$ defined by such a set P is called a G_∞*-space* ([16]). In particular, a power series space of infinite type is a G_∞-space.

The *diametral dimension* $\Delta(X)$ of a locally convex space X is the set of all sequences (ξ_n) such that for each absolutely convex neighborhood U in X there is another such neighborhood V in X with $\lim \xi_n d_n(V, U) = 0$, where $d_n(V, U)$ is the nth Kolmogorov diameter of V with respect to U ([12], [16]). If X is nuclear, one can replace the Kolmogorov diameter d_n in the above definition by the *n-th Gelfand diameter* ([16]). This number is defined by $\gamma_n(V, U) = \inf \inf\{\gamma > 0\colon V \cap L \subset \gamma U\}$ where the second infimum is taken over all subspaces L of X with codimension not exceeding n.

A nuclear G_∞-space $\lambda(P)$ is isomorphic to a subspace of (s) ([4]) and its diametral dimension is equal to $\lambda(P)'$ ([16]).

Proof of Theorem 1. Let $(\| \ \|_k)$ be a sequence of norms on X satisfying the (DN)-inequality, which can be written in the form

$$\frac{\|x\|_k}{\|x\|_{k+1}} \leqslant \frac{\|x\|_{k-1}}{\|x\|_k}$$

for $x \neq 0$. Repeated applications of this inequality yield

$$\frac{\|x\|_{k+m}}{\|x\|_{k+j+m}} \leqslant \frac{\|x\|_k}{\|x\|_{k+j}}$$

for each k, j and m. If $U_k = \{x \in X\colon \|x\|_k \leqslant 1\}$ and if $U_{k+j} \cap L \subset \gamma U_k$ for some subspace L and $\gamma > 0$, we have for $x \in L$,

$$\frac{\|x\|_{k+j}}{\|x\|_{k+2j}} \leqslant \frac{\|x\|_k}{\|x\|_{k+j}} \leqslant \gamma$$

from the above inequality. Hence

$$U_{k+2j} \cap L \subset \gamma(U_{k+j} \cap L) \subset \gamma^2 U_k.$$

Thus

$$\gamma_n(U_{k+2j}, U_k) \leqslant \gamma_n(U_{k+j}, U_k)^2.$$

Since X is assumed to be isomorphic to a subspace of $\Lambda_1(\alpha)$, for each k, m we can find an $\varepsilon > 0$ with

$$\gamma_n(U_{k+m}, U_k) \leqslant \exp(-\varepsilon\alpha_n).$$

Given $R > 0$, we choose q such that $2^q\varepsilon > R$. Since

$$\gamma_n(U_{k+2^q m}, U_k) \leqslant \gamma_n(U_{k+m}, U_k)^{2^q} \leqslant \exp(-R\alpha_n),$$

we have $\Lambda_\infty(\alpha)' = \Delta(\Lambda_\infty(\alpha)) \subset \Delta(X)$. Hence X is a $\Lambda_N(\alpha)$-nuclear space with basis and property (DN). M. Alpseymen has proved that such a space is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ ([1]). Now, we assume X does not have property (DN) and show that it has a complemented subspace isomorphic to a power series space of finite type.

Let (y_n) be a basic sequence in $\Lambda_1(\alpha)$ such that X is isomorphic to the closed linear span of (y_n). If (e_i) denotes the canonical basis of $\Lambda_1(\alpha)$ and if

$$y_n = \sum_i t_i^n e_i,$$

let

$$a_n^k = \|y_n\|_k = \sup_i |t_i^n| e^{-(\alpha_i/k)}.$$

If $A = \{(a_n^k)_{n \in N}\}$, then X is isomorphic to the Köthe space $\lambda(A)$ by the basis theorem for nuclear spaces ([12]). $\lambda(A)$ has property (DN) if there is a k_0 such that for each k there is an m with

$$(a_n^k)^2 = O(a_n^{k_0} a_n^m)$$

([18]). So, by our assumption, there is a p_1 with

$$\sup_n \frac{(a_n^{p_1})^2}{a_n^i a^m} = \infty$$

for each $m \in N$. We now choose a subsequence $I_1 = (n_j)$ of N with

$$(a_{n_j}^{p_1})^2 \geqslant j\, a_{n_j}^1 a_{n_j}^j.$$

Then

(1) $$\lim_{n \in I_1} (a_n^1 a_n^m / (a_n^{p_1})^2) = 0$$

for every $m \in N$. We now consider the Köthe space $\lambda(A_{I_1})$ where

$$A_{I_1} = \{(a_n^k)_{n \in I_1}\colon k \in N\},$$

and claim that it does not have property (DN).

If we assume the contary, we can choose an increasing sequence (k_i) with $k_0 = p_1$ and $\lim k_i = \infty$ such that

$$(2) \qquad (a_n^{k_i+1})^2 = O(a_n^{k_i} a_n^{k_i+2}), \quad n \in I_1,$$

for every $i \in N$ ([18]; 2.1. Satz). Let

$$q_n^k = \max\{q\colon a_n^k = |t_q^n| \exp(-\alpha_q/k)\}.$$

By the fundamental inequality in [14] we have

$$(3) \qquad \exp\left(-\alpha_{q_n^{k+j}}\left(\frac{1}{k} - \frac{1}{k+j}\right)\right) \leqslant \frac{a_n^k}{a_n^{k+j}} \leqslant \exp\left(-\alpha_{q_n^k}\left(\frac{1}{k} - \frac{1}{k+j}\right)\right)$$

for every $(n, k, j) \in N^3$. It follows from (3) that

$$\exp(-\alpha_{q_n^{k_0}}) \leqslant \left(\frac{a_n^1}{a_n^{k_0}}\right)^{k_0/(k_0-1)}$$

and

$$\left(\frac{a_n^{k_0}}{a_n^{k_i}}\right)^{k_0 k_i/(k_1-k_0)} \leqslant \exp(-\alpha_{q_n^{k_0}}).$$

So we have

$$\left(\frac{a_n^{k_0}}{a_n^{k_i}}\right)^{k_1(k_0-1)/(k_i-k_0)} \leqslant \frac{a_n^1}{a_n^{k_0}}.$$

Since $k_1 k_0 (k-k_0) > k_1(k_0-1)$, from (1) we obtain therefore

$$(4) \qquad \lim_{n\in I_1} \left(\frac{a_n^{k_0}}{a_n^{k_i}}\right)^{k_1 k_0} \frac{a_n^{k_i}}{a_n^{k_0}} = 0$$

for every $i \in N$. On the other hand, applying (2) repeatedly, we find an $i \in N$ with

$$\left(\frac{a_n^{k_1}}{a_n^{k_0}}\right)^{k_1 k_0} = O\left(\frac{a_n^{k_i}}{a_n^{k_0}}\right)$$

and this contradicts (4). Hence the claim is verified. So now we can choose p_2 with

$$\sup_{n\in I_1} \frac{(a_n^{p_2})^2}{a_n^2 a_n^m} = \infty \quad \text{for each } m \in N$$

and a subsequence I_2 of I_1 such that

$$\lim_{n\in I_2} (a_n^2 a_n^m/(a_n^{p_2})^2) = 0.$$

Thus, we can choose by induction subsequences I_k and $p_k \in N$ with

(a) I_k is a subsequence of I_{k-1},

(b) $\lim\limits_{n\in I_k} \dfrac{a_n^k a_n^m}{(a_n^{p_k})^2} = 0$ for each $m \in N$.

By the diagonal process we choose a sequence I which is a subsequence of each I_k. It follows that, for every $k \in N$ there is a $p_k \in N$ such that

$$\lim_{n\in I} \frac{a_n^k a_n^m}{(a_n^{p_k})^2} = 0$$

for every $m \in N$. This means that $\lambda(A_I)$, the Köthe space generated by

$$A_I = \{(a_n^k)_{n\in I}\colon k \in N\},$$

is a (d_2)-space and by a result of Dubinsky ([5]; Proposition 6) it is isomorphic to a finite type power series space.

Remark. Our thanks are due to the referee for suggesting a simplification of the first part of the proof. Indeed, a slightly more elaborate argument along the same line, gives that the diametral dimension of a nuclear Fréchet space X with property (DN) is equal to the dual of a suitable G_∞-space.

Applications. We let $O(M)$ denote the space of analytic functions on a Stein complex space M with the topology of uniform convergence on compacta. These spaces provide one with important and interesting examples of nuclear Fréchet spaces and are studied from that point of view by various authors ([15], [11], [21]). By passing to its envelope of holomorphy if necessary, we see that the space of analytic functions defined on an open connected subset of C^d is isomorphic to some $O(M)$ for a suitable Stein manifold M ([9]; III Theorem, p. 49). (Although we shall not mention it again, we shall only consider connected, Hausdorff, separable complex manifolds.) Our first result shows that $O(M)$ is isomorphic to a subspace of a power series space of finite type.

LEMMA 4. *Let M be a complex manifold of dimension d. Then $O(M)$ is isomorphic to a subspace of $O(\Delta^d)$ where Δ^d denotes the unit polycylinder in C^d.*

Proof. In view of a result by Fornaess and Stout ([8], Main Theorem) there is a holomorphic map φ from Δ^d onto M. If $T\colon O(M) \to O(\Delta^d)$ is defined by $Tf(z) = f(\varphi(z))$, $z \in \Delta^d$, then it is easily that T is injective, continuous and has closed range.

We recall that $O(\Delta^d)$ resp. $O(C^d)$ is isomorphic to the finite type resp. infinite type power series space generated by the sequence $(n^{1/d})$ ([15]). We are now ready to prove Theorem 2 stated in the introduction.

Proof of Theorem 2. By Lemma 4 we know that $O(M)$ is isomorphic to a subspace of $\Lambda_1(n^{1/d})$ and we assumed $O(M)$ has a basis. By Theorem 1 we have to show that if $O(M)$ is isomorphic to a subspace of $\Lambda_\infty(n^{1/d})$ (i.e. if $O(M)$ has property (DN)), then $O(M)$ must be isomorphic to $\Lambda_\infty(n^{1/d})$ itself.

Without loss of generality we can assume, via the imbedding theorem for Stein manifolds ([9]; p. 225), that M is a closed subvariety of some C^j. It follows from the Oka–Cartan theory that the restriction operator from $O(C^j)$ to $O(M)$ is a continuous linear surjection. Since $O(M)$ is assumed to have property (DN), by Theorem 3.3 of [7], we conclude that $O(M)$ is isomorphic to a power series space $\Lambda_\infty(\beta)$ for some β. By following a line of argument due to Mitiagin and Henkin ([11]), we shall show that $\Lambda_\infty(\beta)$ is isomorphic to $\Lambda_\infty(n^{1/d})$.

We fix an analytic function f from M to C^d such that the interior of $f(M)$, which we will denote by $\mathscr{M}$, is non-empty. We find a sequence $z_i \in C^d$ such that the sets $\{z_i+\mathscr{M}\}_i$ cover C^d. $O(z_i+\mathscr{M})$ can be naturally identified with a subspace of $O(M)$ for each $i \in N$. We define T from $O(C^d)$ into $O(M)^\infty$, the cartesian product of countably many copies of $O(M)$, by $T(g) = (g|z_i+\mathscr{M})_{i\in N}$. It is easy to see that T is an isomorphism of $O(C^d)$ onto a closed subspace of $O(M)^\infty$. In particular, we have

$$(1)\qquad \Delta\big(O(M)^\infty\big) \subset \Delta\big(O(C^d)\big).$$

On the other hand, $O(M)^\infty$ is isomorphic to a subspace of $O(C^d)^\infty$, and since $\Delta\big(O(C^d)\big) = \Delta\big(O(C^d)^\infty\big)$ by stability of $O(C^d)$, we have $\Delta\big(O(C^d)\big) \subset \Delta\big(O(M)^\infty\big)$ and so from (1) we get

$$(2)\qquad \Delta\big(O(M)^\infty\big) = \Delta\big(O(C^d)\big).$$

We know $\Delta\big(O(M)\big) = \Lambda_\infty(\beta)'$ and if $\beta^{(k)}$ denotes the sequence obtained from β by repeating β_1 k-times, β_2 k-times, etc., we have $\Lambda_\infty(\beta^{(k)})$ is isomorphic to the cartesian product of k-copies of $\Lambda_\infty(\beta)$ ([17]). Hence from (2) we have

$$(3)\qquad \Delta\big(O(M)^\infty\big) = \bigcap_{k=1}^{\infty} \Lambda_\infty(\beta^{(k)})' = \Lambda_\infty(n^{1/d})'.$$

If b is the sequence whose nth term is $\exp(-\beta_n)$, we let $B = \{b^{(k)}: k = 1, 2, \ldots\}$. By (3) the Köthe space $\lambda(B)$ is contained in $\Lambda_\infty(n^{1/d})'$ and the inclusion map of $\lambda(B)$ into $\Lambda_\infty(n^{1/d})'_b$ is bounded. This means that there is a k and a j with

$$1/b_n^{(k)} = O\big(\exp(j\, n^{1/d})\big).$$

Using the stability of $(n^{1/d})$, we get from above $\beta_n = O(n^{1/d})$ and so $\Lambda_\infty(\beta)' \subset \Lambda_\infty(n^{1/d})'$. Since the other inclusion is already shown, we have that $\Lambda_\infty(\beta)$ is isomorphic to $\Lambda_\infty(n^{1/d})$.

Remark. If we combine Lemma 4 again with Proposition 6 of [5] and Lemma 2.1 a of [11], we obtain that $O(M)$ is isomorphic to $O(\Delta^d)$ if and only if $O(M)$ has a basis and the (d_2)-property.

In view of the Oka–Cartan theory for a Stein manifold M, we have an exact sequence of the form

$$(*)\qquad 0 \to I \xrightarrow{S} O(C^j) \xrightarrow{R} O(M) \to 0$$

for some j, where S and R are continuous linear operators. It is of interest to know when an exact sequence of type $(*)$ splits, i.e., when there is a continuous linear operator $T\colon O(M) \to O(C^j)$ such that RT is the identity on $O(M)$ ([11]; Section 6.5).

COROLLARY. *If M is a Stein manifold of dimension d and if $O(M)$ has a basis, then an exact sequence of type $(*)$ splits if and only if $O(M)$ is isomorphic to $O(C^d)$.*

Proof. If $(*)$ splits, then $O(M)$ cannot contain a complemented subspace isomorphic to a power series space of finite type. This follows from a result of Zahariuta [20] which states that a continuous linear operator from a finite type power series space into an infinite type power series space is always compact. Hence $O(M)$ is isomorphic to $O(C^d)$ by Theorem 2. The converse is given in [11]; Prop. 6.4.

Remark: Let V be an algebraic variety in C^n, i.e. $V = \{z \in C^n: f_i(z) = 0,\ i = 1, 2, \ldots, k\}$ where each f_i is a polynomial in n variables. Djakov and Mitiagin [2] proved that $O(V)$ is isomorphic to $O(C^d)$, where d is equal to the dimension of V. An alternate proof can be given as follows:

We write V as a finite union of manifolds M_i with $\max(\dim M_i) = d$ ([19]; Theorem 66, p. 93). Applying Lemma 4 to each M_i and using the stability of $O(\Delta^s)$, we imbed $O(V)$ into $O(\Delta^d)$. It is shown in [2] that $O(V)$ is isomorphic to a subspace of $O(C^n)$ with basis and so by Theorem 1 $O(V)$ is isomorphic to a subspace of $O(C^d)$. The diametral dimension argument used in the proof of Theorem 2 can be repeated to obtain that $O(V)$ is isomorphic to $O(C^d)$ itself.

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An extension theorem of functionals on commutative semigroups

by

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Abstract. Generalizing the Sandwich Theorem, we give an extension theorem of additive functions into $[-\infty, +\infty]$ on commutative semigroups. Several results including some of R. Kaufman's results are derived from it.

The Hahn–Banach type theorem on commutative semigroups has been studied by many authors. The most general and efficient result might be what is called the *Sandwich Theorem*, which is a generalization of the Mazur–Orlicz theorem [7] and was proved originally by Kaufman [3] and established by Kranz [6] and Fuchssteiner [1]. Especially Fuchssteiner deduced many related results from it.

In this note we will generalize the Sandwich Theorem and give an extension theorem of functionals on commutative semigroups, from which several known results follow naturally.

Let G be a commutative semigroup with a compatible quasiorder σ, that is, σ is a reflexive and transitive relation satisfying that $x\sigma y \Rightarrow xz\sigma yz$ for $x, y, z \in G$. $\boldsymbol{R}$ denotes the real line and $\overline{\boldsymbol{R}} = \boldsymbol{R} \cup \{-\infty, +\infty\} = [-\infty, +\infty]$ is the additive (partial) semigroup of the real numbers equipped with the negative and the positive infinite; the addition in $\boldsymbol{R}$ is extended naturally to $\overline{\boldsymbol{R}}$, but note that $(+\infty)+(-\infty)$ and $(-\infty)+(+\infty)$ are not defined.

An equation or an inequality in $\overline{\boldsymbol{R}}$ is understood to hold if it holds as far as every addition contained in it is defined. For example, we say that $a+b = c$ $(a, b, c \in \overline{\boldsymbol{R}})$ holds if either $a+b$ is defined and the equation holds or $a+b$ is not defined.

Let f be a function of G into $\overline{\boldsymbol{R}}$. f is called *additive* if $f(xy) = f(x)+f(y)$ for all $x, y \in G$ (as far as $f(x)+f(y)$ is defined; we will omit this kind of comment hereafter). f is called *subadditive* (resp. *superadditive*) if $f(xy) \leqslant f(x)+f(y)$ (resp. $f(xy) \geqslant f(x)+f(y)$) for all $x, y \in G$. f is called *monotone* if $x\sigma y \Rightarrow f(x) \leqslant f(y)$, for all $x, y \in G$.

The pointwise order of functions of G is denoted by $\leqslant$, that is, for functions f and g of G, $f \leqslant g$ means $f(x) \leqslant g(x)$ for all $x \in G$. The constant function with a value $a \in \overline{\boldsymbol{R}}$ is simply denoted by a.