STUDIA MATHEMATICA, T. LXX. (1981)

Characterization of (p, q, r)-absolutely summing operators on Hilbert space

by

DEBORAH F. ALLINGER (Cambridge, Mass.)

Abstract. A complete description for all classes of (p, q, r)-absolutely summing operators on Hilbert space is given. This completes a problem of Pietsch [7].

0. Introduction. Following Pietsch [7], we say that a bounded linear operator T between Banach spaces X and Y is (p, q, r)-*absolutely summing*, $1 \leqslant p, q, r \leqslant \infty$ and $1/p \leqslant 1/q + 1/r$, provided that the following condition holds.

(0.1) There exists a constant M, independent of n, such that for all finite subsets $\{x_1, \ldots, x_n\} \subset X$ and $\{g_1, \ldots, g_n\} \subset Y^*$ we have

$$\Big(\sum_{i=1}^{n} |\langle Tx_i, g_i\rangle|^p\Big)^{1/p} \leqslant M \cdot \sup_{\|f\|_{X^*} \leqslant 1} \Big(\sum_{i=1}^{n} |\langle x_i, f\rangle|^q\Big)^{1/q} \cdot \sup_{\|y\|_Y \leqslant 1} \Big(\sum_{i=1}^{n} |\langle y, g_i\rangle|^r\Big)^{1/r}.$$

Condition (0.1) is equivalent to

(0.2) $\Big(\sum_{i=1}^{\infty} |\langle Tx_i, g_i\rangle|^p\Big)^{1/p} < \infty$ whenever $\{x_i\}$ and $\{g_i\}$ are sequences from X and Y^* which satisfy

$$\Big(\sum_{i=1}^{\infty} |\langle x_i, f\rangle|^q\Big)^{1/q} < \infty \quad \text{for each } f \text{ in } X^*$$

and

$$\Big(\sum_{i=1}^{\infty} |\langle y, g_i\rangle|^r\Big)^{1/r} < \infty \quad \text{for each } y \text{ in } Y.$$

Interest in these operators developed from the study of (p, q)-absolutely summing operators [5], [4], [3], [1], and the problem we present is the characterization of (p, q, r)-absolutely summing operators on Hilbert space. Our results are contained in Section 1, but we describe known results below. For the statement of these facts, we denote by $\boldsymbol{G}_r$ ($\boldsymbol{G}_{u,v}$) the set of all bounded linear operators on l_2 which admit a factorization, UVW, where W is a unitary operator, U is an isometry on the range of V, and V

is a diagonal operator from l_r $(l_{u,v})$. The ideal, $\boldsymbol{G}_r$, is the *Schatten r-class* [8], and $\boldsymbol{G}_{u,v}$ was described by Bennett [1]. The collection of all (p, q, r)- and (p, q)-absolutely summing operators from X into Y are $\Pi_{p,q,r}(X, Y)$ and $\Pi_{p,q}(X, Y)$, respectively, while $\mathscr{B}(X, Y)$ represents all bounded linear operators.

(i) $\Pi_{p_1,q_1,r_1}(X, Y) \subseteq \Pi_{p_2,q_2,r_2}(X, Y)$ whenever $p_1 \leqslant p_2$, $q_1 \geqslant q_2$, $r_1 \geqslant r_2$ or $p_1 \leqslant p_2$, $q_1 \leqslant q_2$, $r_1 \leqslant r_2$ and $1/p_1 - 1/q_1 - 1/r_1 \geqslant 1/p_2 - - 1/q_2 - 1/r_2$.

(ii) $\Pi_{p,q,r}(l_2, l_2) = \Pi_{p,r,q}(l_2, l_2)$.

(iii) $\Pi_{p,q,r}(l_2, l_2) = \Pi_{S,2,r}(l_2, l_2)$ for $1 \leqslant q \leqslant 2$, $1 \leqslant r \leqslant \infty$ and $1/s = 1/p - 1/q + 1/2$.

(iv) $$\Pi_{p,q}(l_2, l_2) = \begin{cases} \mathscr{B}(l_2, l_2) & \text{for} \quad 1 \leqslant q \leqslant 2, 1/p - 1/q + 1/2 \leqslant 0, \\ \boldsymbol{G}_k & \text{for} \quad 1 \leqslant q \leqslant 2, 1/k = 1/p - 1/q + \\ & \qquad\qquad + 1/2 > 0, \\ \boldsymbol{G}_{2p/q,p} & \text{for} \quad 2 < q < p < \infty. \end{cases}$$

(v) $$\Pi_{k,2,2}(l_2, l_2) = \begin{cases} \mathscr{B}(l_2, l_2) & \text{for} \quad k = \infty, \\ \boldsymbol{G}_k(l_2, l_2) & \text{for} \quad 1 \leqslant k < \infty. \end{cases}$$

Combining (iv) and (v) we have

(vi) $\Pi_{p,q}(l_2, l_2) = \Pi_{k,2,2}(l_2, l_2)$ for $1 \leqslant q \leqslant 2$, $1/k = \max\{1/p - 1/q + +1/2;\ 0\}$.

Results (i), (ii), (v) are due to Pietsch [6], [7]; (iii) to Mitiagin [3]; (iv) to Kwapień [3] and Bennett [1].

1. Characterization of $\Pi_{p,q,r}(l_2, l_2)$.

I. *Suppose* $q \leqslant 2$, $r \leqslant 2$.

(a) *If* $1 + 1/p \leqslant 1/r + 1/q$, $\Pi_{p,q,r} = \mathscr{B}(l_2, l_2)$.

(b) *If* $1 + 1/p > 1/r + 1/q$, $\Pi_{p,q,r} = \boldsymbol{G}_k$ *where* $1/k = 1/p - 1/q + 1/r^*$.

II. *Suppose* $q \leqslant 2$, $r > 2$.

(a) *If* $1/p \leqslant 1/q - 1/2$, $\Pi_{p,q,r} = \mathscr{B}(l_2, l_2)$.

(b) *If* $1/p > 1/q - 1/2$, $p \geqslant q$, $\Pi_{p,q,r} = \boldsymbol{G}_s$ *where* $1/s = 1/p - 1/q + 1/2$.

(c) *If* $1/p > 1/q - 1/2$, $p < q$, $\Pi_{p,q,r} = \boldsymbol{G}_{k,s}$ *where* $1/s = 1/p - 1/q + 1/2$ *and* $1/k = 1/2 + \frac{1}{2}r(1/s - 1/2)$.

The results for $q > 2$, $r \leqslant 2$ are obtained from II and (ii).

III. *Suppose* $q > 2$, $r > 2$ *and let* $m = \min(q, r)$, $M = \max(q, r)$.

(a) *If* $p \leqslant m$, *then* $\Pi_{p,q,r} = \boldsymbol{G}_{k,p}$ *where* $1/k = 1/2 + \frac{1}{2}M(1/p - 1/m)$.

(b) *If* $p > m$, *then* $\Pi_{p,q,r} = \boldsymbol{G}_{2p/m,p}$.

Proof of I. For (a), let p_1 and p_2 be given by the the equations $1/p_1 = 1/q - 1/2$, $1/p_2 = 1/r - 1/2$. Since $1/p \leqslant 1/p_1 + 1/p_2$, we apply Hölder's inequality to (0.1) to show that $T \in \Pi_{p,q,r}$ whenever $T \in \Pi_{p_1,q}$ and $I \in \Pi_{p_2,r}$. Hence I(a) follows from (iv).

For (b), apply (ii) and (iii) to see that

$$\Pi_{p,q,r}(l_2, l_2) = \Pi_{(s,2,r)} = \Pi_{(s,r,2)} = \Pi_{k,2,2}$$
$$\text{where} \quad 1/k = 1/s - 1/r + 1/2;\ 1/S = 1/p - 1/q + 1/2.$$

The assertion now follows from (v).

Proof of II. Results (a) and (b) are a consequence of (i), (iii), and (vi) because

$$\Pi_{p,q}(l_2, l_2) \subseteq \Pi_{(p,q,r)} \subseteq \Pi_{K,2,r} \subseteq \Pi_{K,2,2}; \qquad 1/K = 1/p - 1/q + 1/2.$$

II (c) follows from (iii) and III (a).

The interesting case from the technical point of view is III (a) which we prove now.

Proof of III (a). We begin by showing $\boldsymbol{G}_{K,p} \subseteq \Pi_{p,q,r}$. Without loss of generality, we may assume that $T \in \boldsymbol{G}_{K,p}$ is a diagonal operator, λ, where λ is a decreasing sequence of non-negative real numbers in $l_{K,p}$. We observe from (0.2) that $T = \lambda$ is (p, q, r)—absolutely summing if the *Schur product*, $\mathscr{X} * \mathscr{G}$ is a bounded linear operator from $l_{k,p}$ into l_p whenever $\mathscr{X} \in \mathscr{B}(l_2, l_q)$ and $\mathscr{G} \in \mathscr{B}(l_2, l_r)$. The Schur product, $\mathscr{X} * \mathscr{G}$, is a matrix whose (i, j) entry is $x_{ij} g_{ij}$. Under our hypotheses, $1/m \leqslant 1/p \leqslant 1/M + 1/m$, and using Hölder's inequality it follows that

$$\mathscr{X} * \mathscr{G} \in \mathscr{B}(l_1, l_{Mm/(M+m)})$$

and

$$\mathscr{X} * \mathscr{G} \in \mathscr{B}(l_2, l_m).$$

Interpolation yields

$$\mathscr{X} * \mathscr{G} \in \mathscr{B}(l_{k,p}, l_p)$$

with $1/k = 1/2 + \frac{1}{2}M(1/p - 1/m)$. Therefore $\boldsymbol{G}_{k,p} \subseteq \Pi_{p,q,r}$.

To establish the inclusion $\Pi_{p,q,r} \subseteq \boldsymbol{G}_{k,p}$, we construct matrices $\mathscr{X}$ and $\mathscr{G}$ such that

$$\sum_{t=1}^{\infty} t^{p/k-1} (\lambda_t)^p < \infty \tag{1.1}$$

because (1.1) implies that $\lambda \in l_{k,p}$ [1]. As a consequence of Theorem 1 [2], we may choose for each positive integer t, $m_t \times n_t$ finite matrices A_t and B_t of ± 1's with the following properties:

(a) $m_t = [2^{tM/2}]$; $n_t = 2^t$,

(b) $A_t = B_t$,

(c) $\|A_t\|_{2,M} \leqslant K(M) \max(m_t^{1/M}, n_t^{1/2}) \leqslant K(M) \cdot 2^{t/2}$,

(d) $\|B_t\|_{2,m} \leqslant K(m) \cdot \max(m_t^{1/m}, n_t^{1/2}) \leqslant K(m) \cdot 2^{t/2} \cdot M/m$.

Note that the constants $K(M)$, $K(m)$ are independent of t.

Define

$$\mathscr{X}_t = A_t/2^{t/2}$$

and

$$\mathscr{G}_t = B_t/2^{t/2}\cdot M/m$$

and consider the associated *block diagonal* matrices $\mathscr{X} = \sum_{t=1}^{\infty} \mathscr{X}_t$, $\mathscr{G} = \sum_{t=1}^{\infty} \mathscr{G}_t$ where the summation is performed coordinate-wise. From [1], it follows that

$$\|\mathscr{X}\|_{2,M} \leqslant K(M)$$

and

$$\|\mathscr{G}\|_{2,M} \leqslant K(m).$$

Moreover, (0.2) becomes

$$\infty > \sum_{t=0}^{\infty} 2^{t(M/2)}(2^{M/2}-1)\left(\sum_{K=2^t}^{2^{t+1}-1} \lambda_K \frac{1}{2^{t/2}}\cdot\frac{1}{2^{tM/2m}}\right)^p$$

and, because λ_K is decreasing, we conclude that

$$\sum_{t=1}^{\infty} 2^{tp/k}\lambda_{2^t}^p < \infty \tag{1.2}$$

with $1/K = 1/2+\frac{1}{2}M(1/P-1/m)$. Now (1.1) follows from (1.2) because, for each t, the partial sum

$$2^{t(p/k-1)}(\lambda_{2^t})^p + \ldots + (2^{t+1}-1)^{(p/k-1)}(\lambda_{2^{t+1}-1})^p$$

from (1.1) is bounded by the term

$$2^{tp/k}(\lambda_{2^t})^p\cdot 2^{p/k-1}$$

of (1.2). This completes the inclusion.

Proof of III (b). Letting $m = \min(q, r)$, we have from (i), (iii) and (iv) that

$$G_{2p/m,p} = \Pi_{(p,m)}(l_2, l_2) = \Pi_{(p,m,\infty)} = \Pi_{(p,\infty,m)} \subseteq \Pi(p, q, r).$$

Constructing the block diagonal matrices $\mathscr{X}$ and $\mathscr{G}$ with $m_t = [2^{tm/2}]$, $n_t = 2^t$ permits the inclusion $\Pi_{p,q,r} \subset G_{2p/m,p}$. Our characterization is completed.

References

[1] G. Bennett, *Some ideals of operators on Hilbert space*, Studia Math. 55 (1967), pp. 27–40.

[2] G. Bennett, V. Goodman, and C. M. Newman, *Norms of random matrices*, Pacific J. Math. 59 (1975), pp. 359–365.

[3] S. Kwapień, *Some remarks on (p, q)-absolutely summing operators in l_p-spaces*, Studia Math. 21 (1968), pp. 327–337.

[4] A. Pełczyński, *A characterization of Hilbert Schmidt operators*, ibid. 28 (1967), pp. 355–360.

[5] A. Pietsch, *Absolut p-summierende Abbildungen in normierten Raümen*, ibid. 28 (1967), pp. 333–353.

[6] — *Operator ideals*, Berlin 1978.

[7] — *Theorie der Operatoren Ideale*, Jena 1972.

[8] R. Schatten, *Norm ideals of completely continuous operators*, Berlin 1960.

Received October 6, 1978 (1470)
Revised version March 6, 1979