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L^p-integrability $(1 \leqslant p \leqslant \infty)$ of a class of integral transforms

by

YOSHIMITSU HASEGAWA (Hirosaki)

Abstract. For $1 \leqslant p < \infty$, K. Soni and R. P. Soni [5] and the author [3] proved some L^p-integrability theorems for a class of integral transforms, where it includes the Hankel transform and so on. In the case $1 \leqslant p \leqslant \infty$, we generalize their theorems.

1. Definitions. Throughoutt his paper, we assume that the function $k(t)$ is real-valued, measurable and uniformly bounded in $0 \leqslant t < \infty$, and that

$$(1.1)\quad k(t) = \begin{cases} k(0) + Bt^\beta + o(t^\beta) & \text{as} \quad t \to +0 \quad \text{for} \quad \beta > 0, \text{ where } B \neq 0, \\ k(0) + o(1) & \text{as} \quad t \to +0 \quad \text{for} \quad \beta = 0. \end{cases}$$

As in [3], we define the k-transform as follows: if the function $f(t)$ is real-valued in $0 < t < \infty$ and is of bounded variation in $T \leqslant t < \infty$ for every $T > 0$, and if $\int_0^1 |k(t) - k(0)|\,|df(t)| < \infty$, then the function $F(x)$ is defined by

$$F(x) = \int_0^\infty \{k(xt) - k(0)\}\, df(t), \qquad 0 < x < \infty,$$

and denotes the k-transform of $f(t)$.

For $\beta > 0$, the condition $\int_0^1 |k(t) - k(0)|\,|df(t)| < \infty$ is equivalent to

$$\int_0^1 t^\beta |df(t)| < \infty$$

by (1.1).

It is known by K. Soni and R. P. Soni [6] (also see [3] and [5]) that the k-transform includes the Hankel transform and so on.

In [3], we considered the three cases that $k(t)$ satisfies respectively the additional conditions (K1), (K2) or (K3) as follows.

CONDITION (K1):

(i) $k(0) = 0$ for $\beta > 0$, and $k(0) \neq 0$ for $\beta = 0$,

(ii) $k_1(t) = \int_0^t k(u)\,du$ is uniformly bounded in $0 \leqslant t < \infty$.

CONDITION (K2): there exists a function $\omega(x)$ such that

(i) $\omega(x) \in L(0,1)$, $x^\beta \omega(x) \in L(1,\infty)$ and $\int_0^\infty x^\beta \omega(x)\,dx \neq 0$ for $\beta > 0$,

(ii) $k^*(y) - k^*(0)$ has no change of sign in $0 < y < \infty$, where

$$k^*(y) = \int_0^\infty \omega(x) k(xy)\,dx, \qquad 0 \leqslant y < \infty. \tag{1.2}$$

CONDITION (K3):

(i) $k(t)$ satisfies Condition (K1),

(ii) there exists a function $\omega_1(x)$ such that

(a) $\omega_1(x) \in L(0,1)$, $x^{\beta+1}\omega_1(x) \in L(1,\infty)$ and $\int_0^\infty x^{\beta+1}\omega_1(x)\,dx \neq 0$ for $\beta \geqslant 0$,

(b) $k_1^*(y)$ has no change of sign in $0 < y < \infty$, where

$$k_1^*(y) = \int_0^\infty \omega_1(x) k_1(xy)\,dx, \qquad 0 \leqslant y < \infty.$$

Throughout this paper, we put

$$1/p + 1/q = 1, \qquad 1 \leqslant p, q \leqslant \infty.$$

A positive and measurable function $M(u)$ in $0 < u < \infty$ is said to *belong to the class* S, i.e. $M(u) \in S$, if there exist two positive constants $H_1 = H_1(\delta, \Delta)$ and $H_2 = H_2(\delta, \Delta)$ satisfying

$$H_1 \leqslant M(u) \leqslant H_2 \quad \text{in} \quad \delta \leqslant u \leqslant \Delta \text{ for every } \delta \text{ and } \Delta,\ 0 < \delta < \Delta,$$

and if

$$\lim_{u\to+0} \frac{M(\zeta u)}{M(u)} = \lim_{u\to\infty} \frac{M(\zeta u)}{M(u)} = 1 \quad \text{for every } \zeta > 0.$$

It is clear that $M(u)$ and $M(1/u)$ are slowly varying functions in $1 \leqslant u < \infty$ (see [2]). Therefore, for fixed δ and Δ $(0 < \delta < \Delta)$,

$$\lim_{u\to+0} \frac{M(\zeta u)}{M(u)} = \lim_{u\to\infty} \frac{M(\zeta u)}{M(u)} = 1 \quad \text{uniformly for } \zeta \in [\delta, \Delta], \tag{1.3}$$

and, for every $\alpha > 0$,

$$\lim_{u\to+0} u^{-\alpha} M(u) = \lim_{u\to\infty} u^\alpha M(u) = \infty \quad \text{and} \tag{1.4}$$

$$\lim_{u\to+0} u^\alpha M(u) = \lim_{n\to\infty} u^{-\alpha} M(u) = 0$$

(see [2]).

The letter C, with or without subscript, denotes a positive constant, not necessarily the same at each appearance.

2. Main results.

THEOREM 1. *Let $\beta > 0$. Suppose that $f(t)$ is defined in $0 < t < \infty$ and is of bounded variation in $T \leqslant t < \infty$ for every $T > 0$, and that $\int_0^1 t^\beta |df(t)| < \infty$. Moreover, suppose that $\varphi(u)$ is a measurable function in $0 < u < \infty$ such that*

$$\sup_{r>0} \Big\{\int_0^r |\varphi(u) u^\beta|^p du\Big\}^{1/p} \Big\{\int_r^\infty |\varphi(u) u^{\beta+1}|^{-q} du\Big\}^{1/q} < \infty. \tag{2.1}$$

If

$$\varphi(t) \int_0^t x^\beta |df(x)| \in L^p(0,\infty),$$

then $\varphi(1/x)^{-\beta-2/p} F(x) \in L^p(0,\infty)$.

Remark 1. In Theorem 1 and throughout this paper, $0\cdot\infty$ is to be taken as 0 (for instance in (2.1)), and the symbol L^∞ (the case $p = \infty$) is to be taken as ess sup.

Remark 2. It is easily seen that Theorem 1 holds for $\varphi(t) = t^{\gamma-\beta-2/p} M(t)$, where $1/p < \gamma < \beta + 1/p$ and $M(t) \in S$. In particular, in case $M(t) = 1$ for $t > 0$, that is, $\varphi(t) = t^{\gamma-\beta-2/p}$, Theorem 1 for $1 \leqslant p < \infty$ coincides with [3], Theorem 1.

As a corollary of Theorem 1, we have a theorem as follows.

THEOREM 2. *Let $\beta \geqslant 0$. Let $k(t)$ satisfy Condition* (K1). *Suppose that $g(t)$ decreases to zero in $0 < t < \infty$, and that $t^\beta g(t) \in L(0,1)$. Let*

$$\bar{G}(x) = \int_0^\infty k(xt) g(t)\,dt, \qquad 0 < x < \infty. \tag{2.2}$$

Moreover, suppose that $\varphi_1(u)$ is a measurable function in $0 < u < \infty$ such that

$$\sup_{r>0} \Big\{\int_r^\infty |\varphi_1(u) u^{-\beta-1}|^p du\Big\}^{1/p} \Big\{\int_0^r |\varphi_1(u) u^{-\beta}|^{-q} du\Big\}^{1/q} < \infty \tag{2.3}$$

and

$$(2.4)\qquad \sup_{r>0}\Big\{\int_0^r |\varphi_1(u)|^p\,du\Big\}^{1/p}\Big\{\int_r^\infty |\varphi_1(u)u|^{-q}\,du\Big\}^{1/q} < \infty.$$

If $\varphi_1(t)g(t)\in L^p(0,\infty)$, *then* $\varphi_1(1/x)x^{1-2/p}\bar{G}(x)\in L^p(0,\infty)$.

Remark 3. It was proved by the author [3], Theorem 3, that the integral in (2.2) converges for every $x>0$.

Remark 4. It is easily seen that Theorem 2 holds for

$$\varphi_1(t) = t^{\gamma+1-2/p}M(t),$$

where $-1/q<\gamma<\beta+1/p$ and $M(t)\in S$. In particular, in case $M(t)=1$ for $t>0$, that is, $\varphi_1(t) = t^{\gamma+1-2/p}$, Theorem 2 for $1\leqslant p<\infty$ coincides with [3], Theorem 3.

As the inverse case to Theorem 1, we have the following theorem.

THEOREM 3. *Let* $\beta>0$. *Let* $k(t)$ *satisfy Condition* (K2). *Suppose that* $f(t)$ *is monotone in* $0<t<\infty$ *and tends to a finite value as* $t\to\infty$, *and that* $\int_0^1 t^\beta|df(t)|<\infty$. *Moreover, suppose that* $\psi(x)$ *is a measurable function in* $0<x<\infty$ *such that*

(i) $|\psi(x)| = Cx^{-\gamma}M(x)+o\big(x^{-\gamma}M(x)\big)$ *as* $x\to+0$ *or* $x\to\infty$ *for* $1/p<\gamma<\beta+1/p$ *and* $M(x)\in S$,

(ii) *there exist two positive constants* $H_3=H_3(\delta,\Delta)$ *and* $H_4=H_4(\delta,\Delta)$ *satisfying* $H_3\leqslant|\psi(x)|\leqslant H_4$ *in* $\delta\leqslant x\leqslant\Delta$ *for every* δ *and* Δ, $0<\delta<\Delta$.

If $\psi(x)F(x)\in L^p(0,\infty)$, *then* $\psi(1/t)t^{-\beta-2/p}\int_0^t x^\beta\,df(x)\in L^p(0,\infty)$.

As a corollary of Theorem 3, we have the following theorem inverse to Theorem 2.

THEOREM 4. *Let* $\beta\geqslant0$. *Let* $k(t)$ *satisfy Conditions* (K1) *and* (K3). *Suppose that* $g(t)$ *decreases to zero in* $0<t<\infty$, *and that* $t^\beta g(t)\in L(0,1)$. *Let* $\bar{G}(x)$ *be defined as in* (2.2). *Moreover, suppose that* $\psi_1(x)$ *is a measurable function in* $0<x<\infty$ *such that*

(i) $|\psi_1(x)| = Cx^{-\gamma}M(x)+o\big(x^{-\gamma}M(x)\big)$ *as* $x\to+0$ *or* $x\to\infty$ *for* $-1/q<\gamma<\beta+1/p$ *and* $M(x)\in S$,

(ii) *there exist two positive constants* $H_5=H_5(\delta,\Delta)$ *and* $H_6=H_6(\delta,\Delta)$ *such that*

$$H_5\leqslant|\psi_1(x)|\leqslant H_6\quad \text{in}\quad \delta\leqslant x\leqslant\Delta \text{ for every } \delta \text{ and } \Delta,\ 0<\delta<\Delta.$$

If $\psi_1(x)\bar{G}(x)\in L^p(0,\infty)$, *then* $\psi_1(1/t)t^{1-2/p}g(t)\in L^p(0,\infty)$.

Remark 5. In case $M(x)=1$ for $x>0$, Theorems 3 and 4 for $1\leqslant p<\infty$ are equivalent to [3], Theorems 4 and 5, respectively.

3. Proofs of Theorems 1 and 2. In order to prove Theorems 1 and 2, we need first the basic lemma as follows.

LEMMA 1. *Let three functions* $h(x)$, $\varrho(x)$ *and* $\sigma(x)$ *be measurable in* $0<t<\infty$, *and let* $\sigma(x)h(x)\in L^p(0,\infty)$. *Then we have the following two results.*

(i) *There is a finite* D *for which the inequality*

$$(3.1)\qquad \Big\{\int_0^\infty\Big|\varrho(x)\int_x^\infty h(t)\,dt\Big|^p dx\Big\}^{1/p}\leqslant D\Big\{\int_0^\infty|\sigma(x)h(x)|^p\,dx\Big\}^{1/p}$$

holds if and only if

$$\Omega = \sup_{r>0}\Big\{\int_0^r|\varrho(x)|^p\,dx\Big\}^{1/p}\Big\{\int_r^\infty|\sigma(x)|^{-q}\,dx\Big\}^{1/q}<\infty.$$

Furthermore, if D *is the least constant for which* (3.1) *holds, then* $\Omega\leqslant D\leqslant p^{1/p}q^{1/q}\Omega$ *for* $1<p<\infty$ *and* $\Omega=D$ *for* $p=1$ *or* ∞.

(ii) *There is a finite* D' *for which the inequality*

$$(3.2)\qquad \Big\{\int_0^\infty\Big|\varrho(x)\int_0^x h(t)\,dt\Big|^p dx\Big\}^{1/p}\leqslant D'\Big\{\int_0^\infty|\sigma(x)h(x)|^p\,dx\Big\}^{1/p}$$

holds if and only if

$$\Omega' = \sup_{r>0}\Big\{\int_r^\infty|\varrho(x)|^p\,dx\Big\}^{1/p}\Big\{\int_0^r|\sigma(x)|^{-q}\,dx\Big\}^{1/q}<\infty.$$

Furthermore, if D' *is the least constant for which* (3.2) *holds, then* $\Omega'\leqslant D'\leqslant p^{1/p}q^{1/q}\Omega'$ *for* $1<p<\infty$ *and* $\Omega'=D'$ *for* $p=1$ *or* ∞.

Lemma 1 is due to B. Muckenhoupt [4]. In order to prove Theorems 1 and 4, we need the following lemma.

LEMMA 2. *Let* $s>0$. *Suppose that* $\lambda(u)$ *increases in* $0\leqslant u<\infty$, *and that* $\lambda(+0)=0$. *Moreover, suppose that* $U(x)$ *and* $V(x)$ *are two non-negative and measurable functions in* $0<x<\infty$ *such that*

$$(3.3)\qquad \sup_{r>0}\Big\{\int_0^r U(x)^p\,dx\Big\}^{1/p}\Big\{\int_r^\infty\big(V(x)x^{s+1}\big)^{-q}\,dx\Big\}^{1/q}<\infty.$$

If $V(u)\lambda(u)\in L^p(0,\infty)$, *then the inequality*

$$\Big\{\int_0^\infty\Big(U(u)\int_u^\infty x^{-s}\,d\lambda(x)\Big)^p du\Big\}^{1/p}\leqslant C\Big\{\int_0^\infty\big(V(u)\lambda(u)\big)^p\,du\Big\}^{1/p}<\infty$$

holds.

Proof. If $\big(V(u)u^{s+1}\big)^{-1}\notin L^q(R,\infty)$ for any $R>0$, then (3.3) implies that $U(x)=0$ almost everywhere in $0<x<\infty$ and the lemma is trivial. Now, we assume $\big(V(u)u^{s+1}\big)^{-1}\in L^q(R,\infty)$ for any $R>0$. For $u>R$,

$V(u) > 0$ almost everywhere and the integrability of $|V(u)\lambda(u)|^p$ shows that $\lambda(u) = 0$ at almost every point where $V(u) = \infty$. Therefore, for almost every $u > R$, the equality $\lambda(u) = V(u)\lambda(u)(V(u))^{-1}$ holds. By assumption, $\lambda(u)$ is non-negative and increasing in $0 < u < \infty$. Then, for $u > R$,

$$u^{-s}\lambda(u) \leqslant s\int_u^\infty u^{-s-1}\lambda(u)\,du = s\int_u^\infty \big(V(u)\lambda(u)\big)\big(V(u)u^{s+1}\big)^{-1}du$$

$$\leqslant s\Big\{\int_u^\infty \big(V(u)\lambda(u)\big)^p du\Big\}^{1/p}\Big\{\int_u^\infty \big(V(u)u^{s+1}\big)^{-q}du\Big\}^{1/q} \to 0 \quad \text{as} \quad u\to\infty$$

(use Hölder's inequality for $1 < p < \infty$). Hence, by integration by parts,

$$\int_u^\infty x^{-s}d\lambda(x) = -u^{-s}\lambda(u) + s\int_u^\infty x^{-s-1}\lambda(x)\,dx \leqslant s\int_u^\infty x^{-s-1}\lambda(x)\,dx.$$

Now, when we put

$$\varrho(x) = U(x), \qquad \sigma(x) = V(x)x^{s+1} \qquad \text{and} \qquad h(x) = x^{-s-1}\lambda(x)$$

in Lemma 1(i), we get, under condition (3.3),

$$\Big\{\int_0^\infty \Big(U(u)\int_u^\infty x^{-s}d\lambda(x)\Big)^p du\Big\}^{1/p} \leqslant s\Big\{\int_0^\infty \Big(U(u)\int_u^\infty x^{-s-1}\lambda(x)\,dx\Big)^p du\Big\}^{1/p}$$

$$\leqslant C\Big\{\int_0^\infty \big(V(u)\lambda(u)\big)^p du\Big\}^{1/p}.$$

Thus Lemma 2 is proved.

Proof of Theorem 1. When we set

$$\lambda(u) = \int_0^u x^\beta|df(x)|, \qquad s = \beta, \qquad U(u) = |\varphi(u)u^\beta| \qquad \text{and} \qquad V(u) = |\varphi(u)|$$

in Lemma 2, we have, under condition (2.1),

$$\Big\{\int_0^\infty \Big(|\varphi(u)u^\beta|\int_u^\infty |df(t)|\Big)^p du\Big\}^{1/p} \leqslant C\Big\{\int_0^\infty \Big(|\varphi(u)|\int_0^u t^\beta|df(t)|\Big)^p du\Big\}^{1/p}.$$

Hence, by assumptions on $k(t)$,

$$\Big\{\int_0^\infty |\varphi(1/x)x^{-\beta-2/p}F(x)|^p dx\Big\}^{1/p}$$

$$\leqslant \Big\{\int_0^\infty \Big(|\varphi(1/x)x^{-\beta-2/p}|\int_0^\infty |k(xt)-k(0)|\,|df(t)|\Big)^p dx\Big\}^{1/p}$$

$$\leqslant C_1\Big\{\int_0^\infty \Big(|\varphi(1/x)x^{-2/p}|\int_0^{1/x} t^\beta|df(t)|\Big)^p dx\Big\}^{1/p} +$$

$$+C_2\Big\{\int_0^\infty \Big(|\varphi(1/x)x^{-\beta-2/p}|\int_{1/x}^\infty |df(t)|\Big)^p dx\Big\}^{1/p}$$

$$= C_1\Big\{\int_0^\infty \Big(|\varphi(u)|\int_0^u t^\beta|df(t)|\Big)^p du\Big\}^{1/p} +$$

$$+C_2\Big\{\int_0^\infty \Big(|\varphi(u)u^\beta|\int_u^\infty |df(t)|\Big)^p du\Big\}^{1/p}$$

$$\leqslant C_3\Big\{\int_0^\infty \Big(|\varphi(u)|\int_0^\infty t^\beta|df(t)|\Big)^p du\Big\}^{1/p} < \infty.$$

Thus Theorem 1 is proved.

Proof of Theorem 2. In the proofs of [3], Theorems 2 and 3, we proved that $k_1(0) = 0$ and $k_1(t) = B_1t^{\beta+1} + o(t^{\beta+1})$ as $t \to +0$, where $B_1 \neq 0$, that

$$\int_0^t u^\beta g(u)\,du \geqslant \frac{1}{\beta+1}\int_0^t u^{\beta+1}|dg(u)|, \qquad 0 < t < \infty,$$

and that

$$x\bar{G}(x) = -\int_0^\infty k_1(xt)\,dg(t), \qquad 0 < x < \infty.$$

When we set

$$h(x) = x^\beta g(x), \qquad \varrho(x) = \varphi_1(x)x^{-\beta-1} \qquad \text{and} \qquad \sigma(x) = \varphi_1(x)x^{-\beta}$$

n Lemma 1(ii), we have, under condition (2.3),

$$\Big\{\int_0^\infty \Big(|\varphi_1(t)t^{-\beta-1}|\int_0^t x^\beta g(x)\,dx\Big)^p dt\Big\}^{1/p} \leqslant C\Big\{\int_0^\infty \big(|\varphi_1(t)|\,g(t)\big)^p dt\Big\}^{1/p}.$$

Hence

$$\varphi_1(t)t^{-\beta-1}\int_0^t x^{\beta+1}|dg(x)| \in L^p(0,\infty).$$

Now, when we replace β by $\beta+1$ and then put

$$f(t) = g(t), \qquad \varphi(t) = \varphi_1(t)t^{-\beta-1} \qquad \text{and} \qquad k(x) = k_1(x)$$

in Theorem 1, we have, under condition (2.4),

$$\varphi_1(1/x)x^{\beta+1}x^{-(\beta+1)-2/p}x\bar{G}(x) = \varphi_1(1/x)x^{1-2/p}\bar{G}(x) \in L^p(0,\infty).$$

Thus Theorem 2 is proved.

4. Proofs of Theorems 3 and 4. In order to prove Theorem 3, we need the following four lemmas.

LEMMA 3. *Suppose that $f(t)$ is non-negative and monotone in $0 < t \leqslant \varepsilon$ for some $\varepsilon > 0$, and that it is of bounded variation in $\varepsilon \leqslant t < \infty$. If $\int_0^1 t^\beta |df(t)| < \infty$ for $\beta > 0$, then*

$$\int_0^1 |k(xt)-k(0)|\,|df(t)| = \begin{cases} O(x^\beta) & \text{as} \quad x \to +0, \\ o(x^\beta) & \text{as} \quad x \to \infty; \end{cases}$$

$$\int_1^\infty |k(xt)-k(0)|\,|df(t)| = O(1) \quad \text{as} \quad x \to +0 \text{ or } x \to \infty.$$

LEMMA 4. *Let $\beta > 0$. Let $\omega(x)$ satisfy* (i) *of Condition* (K2), *and let $k^*(y)$ define as in* (1.2). *Then*

(i) *$k^*(y)$ is uniformly bounded in $0 \leqslant y < \infty$,*

(ii) *$k^*(y) - k^*(0) \sim B_2 y^\beta$ as $y \to +0$, where $B_2 \neq 0$.*

Lemmas 3 and 4 are due to [3], Lemmas 2 and 3, respectively.

LEMMA 5. *Let $M(u) \in S$. Then there is a function $M_0(u) \in S$ satisfying the following four properties:*

(i) *$M_0(u)$ is continuously differentiable in $0 < u \leqslant A_1$ and in $A_2 \leqslant u < \infty$ for some A_1 and A_2, $0 < A_1 \leqslant 1 \leqslant A_2$, and further*

$$\lim_{u\to+0} u\big(M_0(u)\big)^{-1}\frac{d}{du}M_0(u) = \lim_{u\to\infty} u\big(M_0(u)\big)^{-1}\frac{d}{du}M_0(u) = 0.$$

(ii) $$\lim_{u\to+0}\frac{M_0(u)}{M(u)} = \lim_{u\to\infty}\frac{M_0(u)}{M(u)} = 1.$$

(iii) *For every $\tau > 0$, $u^\tau M_0(u)$ is increasing and $u^{-\tau}M_0(u)$ is decreasing in $0 < u \leqslant \delta$ for sufficiently small $\delta = \delta(\tau, A_1) > 0$.*

(iv) *For every $\tau > 0$, $u^\tau M_0(u)$ is increasing and $u^{-\tau}M_0(u)$ is decreasing in $\Delta \leqslant u < \infty$ for sufficiently large $\Delta = \Delta(\tau, A_2) > 0$.*

We remark that $M(u)$ and $M(1/u)$ for $u \geqslant 1$ are slowly varying functions. J. Galambos and E. Seneta [2], pp. 111, pointed out that there is $M_0(u)$ in $1 \leqslant u < \infty$ such that it satisfies (i) and (ii) of Lemma 5. Further, R. Bojanić and J. Karamata [1], p. 14 and pp. 36–37, proved that such $M_0(u)$ satisfies Lemma 5(iv).

LEMMA 6. *Let $\beta > 0$ and $1/p < \gamma < \beta + 1/p$. Let $\psi(x)$ be a measurable function in $0 < x < \infty$ satisfying the conditions* (i) *and* (ii) *of Theorem* 3. *Moreover, let $\tau = \min\big((\gamma - 1/p)/2, (\beta + 1/p - \gamma)/2\big)$, and let A_1 and A_2 be defined as in Lemma 5. Then, for sufficiently small $\delta = \delta(\tau, A_1) > 0$ and sufficiently large $\Delta = \Delta(\tau, A_2) > 0$, we have the following four inequalities:*

$$(4.1) \qquad E_1 v^{-\gamma+\tau}|\psi(x)| \leqslant |\psi(xv)| \leqslant E_2 v^{-\gamma-\tau}|\psi(x)| \quad \text{in } 0 < x \leqslant \delta$$
and $0 < v \leqslant 1$, in $x > \Delta/v$ and $0 < v \leqslant 1$, and in $\Delta < x \leqslant \delta/v$,

$$(4.2) \qquad E_3 v^{-\gamma-\tau}|\psi(x)| \leqslant |\psi(xv)| \leqslant E_4 v^{-\gamma+\tau}|\psi(x)| \quad \text{in } 0 < x \leqslant \delta/v$$
and $v > 1$, in $x > \Delta$ and $v > 1$, and in $0 < x \leqslant \delta$ and $\Delta/x < v$,

$$(4.3) \qquad E_5 v^{-\gamma}\big(M(1/v)\big)^{-1}|\psi(x)| \leqslant |\psi(xv)| \leqslant E_6 v^{-\gamma}\big(M(1/v)\big)^{-1}|\psi(x)|$$
in $0 < x \leqslant \delta$ and $\delta/x < v \leqslant \Delta/x$, and in $x > \Delta$ and $\delta/x < v \leqslant \Delta/x$,

$$(4.4) \qquad E_7 v^{-\gamma}M(v)|\psi(x)| \leqslant |\psi(xv)| \leqslant E_8 v^{-\gamma}M(v)|\psi(x)| \quad \text{in } \delta < x \leqslant \Delta$$
and $v > 0$, where $E_i = E_i(\delta, \Delta) > 0$, $j = 1, 2, \ldots, 8$.

Proof. Throughout the proof of this lemma, let $\delta = \delta(\tau, A_1) > 0$ and $\Delta = \Delta(\tau, A_2) > 0$ be taken sufficiently small and sufficiently large, respectively. Let $M_0(u) \in S$ be the function as in Lemma 5. From Lemma 5(ii) and the definition of the class S, we have

$$(4.5) \qquad E_1' M_0(u) \leqslant M(u) \leqslant E_2' M_0(u) \quad \text{in} \quad 0 < u < \infty,$$

where $E_j' = E_j'(\delta, \Delta) > 0$, $j = 1, 2$.

First we show (4.1) in $\Delta < x \leqslant \delta/v$. By the condition (i) of Theorem 3, (4.5), and (iii) and (iv) of Lemma 5, we have, in $\Delta < x \leqslant \delta/v$,

$$\begin{aligned}|\psi(xv)| &\geqslant E_1'(xv)^{-\gamma+\tau}\{(xv)^{-\tau}M_0(xv)\} \geqslant E_1'(xv)^{-\gamma+\tau}\big(\delta^{-\tau}M_0(\delta)\big)\\ &= E_1'\delta^{-\tau}M_0(\delta)\big(x^{-\tau}M_0(x)\big)^{-1}v^{-\gamma+\tau}x^{-\gamma}M_0(x)\\ &\geqslant E_1'\delta^{-\tau}M_0(\delta)\big(\Delta^{-\tau}M_0(\Delta)\big)^{-1}v^{-\gamma+\tau}x^{-\gamma}M_0(x) \geqslant E_1 v^{-\gamma+\tau}|\psi(x)|\end{aligned}$$

and

$$\begin{aligned}|\psi(xv)| &\leqslant E_2'(xv)^{-\gamma-\tau}\{(xv)^{\tau}M_0(xv)\} \leqslant E_1(xv)^{-\gamma-\tau}\big(\delta^{\tau}M_0(\delta)\big)\\ &= E_2'\delta^{\tau}M_0(\delta)\big(x^{\tau}M_0(x)\big)^{-1}v^{-\gamma-\tau}x^{-\gamma}M_0(x)\\ &\leqslant E_2'\delta^{\tau}M_0(\delta)\big(\Delta^{\tau}M_0(\Delta)\big)^{-1}v^{-\gamma-\tau}x^{-\gamma}M_0(x) \leqslant E_2 v^{-\gamma-\tau}|\psi(x)|.\end{aligned}$$

Thus (4.1) in $\Delta < x \leqslant \delta/v$ is proved. The proof of (4.1) in the other sets are simpler than it. Moreover, the proof of (4.2) is similar to (4.1).

Secondly, we show (4.3) in $0 < x \leqslant \delta$ and $\delta/x < v \leqslant \Delta/x$. By the definition of the class S, we may assume

$$(4.6) \qquad E_3' M_0(x) \leqslant M_0(x/\Delta) \leqslant E_4' M_0(x) \quad \text{in} \quad 0 < x \leqslant \delta,$$

where $E_j' = E_j'(\delta, \Delta) > 0$, $j = 3, 4$. Hence, by (4.6), condition (ii) of

Theorem 3, (4.5) and Lemma 5(iii),

$$|\psi(xv)| \geqslant E_1' v^{-\gamma}\left(M_0\left(\frac{1}{v}\right)\right)^{-1}\left(\frac{xv}{\Delta}\right)^{\tau}\frac{M_0(x/\Delta)}{M_0(x)}\cdot\frac{(1/v)^{\tau}M_0(1/v)}{(x/\Delta)^{\tau}M_0(x/\Delta)}\,x^{-\gamma}M_0(x)$$

$$\geqslant E_1' v^{-\gamma}\left(M_0\left(\frac{1}{v}\right)\right)^{-1}\left(\frac{\delta}{\Delta}\right)^{\tau}E_3' x^{-\gamma}M_0(x) \geqslant E_5 v^{-\gamma}\left(M\left(\frac{1}{v}\right)\right)^{-1}|\psi(x)|$$

and

$$|\psi(xv)| \leqslant E_2' v^{-\gamma}\left(M_0\left(\frac{1}{v}\right)\right)^{-1}\left(\frac{\Delta}{xv}\right)^{\tau}\frac{M_0(x/\Delta)}{M_0(x)}\cdot\frac{(1/v)^{-\tau}M_0(1/v)}{(x/\Delta)^{-\tau}M_0(x/\Delta)}\,x^{-\gamma}M_0(x)$$

$$\leqslant E_2' v^{-\gamma}\left(M_0\left(\frac{1}{v}\right)\right)^{-1}\left(\frac{\Delta}{\delta}\right)^{\tau}E_4' x^{-\gamma}M_0(x) \leqslant E_6 v^{-\gamma}\left(M\left(\frac{1}{v}\right)\right)^{-1}|\psi(x)|.$$

Thus (4.3) in $0 < x \leqslant \delta$ and $\delta/x < v \leqslant \Delta/x$ is proved. The proof of (4.3) in $x > \Delta$ and $\delta/x < v \leqslant \Delta/x$ is similar to it.

Lastly we show (4.4). By the definition of the class S, and (1.3), we have

$$E_5' M(v) \leqslant M(xv) \leqslant E_6' M(v) \quad \text{in} \quad \delta < x \leqslant \Delta \text{ and } v > 0$$

and

$$E_7' \leqslant M(x) \leqslant E_8' \quad \text{in} \quad \delta < x \leqslant \Delta,$$

where $E_j' = E_j'(\delta, \Delta) > 0$, $j = 5, 6, 7, 8$. From this, the definition of the class S, the conditions (i) and (ii) of Theorem 3, and

$$\psi(xv) = \frac{1}{M(x)}\cdot\frac{M(xv)}{M(v)}\cdot\frac{x^{-\gamma}M(x)}{\psi(x)}\cdot\frac{\psi(xv)}{(xv)^{-\gamma}M(xv)}\big(v^{-\gamma}M(v)\big)\psi(x),$$

we get easily (4.4). Thus Lemma 6 is proved.

Proof of Theorem 3. As in the proof of [3], Theorem 4, we have, for $t > 0$,

$$\mu(t) = \psi(t)t^{-1}\int_0^\infty \omega(x/t)F(x)\,dx \tag{4.7}$$

$$= \psi(t)\int_0^\infty \omega(u)\,du\int_0^\infty \{k(tuy)-k(0)\}\,df(y) = \psi(t)\int_0^\infty \{k^*(ty)-k^*(0)\}\,df(y),$$

using Lemma 3, where $k^*(u)$ is defined as in (1.2). Let τ, δ and Δ be taken as in Lemma 6, and further let δ and Δ be fixed. Let

$$\eta(u) = u^{\gamma+\tau-1/p} + u^{\gamma-\tau-1/p} + u^{\gamma-1/p}\big(M(u)\big)^{-1} + u^{\gamma-1/p}M(1/u) \quad \text{in} \quad 0 < u < \infty.$$

It is clear that $0 < \gamma - 1/p < \beta$ and $0 < \gamma \pm \tau - 1/p < \beta$, and that $\big(M(u)\big)^{-1} \in S$ and $M(1/u) \in S$. By (i) of Condition (K2) and (1.4), we get $\omega(u)\eta(u) \in L(0, \infty)$.

First we put $p = \infty$. Then, by (4.7), (4.1)–(4.4), $\psi(x)F(x) \in L^\infty(0, \infty)$ and $\omega(u)\eta(u) \in L(0, \infty)$, we have, for $t > 0$,

$$|\mu(t)| \leqslant |\psi(t)t^{-1}|\int_0^\infty |\omega(x/t)F(x)|\,dx = |\psi(t)t^{-1}|\int_0^\infty \left|\frac{\omega(x/t)}{\psi(x)}\right| |\psi(x)F(x)|\,dx$$

$$\leqslant \left(\int_0^\infty \left|\omega(u)\frac{\psi(t)}{\psi(tu)}\right| du\right)\Big(\sup_{0<x<\infty}|\psi(x)F(x)|\Big)$$

$$\leqslant H\left(\int_0^\infty |\omega(u)\eta(u)|\,du\right)\Big(\sup_{0<x<\infty}|\psi(x)F(x)|\Big),$$

where $H = H(\delta, \Delta) > 0$. Hence $\mu(t) \in L^\infty(0, \infty)$.

Secondly we put $1 \leqslant p < \infty$. Then, by (4.7), (4.1)–(4.4), a generalized form of Minkowski's inequality [7], p. 19, and $\psi(x)F(x) \in L^p(0, \infty)$, we get

$$\left(\int_0^\infty |\mu(t)|^p\,dt\right)^{1/p} \leqslant \left\{\int_0^\infty \left(|\psi(t)t^{-1}|\int_0^\infty |\omega(x/t)F(x)|\,dx\right)^p dt\right\}^{1/p}$$

$$= \left\{\int_0^\infty \left(\int_0^\infty |\omega(u)\psi(t)F(tu)|\,du\right)^p dt\right\}^{1/p} \leqslant \int_0^\infty \left(\int_0^\infty |\omega(u)\psi(t)F(tu)|^p\,dt\right)^{1/p} du$$

$$\leqslant \int_0^\infty |\omega(u)u^{-1/p}|\left(\int_0^\infty |\psi(w/u)F(w)|^p\,dw\right)^{1/p} du$$

$$\leqslant H'\left(\int_0^\infty |\omega(u)\eta(u)|\,du\right)\left(\int_0^\infty |\psi(w)F(w)|^p\,du\right)^{1/p} < \infty,$$

where $H' = H'(\delta, \Delta) > 0$. Hence $\mu(t) \in L^p(0, \infty)$ for $1 \leqslant p < \infty$.

Now, let $1 \leqslant p \leqslant \infty$. From Lemma 4(ii), there exists a positive number $\xi < 1$ such that

$$|k^*(y) - k^*(0)| \geqslant \tfrac{1}{2}|B_2|y^\beta \quad \text{for any } y,\ 0 < y \leqslant \xi. \tag{4.8}$$

Since $M(\xi/u) \in S$ for such ξ, it is easily seen, from the definition of the class S, and the conditions (i) and (ii) of Theorem 3, that

$$E_1^*|\psi(1/u)| \leqslant |\psi(\xi/u)| \leqslant E_2^*|\psi(1/u)| \quad \text{in} \quad 0 < u < \infty, \tag{4.9}$$

where $E_j^* = E_j^*(\delta, \Delta, \xi) > 0$, $j = 1, 2$. Since $f(t)$ is monotone, we have, by $\mu(t) \in L^p(0, \infty)$, (4.7), (ii) of Condition (K2), (4.8) and (4.9),

$$\left(\int_0^\infty |\mu(t)|^p\,dt\right)^{1/p} = \left\{\int_0^\infty \left(|\psi(t)|\int_0^\infty |k^*(ty)-k^*(0)|\,|df(y)|\right)^p dt\right\}^{1/p}$$

$$\geqslant \left\{\int_0^\infty \left(|\psi(t)|\int_0^{\xi/t} |k^*(ty)-k^*(0)|\,|df(y)|\right)^p dt\right\}^{1/p}$$

$$\geqslant \tfrac{1}{2}|B_2|\Big\{\int_0^\infty \Big(|\psi(t)t^\beta|\int_0^{\xi/t} y^\beta |df(y)|\Big)^p dt\Big\}^{1/p}$$

$$= \tfrac{1}{2}|B_2|\,\xi^{\beta+1/p}\Big\{\int_0^\infty \Big(|\psi(\xi/u)u^{-\beta-2/p}|\int_0^u y^\beta |df(y)|\Big)^p du\Big\}^{1/p}$$

$$\geqslant E_3^*\Big(\int_0^\infty |\psi(1/u)u^{-\beta-2/p}\int_0^u y^\beta df(y)|^p du\Big)^{1/p},$$

where $E_3^* = E_3^*(\delta, \Delta, \xi, B_2) > 0$. Thus Theorem 3 is proved.

Proof of Theorem 4. The first part of the proof is the same as in that of Theorem 2. In Theorem 3, we replace β and γ by $\beta+1$ and $\gamma+1$, respectively, and put

$$f(t) = g(t), \quad \psi(x) = \psi_1(x)x^{-1} \quad \text{and} \quad k(x) = k_1(x).$$

Then, since $\psi_1(x)x^{-1}\big(x\bar{G}(x)\big) \in L^p(0, \infty)$, we get

$$\psi_1(1/t)^{-\beta-2/p}\int_0^t x^{\beta+1}dg(x) \in L^p(0, \infty).$$

By the conditions (i) and (ii) of Theorem 4, and the definition of the class S, we see easily that

$$0 < \sup_{r>0}\Big(\int_0^r |\psi_1(1/x)|^p x^{p-2}dx\Big)^{1/p}\Big(\int_r^\infty |\psi_1(1/x)|^{-q}x^{-2}dx\Big)^{1/q}$$

$$\leqslant C\cdot\sup_{r>0}\Big\{\int_0^r x^{\gamma p+p-2}\big(M(1/x)\big)^p dx\Big\}^{1/p}\Big\{\int_r^\infty x^{-\gamma q-2}\big(M(1/x)\big)^{-q}dx\Big\}^{1/q} = C_1.$$

Now, when we put

$$\lambda(t) = \int_0^t x^{\beta+1}|dg(x)|, \quad s = \beta+1$$

and

$$U(x) = |\psi_1(1/x)x^{1-2/p}|, \quad V(x) = |\psi_1(1/x)x^{-\beta-2/p}|$$

in Lemma 2, we have, by assumptions,

$$\psi_1(1/t)^{1-2/p}\int_t^\infty x^{-\beta-1}x^{\beta+1}|dg(x)| = \psi_1(1/t)t^{1-2/p}g(t) \in L^p(0, \infty),$$

since $g(t)$ decreases to zero in $0 < t < \infty$. Thus Theorem 4 is proved.

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HIROSAKI UNIVERSITY
HIROSAKI, 〒 036, JAPAN

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