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A note on weak estimates for oscillating kernels

by

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Abstract. In this paper we prove weak type L estimates for kernels of the form $K = k\hat{g}$ with applications to $K(x) = e^{i|x|^a}/(1+|x|)$ $(0 < a < 1)$. In a later paper, we will use these methods to solve an L^p endpoint problem, $1 < p < 2$.

Introduction. In an earlier paper [5], we discussed the L^p to L^p $(1 < p < \infty)$ mapping properties of the kernels $k(x)e^{if(x)}$ with fairly general conditions on $f(x)$ and $k(x)$. Other authors such as Fefferman [6], [7], Carleson–Sjölin [2], Sjölin [11], [12], and Hörmander [8], have given weak $(1,1)$ estimates for some of these kernels. As a matter of fact, their results apply to kernels with compact support. I mean by that, that the kernels oscillate badly about the origin, but have virtually no oscillation at infinity. Zafran [14] and others have handled similar kernels.

In this paper, we discuss the weak mapping properties for the kernels $k(x)e^{i|x|^a}$, $0 < a < 1$, (see Theorem 3). In fact, we can prove that a class of kernels $k(x)e^{if(x)}$ satisfies a weak type mapping property for a fairly general $f(x)$ and $k(x) = O(|x|^{-1})$.

In §5 we show that for some of these kernels (namely $e^{i|x|^a}/x$), Hörmander's condition fails (see (1)).

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§ 0. Notation and definitions. Here, we discuss kernels $K = K\hat{g}$ on $\boldsymbol{R}^1$ where "little" k satisfies the Hörmander condition,

$$\sup_{y\neq 0} \int_{|x|\geqslant 2|y|} dx\,|k(x-y)-k(x)| = B_1 \quad (< \infty), \tag{1}$$

and $\hat{g}$ oscillates in a somewhat regular fashion and $\hat{g} \in L^\infty$. As far as the notation is concerned, $\hat{g}$ will be a known function, and in those cases where g has a well-defined Fourier transform (e.g. $g \in L^2$), then $\hat{g}$ will be the Fourier transform of g.

We shall further assume that

(2) $$\sup_{0\neq|y|\leq 1}\int_{|x|\geq 2|y|} dx\,|k(x)|\,|\hat{g}(x-y)-\hat{g}(x)| = B_2 < \infty.$$

Our applications in this paper will be to problems where $k(x) = O(|x|^{-1})$, in this case condition (2) resembles a famous expression due to Hardy [3], p. 571, namely

(2′) $$\int_{-\infty}^{\infty} |\hat{g}(x)|\frac{dx}{|x|} \qquad \left(\int g = 0\right).$$

We thus view (2) as a weak Hardy condition. Furthermore, we shall assume

(3) $$\sup_{|y|\geq 1}\int_{|x|\geq 2|y|^r} dx\,|k(x)|\,|\hat{g}(x-y)-\hat{g}(x)| = B_3\ (<\infty) \quad \text{for some } r\geq 1.$$

I should point out that when $r=1$, then if $K = k\hat{g}$ and both (1) and (2) hold, then K would satisfy Hörmander's condition (1). Thus, K would be of Calderón–Zygmund type and thus well understood.

By L_0^∞ we mean the class of L^∞ functions with compact support. Throughout the paper we will assume that for $f\in L_0^\infty$,

(4) $$K^*f(x) = k\hat{g}*f(x) = \lim_{\varepsilon\to 0}\int_{|t|>\varepsilon} k(t)\hat{g}(t)f(x-t)\,dt \quad \text{exists for a.a. } x.$$

Definition 1. If T is a linear operator defined on L_0^∞ into L^2 so that

$$\|Tf\|_2 \leq B\|f\|_2$$

where B is a positive constant independent of f, then we say that T *maps L^2 into L^2 continuously*. We denote the smallest B by $\|T\|_{2,2}$. Thus, e.g. $\|K\|_{2,2} < \infty$, i.e. K maps L^2 into L^2 continuously as a convolution.

Among our results we prove that for $0<a<1$,

$$\left|\left\{x\colon \left|\frac{e^{i|x|^a}}{x}*f(x)\right| > \lambda\right\}\right| \leq \frac{cB_a\log(2+\lambda^{-1})}{\lambda}\|f\|_1,$$

(see Theorem 3).

When we use the letter c, we mean an absolute constant. When we use the letter B, we mean a constant that depends exclusively on any one of or all of the constants $B_1, B_2, B_3, \ldots, \|K\|_{2,2}, \|\hat{g}\|_\infty$.

§ 1. A representation theorem. In this section we prove the key lemma of the paper.

Lemma 1. *Let $K = k\hat{g}$ and $f\in L^1$. If $k, \hat{g}\in L^\infty$, then for each $\lambda>0$ there exist pairwise disjoint intervals $\{I_j\}_1^\infty$ so that*

(5) $$\lambda \leq |I_j|^{-1}\int_{I_j}|f(t)|\,dt \leq 2\lambda,$$

and

(6) $$|f(t)|\leq\lambda \quad \textit{for a.a. } t\notin Q,\ Q = \bigcup_{j=1}^\infty I_j,$$

so that for each x

(7) $$\begin{aligned} K*f(x) = {} & \sum_{j=1}^\infty\int dt\,\big(k(x-t)-k(x-c_j)\big)\hat{g}(x-t)f(t)\chi_j(t) - \\ & -\sum_{j=1}^\infty\int dt\,\big(k(x-t)-k(x-c_j)\big)\chi_j(t)|I_j|^{-1}\int_{I_j}du\,f(u)\hat{g}(x-u) + \\ & +\sum_{j=1}^\infty\int dt\,k(x-t)\chi_j(t)|I_j|^{-1}\int_{I_j}du\,f(u)\hat{g}(x-u) \\ & +\int dt\,K(x-t)f(t)\chi_{\tilde{Q}}(t) \end{aligned}$$

where c_j is the center of I_j, χ_j is the characteristic function of the interval I and $\tilde{Q}$ is the complement of the set Q.

Proof. Given $\lambda>0$, since $f\in L^1$, there exist pairwise disjoint intervals $\{I_j\}_1^\infty$ so that (5) and (6) are satisfied. We note that

(8) $$\begin{aligned} & \int dt\,k(x-t)\hat{g}(x-t)f(t)\,dt \\ & = \Big\{\int dt\,k(x-t)f(t)\chi_Q(t)\hat{g}(x-t) - \\ & \quad -\sum_{j=1}^\infty\int dt\,k(x-t)\chi_j(t)|I_j|^{-1}\int_{I_j}du\,f(u)\hat{g}(x-u)\Big\} + \\ & \quad +\Big\{\int dt\,k(x-t)\hat{g}(x-t)f(t)\chi_{\tilde{Q}}(t) + \\ & \quad +\sum_{j=1}^\infty\int dt\,k(x-t)\chi_j(t)|I_j|^{-1}\int_{I_j}du\,f(u)\hat{g}(x-u)\Big\} \\ & = I + II \end{aligned}$$

where here $\tilde{Q}$ is the complement of the set Q. Since the equation in (8) holds formally, we just need to check that all the integrals on the right make sense. But all the integrals on the right do exist for all x, since

$k, \hat{g} \in L^\infty$ and $f \in L^1$. Now for c_j the center of I_j, we get

$$I = \sum_{j=1}^{\infty} \Big(\int dt\, k(x-t)\hat{g}(x-t) f(t)\chi_j(t) - $$
$$ - \int dt\, k(x-t)\chi_j(t)|I_j|^{-1} \int_{I_j} du\, f(u)\hat{g}(x-u) \Big)$$
$$= \sum_{j=1}^{\infty} \int dt \big(k(x-t) - k(x-c_j)\big)\chi_j(t)$$
$$\{\hat{g}(x-t)f(t) - |I_j|^{-1} \int_{I_j} du\, f(u)\hat{g}(x-u)\}$$

and hence the lemma follows.

§ 2. Weak estimates. As stated earlier we shall be studying kernels $K = k\hat{g}$. We will only be concerned with problems where

(9) $$|k(x)| \leqslant B_4 |x|^{-1}.$$

Now let us begin by defining $(s > 0)$

$$K^s(x) = K(x)\chi_s(x) = k\chi_s\hat{g} = k^s\hat{g}$$

where χ_s is the characteristic function of the interval $(-s^{-1}, s^{-1})$ and we set

$$K_s(x) \equiv K(x) - K^s(x).$$

We note that $|K_s(x)| \leqslant B_4 \|\hat{g}\|_\infty s$, and k^s has support in $(-s^{-1}, s^{-1})$.

And we set

(10) $$h(s) = \sup_{1 \leqslant |v|} \int_{2|v| \leqslant |x| \leqslant 2|v|^r} dx\, |k^s(x)|$$

where $r\ (\geqslant 1)$ and is given in (3). Note that when $s \geqslant 1$, then $h(s) = 0$.

THEOREM 1. *Let $K = k\hat{g}$, $K^s = k^s\hat{g}$ and $f \in L^1$. Suppose that* (1), (2), (3) *and* (9) *hold. If*

(i) $k, \hat{g} \in L^\infty$,

(ii) $\|K\|_{2,2} < \infty$,

hen for each $\lambda > 0$,

$$|\{x\colon |K^s * f(x)| > \lambda\}| \leqslant cB\left(\frac{1+h(s)}{\lambda}\right)\|f\|_1$$

where $h(s)$ is given in (10) *and* $B = (1+B_1+B_2+B_3+B_4+\|K\|_{2,2}+\|\hat{g}\|_\infty)^2$. (Note these constants c, B are independent of s, λ and f.) *To avoid any confusion by the hypothesis we mean that k satisfies* (1) *and* (9) *and the conditions in* (2) *and* (3) *hold.* (Note the role of k and $\hat{g}$ in (2) and (3).)

Proof. Given $\lambda > 0$ and $f \in L^1$, we get from Lemma 1, pairwise disjoint intervals $\{I_j\}_1^\infty$, so that (5) and (6) hold for f and from (7),

(7′) $$K^s * f(x) = \Big\{ \sum_{j=1}^{\infty} \int dt \big(k^s(x-t) - k^s(x-c_j)\big)\hat{g}(x-t)f(t)\chi_j(t) - $$
$$- \sum_{j=1}^{\infty} \int dt \big(k^s(x-t) - k^s(x-c_j)\big)\chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, f(u)\hat{g}(x-u)\Big\} +$$
$$+ \Big\{ \sum_{j=1}^{\infty} \int dt\, k^s(x-t)\chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, f(u)\hat{g}(x-u) +$$
$$+ \int dt\, k^s(x-t)\hat{g}(x-t)f(t)\chi_{\tilde{Q}}(t)\Big\},$$
$$= \mathrm{I} + \mathrm{II}$$

where c_j is the center of I, $Q = \bigcup_{j=1}^{\infty} I_j$ and $\tilde{Q}$ is the complement of Q. Set $I_j^* = (a_j - 2|I_j|, b_j + 2|I_j|)$ with $I_j = (a_j, b_j)$ and set $Q^* = \bigcup_{j=1}^{\infty} I_j^*$, then $|Q^*| \leqslant \frac{5}{\lambda}\int_Q |f|$.

For I in (7′) we note that,

(11) $$\sum_{j=1}^{\infty} \int_{\tilde{Q}^*} dx \int dt\, |k^s(x-t) - k\ (x-c\ )|\chi_j(t)\Big\{|\hat{g}(x-t)||f(t)| +$$
$$+ \frac{1}{|I_j|}\int_{I_j} du\, |f(u)||\hat{g}(x-u)|\Big\}$$
$$\leqslant \sum_{j=1}^{\infty} \int dt\,\chi_j(t) \int_{\tilde{Q}^*} dx\, |k^s(x-t) - k^s(x-c_j)|\Big\{\|\hat{g}\|_\infty\Big(|f(t)| + \frac{1}{|I_j|}\int_{I_j} du\, |f(u)|\Big)\Big\}$$
$$= \|\hat{g}\|_\infty \sum_{j=1}^{\infty} \int dt\,\chi_j(t)\Big\{|f(t)| + \frac{1}{|I_j|}\int_{I_j} du\, |f(u)|\Big\} \int_{\tilde{Q}^*} dx\, |k^s(x-t) - k^s(x-c_j)|.$$

But since k satisfies (1) and (9), it follows that,

$$\leqslant c(B_1+B_4)\|\hat{g}\|_\infty \int_Q |f|.$$

And hence,

(12) $$|\{x \notin Q^*\colon |\mathrm{I}| > 2\lambda\}| \leqslant \frac{c(B_1+B_4)}{\lambda}\|\hat{g}\|_\infty \|f\|_1.$$

For II in (7′) we note that,

$$\mathrm{II} = \sum_{j=1}^{\infty} \int dt\, k^s(x-t)\chi_j(t) \frac{1}{|I_j|} \int_{I_j} du\, f(u)\big(\hat{g}(x-u)-\hat{g}(x-t)\big) + \\ +2\int dt\, k^s(x-t)\,\hat{g}(x-t) f(t)\chi_{\tilde{Q}}(t) \\ = \mathrm{II}_A + \mathrm{II}_B .$$

Since $K = k\hat{g} \in L_2^2$ and k satisfies (9), it follows that

(13) $$\|K^s\|_{2,2} \leqslant 2cB_4 + 4\|K\|_{2,2}.$$

Let us assume (13) for the moment, we will give a proof of it at the end of the section. Hence for II_B, since $|f(t)| \leqslant \lambda$ for $t \notin Q$,

$$\|K^s * \chi_{\tilde{Q}} f\|_2 \leqslant \|K^s\|_{2,2}\|\chi_{\tilde{Q}} f\|_2 \leqslant \lambda^{1/2}\|K^s\|_{2,2}\|f\|_1^{1/2} \\ \leqslant \lambda^{1/2}(2cB_4+4\|K\|_{2,2})\|f\|_1^{1/2}.$$

Hence,

(14) $$\big|\{x:\ |K^s * \chi_{\tilde{Q}} f| > \lambda\}\big| \leqslant \lambda^{-1}(2cB_4+4\|K\|_{2,2})^2\|f\|_1.$$

Now to estimate the critical term II_A, i.e.

$$\sum_{\substack{j=1\\|I_j|\leqslant 1}}^{\infty}{}' \int dt\, k^s(x-t)\chi_j(t) \frac{1}{|I_j|}\int_{I_j} du\, f(u)\big(\hat{g}(x-u)-\hat{g}(x-t)\big) + \\ + \sum_{|I_j|>1}{}'' \int dt\, k^s(x-t)\chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, f(u)\big(\hat{g}(x-u)-\hat{g}(x-t)\big).$$

Here, $\sum'$ sums over those intervals I_j for which $|I_j| \leqslant 1$ and $\sum''$ sums over those intervals I_j for which $|I_j| > 1$.

But we note that,

(15) $$\sum{}' \int_{\tilde{Q}^*} dx \int dt\, |k^s(x-t)|\chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, |f(u)|\,|\hat{g}(x-u)-\hat{g}(x-t)| \\ \leqslant \sum{}' \int dt\, \chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, |f(u)| \int_{\tilde{Q}^*} dx\, |k^s(x-t)|\,|\hat{g}(x-u)-\hat{g}(x-t)|$$

(since for $x \notin Q^*$, $t \in I_j \Rightarrow |x-t| \geqslant 2|I_j| \geqslant 2|u-t|$, for $u,\ t \in I_j$, and $|I_j| \leqslant 1$, this implies by (2) that)

$$\leqslant B_2 \sum{}' \int dt\, \chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, |f(u)| \leqslant B_2 \int_Q |f|.$$

And now to handle the term with the "large intervals". For the intervals I_j, where $|I_j| > 1$, we set $E_j = \{(u,t):\ u, t \in I_j \text{ and } |u-t| \leqslant 1\}$ and $F_j = \{(u,t):\ u,t \in I_j \text{ and } |u-t| > 1\}$. We are left with,

$$\sum{}'' \int k^s(x-t)\chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, f(u)\big(\hat{g}(x-u)-\hat{g}(x-t)\big).$$

Now,

$$\int_{\tilde{Q}^*} dx \Big|\sum{}'' \int k^s(x-t)\chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, f(u)\big(\hat{g}(x-u)-\hat{g}(x-t)\big)\Big| \\ \leqslant \sum{}'' \Big(\int dt\, \chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, |f(u)| \times \\ \times \int_{\tilde{Q}^*} dx\, |k^s(x-t)|\,|\hat{g}(x-u)-\hat{g}(x-t)|\,\chi_{E_j}(u,t) + \\ + \int dt\, \chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, |f(u)| \int_{\tilde{Q}^*} dx\, |k^s(x-t)|\,|\hat{g}(x-u)-\hat{g}(x-t)| + \chi_{F_j}(u,t)\Big) \\ \leqslant B_2 \sum{}'' \int_{I_j} du\, |f(u)| + \\ + \sum{}'' \int dt\, \chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, |f(u)| \int_{\tilde{Q}^*} dx\, |k^s(x-t)|\,|\hat{g}(x-u)-\hat{g}(x-t)| \times \\ \times \chi_{F_j}(u,t) \\ \leqslant B_2 \int_Q |f| + \\ + \sum{}'' \int dt\, \chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, |f(u)| \int_{\substack{2|y|\leqslant|p|\leqslant 2|y|^r\\(|y|\geqslant 1)}} dp\, |k^s(p)|\,|\hat{g}(p)-\hat{g}(p-y)| + \\ + \sum{}'' \int dt\, \chi_j(t)\frac{1}{|I_j|}\int_{I_j} du\, |f(u)| \int_{\substack{2|y|^r\leqslant|p|\\(|y|\geqslant 1)}} dp\, |k^s(p)|\,|\hat{g}(p)-\hat{g}(p-y)|$$

where $p = x-t$ and $x-u = x-t-(u-t) = p-y$ and $x \notin Q^*$,

$$\leqslant B_2 \int_Q |f| + 2\|\hat{g}\|_\infty h(s)\int_Q |f| + B_3 \int_Q |f|.$$

Thus,

(16) $$|\{x \notin Q^*:\ |\mathrm{II}_A| > \lambda\}| \leqslant \left(\frac{B_2 + 2h(s)\|\hat{g}\|_\infty + B_3}{\lambda}\right)\|f\|_1.$$

Hence, from (14), (15) and (16) we get

$$(17)\qquad |\{x \notin Q^*:\ |\mathrm{II}| > \lambda\}| \leqslant \frac{1}{\lambda} c\big(B_2+B_3+B_4+\|\hat{g}\|_\infty h(s)+ +(B_4+\|K\|_{2,2})^2\big)\|f\|_1.$$

And since $|Q^*| \leqslant 5/\lambda \|f\|_1$, then from (12) and (17) we get our result. We are left with showing (13).

LEMMA 2. *If*

(i) $\sup_\varrho \int_{\varrho\leqslant|t|\leqslant 2\varrho} |K(t)|\,dt \leqslant B\ (<\infty)$,

(ii) $\|K\|_{2,2} < \infty$,

(iii) $K \in L_{\mathrm{loc}}$,

then

$$(18)\qquad \sup_{m,a}\Big|\int K(t)\chi_m(t)e^{iat}\,dt\Big| \leqslant 2B+4\|K\|_{2,2}$$

for all real numbers m and a. Here, χ_m is the characteristic function of the set $[-m, m]$.

Let a and m be given. Then,

$$\begin{aligned} |Ke^{iat}*\chi_{2m}(x)| &= \Big|\int K(t)e^{-ia(x-t)}\chi_{2m}(x-t)\,dt\Big| \\ &= \Big|\int_{|t|\leqslant m} K(t)e^{-ia(x-t)}\,dt + \int_{|t|>m} K(t)e^{-ia(x-t)}\chi_{2m}(x-t)\,dt\Big| \\ &\leqslant \Big|\int_{|t|\leqslant m} K(t)e^{-ia(x-t)}\,dt\Big| + B = K(a, m)+B. \end{aligned}$$

If

$$(19)\qquad K(a, m) \leqslant 2B,$$

then we are finished. Otherwise,

$$K(a, m) > 2B$$

and this implies,

$$|Ke^{iat}*\chi_{2m}(x)| \geqslant \tfrac{1}{2}K(a, m) \quad \text{for} \quad |x| \leqslant m.$$

Hence,

$$2m \leqslant \big|\{x:\ |Ke^{iat}*\chi_{2m}(x)| > \tfrac{1}{2}K(a, m)\}\big| \leqslant \frac{4\|K\|_{2,2}^2}{(K(a, m))^2}\cdot 4m$$

and hence

$$(20)\qquad K(a, m) \leqslant 4\|K\|_{2,2},$$

and (18) follows from (19) and (20). Thus, the lemma follows.

§ 3. More on weak estimates. Again we have $K = k\hat{g}$. In Theorem 1 we deal only with a part of K, i.e. $K = K^s + K_s$ and it (Theorem 1) only applies to K^s. In this section, we will get weak estimates for the entire kernel K.

THEOREM 2. *Let $K = k\hat{g}$, $K^s = k\chi_s\hat{g}$ and $K = K^s+K_s$. Suppose that k and $\hat{g}$ satisfy conditions* (1), (2), (3) *and* (9). *If*

(i) $k, \hat{g} \in L^\infty$,

(ii) $\|K\|_{2,2} < \infty$,

and

(iii) *there exists a positive constant $\beta > 0$ so that*

$$|\hat{K}_s(x)| \leqslant \frac{B_5}{|x|+s^{-\beta}} \quad \text{for all } x \text{ and all } s \leqslant 1,$$

then, for each $\lambda > 0$,

$$\big|\{x:\ |K*\chi_E(x)| > \lambda\}\big| \leqslant \frac{cB}{\lambda}\big(1+h(\lambda^{1/\beta})\big)|E|,$$

for each finite measurable set E, χ_E is the characteristic function of the set E.

Proof. Let $\lambda \geqslant 1$. Since

$$\int_{-\infty}^{\infty} |K*\chi_E(x)|^2\,dx \leqslant \|K\|_{2,2}^2|E|,$$

this implies that

$$\lambda^2\big|\{x:\ |K*\chi_E(x)| > \lambda\}\big| \leqslant \|K\|_{2,2}^2|E|;$$

this implies

$$(21)\qquad \big|\{x:\ |K*\chi_E(x)| > \lambda\}\big| \leqslant \frac{\|K\|_{2,2}^2}{\lambda}|E| \quad \text{for} \quad \lambda \geqslant 1.$$

Now let $0 < \lambda \leqslant 1$. Choose $s = \lambda^{1/\beta}$, then

$$\begin{aligned} &\big|\{x:\ |K*\chi_E(x)| > 2\lambda\}\big| \\ &\qquad \leqslant \big|\{x:\ |K^s*\chi_E(x)| > \lambda\}\big| + \big|\{x:\ |K_s*\chi_E(x)| > \lambda\}\big|. \end{aligned}$$

By Theorem 1 we get that,

$$\leqslant \frac{cB}{\lambda}\big(1+h(\lambda^{1/\beta})\big)|E| + \big|\{x:\ |K_s*\chi_E(x)| > \lambda\}\big|.$$

But,

$$\begin{aligned} \int_{-\infty}^{\infty} |K_s*\chi_E(x)|^2dx &= \int_{-\infty}^{\infty} |\hat{K}_s(x)|^2|\hat{\chi}_E(x)|^2\,dx \\ &\leqslant B_5^2\int_{-\infty}^{\infty}\frac{1}{s^{-2\beta}+x^2}|\hat{\chi}_E(x)|^2\,dx = B^2\int_{-\infty}^{\infty}\frac{1}{\lambda^{-2}+x^2}|\hat{\chi}_E(x)|^2\,dx \end{aligned}$$

$$\overset{(u=\lambda^{-1})}{=} B^2 \int_{-\infty}^{\infty} dx \frac{1}{u^2+x^2} \int_{-\infty}^{\infty} \chi_E(v)\, e^{-ivx}\, dv \int_{-\infty}^{\infty} \chi_E(w)\, e^{iwx}\, dw$$

$$= B^2 \int_{-\infty}^{\infty} dv\, \chi_E(v) \int_{-\infty}^{\infty} dw\, \chi_E(w) \int_{-\infty}^{\infty} dx \frac{e^{ix(w-v)}}{u^2+x^2}$$

$$= \frac{cB^2}{u} \int_{-\infty}^{\infty} dv\, \chi_E(v) \int_{-\infty}^{\infty} dw\, \chi_E(w)\, e^{-|u||w-v|}$$

$$\overset{(u\geqslant 1)}{\leqslant} \frac{cB^2}{u} |E| = cB^2 \lambda |E| \qquad (u = \lambda^{-1}),$$

hence

$$\lambda^2 \big|\{x\colon\ |K_s * \chi_E(x)| > \lambda\}\big| \leqslant cB^2 \lambda |E| \tag{22}$$

and we are through.

In Theorem 2, we showed that a certain class of kernels map characteristic functions χ_E weakly. In fact, we will show that we could replace characteristic functions χ_E by $f \in L_0^\infty$ and still get the same estimates. Actually, we use the idea of K. H. Moon [9].

THEOREM 2′. *Let* $K = k\hat{g}$, $K_s = k\chi_s\hat{g}$ *and* $K = K^s + K_s$. *Suppose that* k *and* $\hat{g}$ *satisfy conditions* (1), (2), (3) *and* (9). *If*

(i) $k, \hat{g} \in L^\infty$,

(ii) $\|K\|_{2,2} < \infty$,

and

(iii) *there exists a positive constant* $\beta > 0$ *so that*

$$|\hat{K}_s(x)| \leqslant \frac{B_5}{|x| + s^{-\beta}}$$

for all x *and all* $s \leqslant 1$, *then, for each* $\lambda > 0$,

$$\big|\{x\colon\ |K * f(x)| > \lambda\}\big| \leqslant \frac{cB}{\lambda} \big(1 + h(\lambda^{1/\beta})\big) \|f\|_1,$$

for all $f \in L^1$.

Proof. By Theorem 1, we get that

$$\big|\{x\colon\ |K^s * f(x)| > \lambda\}\big| \leqslant cB\left(\frac{1+h(s)}{\lambda}\right) \|f\|_1$$

for each $f \in L^1$. Where c, B are independent of s, λ and f. Following the proof of Theorem 3, it suffices to deal with the terms (21) and (22). Since the arguments are similar, we will just treat (22).

Let $n > 1/s$, then by (iii) above we get that

$$|\hat{K}_s^n(x)| = |\hat{K}_s(x) - \hat{K}_{1/n}(x)| \leqslant \frac{2B}{|x| + s^{-\beta}}$$

where

$$K_s^n(x) = \begin{cases} K(x) & 1/s \leqslant |x| \leqslant n, \\ 0 & \text{elsewhere}. \end{cases}$$

And hence as in (22) we get that

$$\big|\{x\colon\ |K_s^n * \chi_E(x)| > \lambda\}\big| \leqslant \frac{cB^2}{\lambda} |E| \tag{22′}$$

for each finite measurable set E. Now we argue as in [9]. Since the argument is well-known and straight-forward we will simply give a sketch of it.

Let $0 \leqslant f \in L^\infty$ and be a simple function. Since $K_s^n \in L^1$, there exists an $h^n \in C_0'$ so that

$$K_s^n * f \approx h^n * f.$$

Now we can decompose the plane (or the line) into compact cubes $\{I_k\}$ where $\mathrm{dia}(I_k)$ is small and $f(x) = a_k$ for $x \in I_k$ (i.e. f is constant on I_k). Now consider compact cubes $F_k \subset I_k$ so that $|F_k| = (a_k/a)|I_k|$ where $a = \sup_k a_k = \|f\|_\infty$ and set $E_n = \bigcup_k F_k$. And then it follows that

$$|K_s^n * f(x)| \leqslant a |K_s^n * \chi_{E_n}| + \lambda$$

and by (22′) the proof is complete.

§ 4. Applications. Here, we will give applications, i.e. show weak estimates for the kernels $k(x) e^{i|x|^a}$, $0 < a < 1$, where $k(x)$ is one of the kernels,

$$\frac{1}{x} \quad (x \neq 0), \quad \frac{\log\big(\log(4+|x|)\big)}{x \log(2+|x|)} \quad (x \neq 0), \quad \frac{1}{x \log(2+|x|)} \quad (x \neq 0),$$

$$\frac{1}{1+|x|}, \quad \frac{\log\big(\log(4+|x|)\big)}{(1+|x|)\log(2+|x|)}, \quad \frac{1}{(1+|x|)\log(2+|x|)}.$$

Actually, we can prove a more general theorem, i.e. for kernels $k(x)e^{if(x)}$, where $k(x) = O(|x|^{-1})$ and f is a real-valued, even function with further conditions on f' and f''. However, the number of conditions placed on k and f would be somewhat prohibitive. Thus, for the sake of simplicity we will just give a proof for the functions

$$K_j(x) = k_j(x) e^{i|x|^a} \qquad (0 < a < 1),\ 1 \leqslant j \leqslant 6,$$

where

$$k_1(x) = 1/x \quad (x \neq 0), \qquad k_2(x) = \frac{\log_2(4+|x|)}{x\log(2+|x|)}$$

and so on.

THEOREM 3. *Let $0 < \alpha < 1$. Then, for the kernels*

$$K_j(x) = k_j(x)e^{i|x|^\alpha}, \quad j = 1, 2, \ldots, 6,$$

we get,

(a) $$\left|\{x\colon |K_j * f(x)| > \lambda\}\right| \leqslant \frac{c_1}{\lambda}\big(1+h_j(\lambda)\big)\|f\|_1, \quad \textit{for} \quad \lambda > 0,$$

and

(b) $$\|K_j * f\|_p \leqslant c_p\|f\|_p$$

for $1 < p < \infty$, $j = 1, \ldots, 6$.

Here, c_p is a positive constant independent of f, while c_1 is independent of both f and λ. Also,

$$h_1(\lambda)+h_4(\lambda) \leqslant c\log(2+\lambda^{-1})$$

$$h_2(\lambda)+h_5(\lambda) \leqslant c\log\big(\log(4+\lambda^{-1})\big)$$

and

$$h_3(\lambda)+h_6(\lambda) \leqslant c.$$

(Note the estimate (a) holds for all $f \in L^1$ and (b) holds for $f \in L^p, 1 < p < \infty$).

Proof. We will work the case $K_1(x) = e^{i|x|^\alpha}/x$ $(x \neq 0)$, the proofs for the other 5 cases being similar.

To show that our convolutions are well-defined say for all $f \in L$, i.e. to show (4) for all $f \in L^1$, we note that,

$$\int_{-\infty}^{\infty}\frac{e^{i|t|^\alpha}}{t}f(x-t)\,dt = \int_{-1}^{1}\frac{(e^{i|t|^\alpha}-1)}{t}f(x-t)\,dt + \\ + \int_{-1}^{1}\frac{1}{t}f(x-t)\,dt + \int_{|t|\geqslant 1}\frac{e^{i|t|^\alpha}}{t}f(x-t)\,dt$$

exists for a.a. x. And,

$$\left|\left\{x\colon \left|\int_{-\infty}^{\infty}\frac{e^{i|t|^\alpha}}{t}f(x-t)\,dt\right| > 3\lambda\right\}\right| \\ \leqslant \frac{c}{\lambda}\|f\|_1 + \left|\left\{x\colon \left|\int_{|t|\geqslant 1}\frac{e^{i|t|^\alpha}}{t}f(x-t)\,dt\right| > \lambda\right\}\right|.$$

We will now show that the kernel

$$\frac{e^{i|t|^\alpha}}{t}\big(1-\chi_1(t)\big) \quad \text{where} \quad \chi_1(t) = \begin{cases}1 & |t| \leqslant 1,\\ 0 & \text{elsewhere}\end{cases}$$

satisfies the conditions of Theorem 2′. Here, $k(t) = \big(1-\chi_1(t)\big)/t$ and $\hat{g}(t) = e^{i|t|^\alpha}$.

It is easy to see that $\big(1-\chi_1(t)\big)/t$ satisfies (1), and in fact, $B_1 \leqslant 2$. To show that $k(t)e^{i|t|^\alpha}$ satisfies (2), we note that,

$$|e^{i|x-y|^\alpha} - e^{i|x|^\alpha}| \leqslant \frac{c_\alpha|y|}{|x|^{1-\alpha}} \quad \text{where} \quad |x| \geqslant 2|y|,$$

$0 < \alpha < 1$, and c_α is a constant that depends only on α. Here, for $|y| \leqslant 1$, we get,

$$\int_{|x|\geqslant 2|y|}\frac{1}{|x|}|e^{i|x-y|^\alpha} - e^{i|x|^\alpha}|\,dx \leqslant c_\alpha|y|\int_{|x|\geqslant 2|y|}\frac{dx}{|x|^{2-\alpha}} \leqslant c_\alpha|y|^\alpha \leqslant c_\alpha,$$

hence $B_2 \leqslant c_\alpha$.

To show (3), we choose $r = 1/(1-\alpha)$ $(r > 1)$ then for $|y| > 1$,

$$\int_{|x|\geqslant 2|y|^{1/(1-\alpha)}}|e^{i|x-y|^\alpha} - e^{i|x|^\alpha}|\frac{dx}{|x|} \leqslant c_\alpha|y|\int_{|x|\geqslant 2|y|^{1/(1-\alpha)}}\frac{dx}{|x|^{2-\alpha}} \\ \leqslant c_\alpha|y|\frac{1}{(|y|^{1/(1-\alpha)})^{1-\alpha}} \leqslant c_\alpha,$$

hence $B_3 \leqslant c_\alpha$.

Now to show that $\|K_1\|_{2,2} < \infty$ or equivalently that

$$\mathfrak{F}(k\hat{g}) \in L^\infty.$$

We note that by Van der Corput,

$$\left|\int_{1<|t|<\infty}\frac{e^{i|t|^\alpha}}{t}e^{ixt}\,dt\right| \leqslant \sum_{m=0}^{\infty}\left|\left(\int_{2^m}^{2^{m+1}} + \int_{-2^{m+1}}^{-2^m}\right)\frac{e^{i|t|^\alpha}e^{ixt}}{t}\,dt\right| \\ \leqslant c\sum_{m=0}^{\infty}\frac{1}{2^m}\cdot 2^{m(1-\alpha/2)} < \infty.$$

We are finished once we show that $\dfrac{1-\chi_1(t)}{t}e^{i|t|^\alpha}$ satisfies (iii) of Theorem 2′.

We choose $s \leqslant 1$ and consider,

$$\left|\int\limits_{|t|\geqslant 1/s} \frac{e^{i|t|^\alpha}}{t} e^{-ixt}\,dt\right| = \left|\sum_{l=0}^{\infty} \int\limits_{2^l/s \leqslant |t| \leqslant 2^{l+1}/s} \frac{e^{i|t|^\alpha}}{t} e^{-ixt}\right| \tag{23}$$

$$\leqslant \sum_{l=0}^{\infty} \left|\int\limits_{2^l/s \leqslant |t| \leqslant 2^{l+1}/s} \frac{e^{i|t|^\alpha}}{t} e^{-ixt}\,dt\right| \leqslant c_\alpha \sum_{l=0}^{\infty} \frac{s}{2^l} (2^l s^{-1})^{1-\alpha/2}$$

$$\leqslant c_\alpha s^{\alpha/2} \sum_{l=0}^{\infty} \frac{1}{2^{l\alpha/2}} \leqslant B_\alpha s^{\alpha/2} \qquad (\alpha \neq 1)\ (\alpha > 0).$$

Thus, the β in Theorem 2′ becomes $\alpha/2$.

Integrating by parts and estimating we get,

$$\left|\int\limits_{1/s \leqslant |t| \leqslant N} \frac{e^{i|t|^\alpha}}{t} e^{-ixt}\,dt\right|$$

$$\leqslant \frac{s}{|x|} + \frac{1}{|x|\cdot N} + \frac{1}{|x|} \int\limits_{1/s \leqslant |t| \leqslant N} |t|^{-2+\alpha} dt + \frac{1}{|x|} \int\limits_{1/s \leqslant |t| \leqslant N} |t|^{-2} dt,$$

and hence,

$$\left|\int\limits_{1/s \leqslant |t|} \frac{e^{i|t|^\alpha}}{t} e^{-ixt}\,dt\right| \leqslant \frac{c}{|x|}. \tag{24}$$

Putting (23) and (24) together, we get

$$\left|\widehat{\frac{e^{i|t|^\alpha}}{t}(1-\chi_s(\cdot))}(x)\right| \leqslant \frac{B_\alpha}{|x|+s^{-\alpha/2}}.$$

Note that

$$\chi_s(t) = \begin{cases} 1 & |t| \leqslant 1/s \\ 0 & \text{elsewhere} \end{cases} \qquad (s \leqslant 1).$$

Now since $\frac{e^{i|t|^\alpha}}{t}\big(1-\chi_1(t)\big)$ satisfies Theorem 2′, that implies for $f \in L^1$

$$\left|\left\{x:\ \left|\frac{e^{i|t|^\alpha}}{t}\big(1-\chi_1(t)\big)*f(x)\right| > \lambda\right\}\right| \leqslant \frac{cB_\alpha}{\lambda}\big(1+h_1(\lambda^{2/\alpha})\big)\|f\|_1.$$

But for $\lambda \leqslant 1$ $(h_1(\lambda^{2/\alpha}) = 0$ for $\lambda > 1)$,

$$h_1(\lambda^{2/\alpha}) = \sup_{1 \leqslant |y| \leqslant \lambda^{-2/\alpha}} \int\limits_{2|y| \leqslant |x| \leqslant 2|y|^{1/(1-\alpha)}} \frac{dx}{|x|} \leqslant c_\alpha \log\frac{1}{\lambda}.$$

We note that in estimating $h_2(\lambda^{2/\alpha})$ which applies to the kernel

$$\frac{\log\big(\log(4+|t|)\big)}{t\log(2+|t|)} e^{i|t|^\alpha}\big(1-\chi_1(t)\big),$$

we get that,

$$h_2(\lambda^{2/\alpha}) = \sup_{1 \leqslant |y| \leqslant \lambda^{-2/\alpha}} \int\limits_{2|y| \leqslant |x| \leqslant 2|y|^{1/(1-\alpha)}} \frac{\log\big(\log(4+|x|)\big)}{|x|\log(2+|x|)}$$

$$\leqslant c_\alpha \sup_{1 \leqslant |y| \leqslant \lambda^{-2/\alpha}} \log\big(\log(4+|y|)\big) \leqslant c_\alpha \log\big(\log(4+\lambda^{-1})\big).$$

The other 4 cases (i.e. solving for h_3, h_4, h_5, h_6) are similar to the ones given and the proofs will be omitted.

Now we shall show part (b) of Theorem 3. We need the following expression:

$$U_\varepsilon = \int_0^1 \frac{g_1(\lambda)}{\lambda^{1-\varepsilon}}\,d\lambda + \int_1^\infty \frac{g_2(\lambda)}{\lambda^{1+\varepsilon}}\,d\lambda \tag{25}$$

where $\varepsilon > 0$ and preassigned and g_1, g_2 are non-negative functions defined on $(0, \infty)$. This U_ε is a non-negative number that plays a role in the next lemma.

Lemma 3. *Let T be an operator defined on all functions $f \in L_0^\infty$. If for each $\lambda > 0$,*

$$\big|\{x:\ |(T\chi_E)(x)| > \lambda\}\big| \leqslant |E|\min\big(g_1(\lambda)/\lambda,\ g_2(\lambda)/\lambda^2\big)$$

then

$$\|T\chi_E\|_p^p \leqslant p\,U_\varepsilon |E|$$

for $1+\varepsilon \leqslant p \leqslant 2-\varepsilon$, $\varepsilon > 0$. Note the lemma is valuable only when $U_\varepsilon < \infty$ and $0 < \varepsilon \leqslant \frac{1}{2}$. Also, no other condition on the operator T is required.

Before we prove the lemma, let's see how it applies to the kernel $e^{i|t|^\alpha}/t$. Since $e^{i|t|^\alpha}/t$ maps L^2 into L^2 continuously and from part (a) of Theorem 3, we get

$$\left|\left\{x:\ \left|\frac{e^{i|t|^\alpha}}{t} * \chi_E(x)\right| > \lambda\right\}\right| \leqslant B_\alpha |E| \min\left(\frac{\log(2+1/\lambda)}{\lambda}, 1/\lambda^2\right).$$

Hence from the lemma we see that $g_1(\lambda) = B_\alpha \log(2+1/\lambda)$ and $g_2(\lambda) = B_\alpha$. Thus, we see that $U_\varepsilon < \infty$ from (25) for each $\varepsilon > 0$. Thus, from the lemma we get that

$$\|T\chi_E\|_p^p \leqslant pB_\alpha U_\varepsilon |E| \qquad \text{for} \qquad 1+\varepsilon \leqslant p \leqslant 2-\varepsilon$$

and by Stein–Weiss [13], we get part (b) of Theorem 3. (Note that Stein–Weiss only applies to linear operators.)

Proof of Lemma 3. Let E be a given measurable set and $1+\varepsilon \leqslant p \leqslant 2-\varepsilon$. Then,

$$\int_0^\infty \lambda^{p-1}|\{x\colon\ |T\chi_E(x)|>\lambda\}|\,d\lambda = \int_0^1 \lambda^{p-1}|\{x\colon\ |T\chi_E(x)|>\lambda\}|\,d\lambda +$$

$$+\int_1^\infty \lambda^{p-1}|\{x\colon\ |T\chi_E(x)|>\lambda\}|\,d\lambda \leqslant |E|\left(\int_0^1 d\lambda\,\frac{g_1(\lambda)}{\lambda^{1-(p-1)}}+\int_1^\infty d\lambda\,\frac{g_2(\lambda)}{\lambda^{1+(2-p)}}\right).$$

Since $1+\varepsilon\leqslant p\leqslant 2-\varepsilon \Rightarrow \varepsilon \leqslant p-1, 2-p$ and hence

$$\|T\chi_E\|_p^p \leqslant p\,U_\varepsilon |E|$$

and the lemma is proved.

§ 5. Hörmander's condition fails. I would like to point out that the kernels K_1, K_2, K_4, K_5 do not satisfy Hörmander's condition, i.e.

$$\sup_{y\neq 0}\int_{|x|\geqslant 2|y|}|K_i(x-y)-K_i(x)|\,dx = +\infty \tag{26}$$

for $i=1,2,4,5$, while for $i=3$ and 6, Hörmander's condition holds.

I will work the case where $K_1(x) = e^{i|x|^{1/2}}/x$ $(x\neq 0)$, all the other cases are similar. The proof will follow once we can show that,

$$\int_{2y\leqslant x\leqslant 2y^2} |e^{ix^{1/2}}-e^{i(x-y)^{1/2}}|\,\frac{dx}{x} \geqslant \frac{\log y}{2} \tag{27}$$

for $y>1$. We note that,

$$\left|\frac{e^{ix^{1/2}}-e^{i(x-y)^{1/2}}}{2}\right| \geqslant \left|\frac{e^{ix^{1/2}}-e^{i(x-y)^{1/2}}}{2}\right|^2 = \frac{1}{2}-\frac{1}{2}\mathrm{Re}\,[e^{ix^{1/2}}e^{-i(x-y)^{1/2}}].$$

Thus, (for $y>1$)

$$\int_{2y}^{2y^2} |e^{ix^{1/2}}-e^{i(x-y)^{1/2}}|\,\frac{dx}{x} \geqslant \log y - \mathrm{Re}\int_{2y}^{2y^2}\frac{dx}{x}\,e^{i(x-y)^{1/2}}e^{-ix^{1/2}}. \tag{28}$$

For fixed y $(y>1)$, set

$$f(x) = (x-y)^{1/2}-x^{1/2} \quad \text{for} \quad x\geqslant 2y.$$

Then, $f'(x) = \frac{1}{2}(x-y)^{-1/2}-\frac{1}{2}x^{-1/2}$, and

$$\int_{2y}^{2y^2}\frac{dx}{x}\,e^{i(x-y)^{1/2}}e^{-ix^{1/2}} = \int_{2y}^{2y^2}\frac{dx}{x}\,\frac{f'(x)}{\frac{1}{2}(x-y)^{-1/2}-\frac{1}{2}x^{-1/2}}\,e^{if(x)}. \tag{29}$$

Now, the function $\dfrac{1}{xf'(x)}$ is increasing as a function of x, for $x>y>0$. In fact,

$$f(x)\frac{d}{dx}\frac{1}{x}\frac{1}{f'(x)} = \frac{1}{2}\frac{1}{x^{1/2}}\frac{1}{(x-y)^{1/2}}\left(x+(x-y)+(x-y)^{1/2}x^{1/2}\right) - \frac{x^{1/2}(x-y)^{1/2}}{x^2} \geqslant 0, \quad \text{for} \quad x>y>0.$$

Thus for $y>1$, by the second mean value theorem for integrals,

$$\int_{2y}^{2y^2} dx\,\frac{1}{xf'(x)}\,f'(x)\,e^{if(x)} = O(1).$$

By (28), (29) and (30) we are through.

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