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TECHNISCHE UNIVERSITÄT
SEKTION MATHEMATIK
DDR-8027 DRESDEN

Received January 12, 1979 (1502)

STUDIA MATHEMATICA, T. LXXI. (1981)

On unconditional polynomial bases of the space L_p

by

Z. A. CHANTURIA (Tiflis)

Abstract. In all spaces $L_p(0,1)$ where $p\in(1,\infty)$ and for each $\varepsilon>0$ we prove the existence of the ONS of algebraic polynomials $\{P_n\}$ which forms an unconditional basis in $L_p(0,1)$ and satisfies the condition

$$\deg P_n = \nu_n \leqslant n^{1+\varepsilon}$$

for $n>n_0(p,\varepsilon)$. An analogous theorem is valid for trigonometrical polynomials.

As is well known, J. Marcinkiewicz was the first to prove the existence of unconditional bases in all spaces $L_p(0,1)\equiv L_p$ for $p>1$. Namely, he has proved that the Haar orthonormal system (ONS) is an unconditional base ([12], see also [14], pp. 397–423). In 1974 S. V. Bochkarev proved that the Franklin orthonormal system is also an unconditional basis in L_p for all $p>1$ [1]. Z. Ciesielski [7] has introduced a new class of orthonormal systems, which contains the systems of both Haar and Franklin, and has proved [8] that these systems are unconditional bases in all L_p for $p>1$ as well.

On the other hand, although the trigonometrical system is a basis in L_p for $p>1$, $p\neq 2$ the basis is not unconditional. What is more, V. F. Gaposhkin [10] has shown that none of the uniformly bounded systems normed in L_p can be an unconditional basis in L_p, $p\neq 2$.

It should be noted that there is no unconditional basis in the spaces $C(0,1)$ and $L(0,1)$ at all. These results are due to S. Karlin [11] and A. Pełczyński [13], respectively.

Let us pose the following question. Does an orthonormal system of polynomials (algebraic or trigonometrical) which forms an unconditional basis in L_p exist for any $p>1$, $p\neq 2$, and if it does, then what minimal growth of powers may that basis have?

In 1971, using the Haar system, we proved [2] that in every $L_p(p>1)$ for each $\varepsilon>0$ there exists an ONS of trigonometrical polynomials $\{T_n\}$ which forms an unconditional basis in L_p and for $n>n_0(p,\varepsilon)$

$$\deg T_n = \nu_n \leqslant \begin{cases} n^{p+\varepsilon} & \text{for} \quad p>2, \\ n^{q+\varepsilon} & \text{for} \quad 1<p<2, \end{cases} \qquad 1/p+1/q=1.$$

In 1972 G. E. Tkebuchava [15] proved that there exist unconditional polynomial (in general, non-orthogonal) bases in the spaces L_p $(1<p<2)$ with the growth of powers as $n^{1+p/2+\varepsilon}$.

Recently, using the Bochkarev theorem, we have essentially reduced the growth of powers of unconditional polynomial orthogonal bases [5].

The purpose of the present paper is to prove the following theorem:

THEOREM 1. *For any $p\in(1,\infty)$ and any $\varepsilon>0$ there exists an ONS of algebraic polynomials $\{P_n\}_{n=0}^{\infty}$ in the space $L_p(0,1)$ which forms an unconditional basis in L_p and satisfies the condition*

$$\deg P_n=\nu_n\leqslant n^{1+\varepsilon}$$

for $n>n_0(p,\varepsilon)$.

The proof of this theorem, as well as those of our previous ones dealing with similar problems, is based on the following theorem on the stability of the process of orthogonalization, which we published in 1971 ([3], see also [4]);

THEOREM A. *Let $\{\chi_n\}$ be an ONS system in the Hilbert space H and let the system $\{\varphi_n\}$ satisfy the condition*

$$\sum_{n=1}^{\infty}\|\chi_n-\varphi_n\|_H^2=\varkappa<1.$$

Moreover, let $\gamma_{in}=(\varphi_i,\chi_n)$, $i\neq n$ and $\gamma_{nn}=(\varphi_n,\chi_n)-1$. Then the normalized system $\{f_n\}$, which is obtained from (φ_n) by means of the Schmidt triangular orthogonalization method $((f_n,\varphi_n)>0,\ n=1,2,\dots)$, satisfies the following inequality:

$$\|\chi_n-f_n\|_H\leqslant c\left(\|\chi_n-\varphi_n\|_H+\sqrt{|\gamma_{nn}|}+\sqrt{\sum_{i=1}^{\infty}\gamma_{in}^2}\right).$$

In addition we shall use some properties of Ciesielski systems.

THEOREM B. *Let $\{f_n^{(m)}\}_{n=0}^{\infty}$ be a Ciesielski system orthonormal in $[0,1]$. Then*

(1) (see [8]). *The system $\{f_n^{(m)}\}_{n=0}^{\infty}$ is an unconditional basis in all L_p for $p>1$.*

(2) (see [8]). *There exist constants $c(m,p)$ and $c'(m,p)$* such that for any finite set of real numbers $\{a_n\}$*

$$c'(m,p)\Big\|\sum a_nf_n^{(m)}\Big\|_p\leqslant\Big\|\Big(\sum(a_nf_n^{(m)})^2\Big)^{1/2}\Big\|_p\leqslant c(m,p)\Big\|\sum a_nf_n^{(m)}\Big\|_p.$$

* Here and in what follows $c(n,m,p,\dots)$ are positive constants (in general distinct in different formulae) which depend on the parameters in brackets only.

(3) (see [7], p. 289). *The derivative of order $k-f_n^{(m,k)}$ of the function $f_n^{(m)}$ satisfies the estimation $(0\leqslant k\leqslant m+1)$*

$$c'(m)n^{1/2+k-1/p}\leqslant\|f_n^{(m,k)}\|_{L_p}\leqslant c(m)n^{1/2+k-1/p}$$

for all $1\leqslant p\leqslant\infty$, $L_\infty\equiv C$.

We shall also use the following

LEMMA. *Let $1<p<2$, let $\{f_n^{(m)}\}_{n=0}^{\infty}$ be the Ciesielski system and let $\{a_n\}$ be an arbitrary finite set of real numbers. Then for any sequence $\{P_n\}_{n=0}^{\infty}$ from L_p we have*

$$\Big\|\sum a_n(f_n^{(m)}-P_n)\Big\|_{L_p}\leqslant c(p)\Big\{\sum_{s=0}^{\infty}2^{s(2/p-1)}\sum_{h=2^s}^{2^{s+1}-1}\|f_n^{(m)}-P_n\|_p^2\Big\}^{1/2}\cdot\Big\|\sum a_nf_n^{(m)}\Big\|_p.$$

By means of properties (2) and (3) of the Ciesielski system this Lemma can be proved in exactly the same way as the analogous lemmas of [2] and [5].

Proof of Theorem 1. Let $p\in(1,2)$ and $\varepsilon\in(0,1)$ be given. Let us take a natural number $m=[10/\varepsilon]+1$ and the corresponding Ciesielski orthonormal basis $\{f_n^{(m)}\}_{n=0}^{\infty}$ for such an m. Assume that σ_n is the partial sum of order ν_n of the Fourier series of the function $f_n^{(m)}$ with respect to the Legendre system $\{\sqrt{2n+1}\,P_n(2x-1)\}_{n=0}^{\infty}$ orthonormal in $[0,1]$, i.e.

$$\sigma_n(x)=\sum_{k=0}^{\nu_n}a_k(f_n^{(m)})\sqrt{2k+1}\,P_k(2x-1),$$

where

$$a_k(f_n^{(m)})=\sqrt{2k+1}\cdot\int_0^1f_n^{(m)}(t)P_n(2t-1)\,dt$$

Using property (3) of the Ciesielski system, we can obtain the following estimate (see [6]):

$$(1)\qquad |\gamma_{in}|=|(\sigma_i,f_n^{(m)})-\delta_{in}|=|(\sigma_i-f_i^{(m)},f_n^{(m)})|\leqslant\begin{cases}c(m)i^{m-1/2}\nu_i^{1-m} & \text{for all } i \text{ and } n,\\ c(m)i^{m-1/2}\nu_i^{m+6}n^{-(m+5/2)} & \text{for } i\neq n.\end{cases}$$

Now let

$$\nu_n=\begin{cases}[n^{1+\varepsilon}] & \text{for}\quad n\geqslant 2^{s_0}\\ [2^{s_0(1+\varepsilon)}] & \text{for}\quad n<2^{s_0},\end{cases}$$

where s_0 is a certain sufficiently large natural number, which we shall define more exactly later. Note that according to (1)

$$(2)\qquad \sum_{n=2^s}^{2^{s+1}-1}\gamma_{in}^2=\sum_{n=2^s}^{2^{s+1}-1}|(\sigma_i-f_i^{(m)},f_n^{(m)})|^2\leqslant\|\sigma_i-f_i^{(m)}\|_{L_2}^2\leqslant c(m)\frac{i^{2m-1}}{\nu_i^{2m-2}}.$$

Now let us estimate the sum

$$I \equiv \sum_{i=1}^{\infty} \sum_{n=2^s}^{2^{s+1}-1} \gamma_{in}^2 .$$

Consider two cases: (a) $s \geqslant 4s_0$ and (b) $s < 4s_0$. In the first case, using (1), (2) and the definition of m and setting $N = [2^{s/4}]$, we have

$$(3) \qquad I = \sum_{i=1}^{N} \sum_{n=2^s}^{2^{s+1}-1} \gamma_{in}^2 + \sum_{i=N+1}^{\infty} \sum_{h=2^s}^{2^{s+1}-1} \gamma_{in}^2$$

$$\leqslant c(m) \left\{ \sum_{i=1}^{N} \sum_{n=2^s}^{2^{s+1}-1} \frac{i^{2m-1} \nu_i^{2m+12}}{n^{2m+5}} + \sum_{i=N+1}^{\infty} \frac{i^{2m-1}}{\nu_i^{2m-2}} \right\}$$

$$\leqslant c(m) \left\{ \sum_{i=1}^{N} \frac{i^{2m-1} \nu_i^{2m+12}}{2^{(2m+4)s}} + \sum_{i=N+1}^{\infty} \frac{1}{i^{(2m-2)s-2}} \right\}$$

$$\leqslant c(m) \left\{ \frac{N^{2m} \nu_N^{2m+12}}{2^{(2m+4)s}} + \frac{1}{N^{(2m-2)s-2}} \right\} \leqslant c(m) 2^{-4s}.$$

In the case where $s < 4s_0$, we have

$$(4) \qquad I = \sum_{i=1}^{\infty} \sum_{n=2^s}^{2^{s+1}-1} \frac{i^{2m-1}}{\nu_i^{2m-2}} \leqslant 2^s \left\{ \sum_{i=1}^{2^{s_0}-1} \frac{i^{2m-1}}{\nu_i^{2m-2}} + \sum_{i=2^{s_0}}^{\infty} \frac{i^{2m-1}}{\nu_i^{2m-2}} \right\}$$

$$\leqslant c(m) 2^{4s_0} \left\{ \frac{2^{2ms_0}}{\nu_{2^{s_0}}^{2m-2}} + 2^{-s_0[(2m-2)(1+\varepsilon)-2m]} \right\}$$

$$\leqslant c(m) 2^{-s_0[\varepsilon(2m-2)-6]} \leqslant c(m) 2^{-12s_0}.$$

Let us take s_0 so large that

$$(5) \qquad \sum_{n=0}^{\infty} \|f_n^{(m)} - \sigma_n\|_{L_2}^2 \leqslant c(m) \sum_{n=1}^{\infty} \frac{n^{2m-1}}{\nu_n^{2m-2}}$$

$$\leqslant c(m) \left\{ \sum_{n=1}^{2^{s_0}-1} \frac{n^{2m-1}}{\nu_{2^{s_0}}^{2m-2}} + \sum_{n=2^{s_0}}^{\infty} \frac{n^{2m-1}}{n^{(2m-2)(1+\varepsilon)}} \right\}$$

$$\leqslant c(m) 2^{-s_0[(2m-2)\varepsilon-2]} \leqslant c(m) 2^{-16s_0} < 1/4.$$

So Theorem A may be used. Therefore, if the system $\{Q_n\}$ is obtained from $\{\sigma_n\}$ by means of the Schmidt method, then using (2), (1) and (3)

we have for $s \geqslant 4s_0$

$$(6) \qquad \sum_{n=2^s}^{2^{s+1}-1} \|f_n^{(m)} - Q_n\|_{L_2}^2 \leqslant c \sum_{n=2^s}^{2^{s+1}-1} \left\{ \|f_n^{(m)} - \sigma_n\|_{L_2}^2 + |\gamma_{nn}| + \sum_{i=1}^{\infty} \gamma_{in}^2 \right\}$$

$$\leqslant c(m) \left\{ \sum_{n=2^s}^{2^{s+1}-1} \left(\frac{n^{2m-1}}{\nu_n^{2m-2}} + \frac{n^{m-1/2}}{\nu_n^{m-1}} \right) + 2^{-4s} \right\}$$

$$\leqslant c(m) \{ 2^{s[(m+1/2)-(m-1)(1+\varepsilon)]} + 2^{-4s} \} \leqslant c(m) 2^{-4s}.$$

Applying (4) we establish the same estimate for $s < 4s_0$:

$$(7) \qquad \sum_{n+2^s}^{2^{s+1}-1} \|f_n^{(m)} - Q_n\|_{L_2}^2 \leqslant c(m) 2^{-4s_0}.$$

Using (6) and (7), we have

$$\sum_{s=0}^{\infty} \left(2^{s(2/p-1)} \sum_{n=2^s}^{2^{s+1}-1} \|f_n^{(m)} - Q_n\|_p^2 \right)$$

$$\leqslant c(m) \left\{ \sum_{s=0}^{4s_0-1} 2^{s(2/p-1)} 2^{-4s_0} + \sum_{s=4s_0}^{\infty} 2^{s(2/p-1)} 2^{-4s} \right\} \leqslant c(m) 2^{-8s_0(1-1/p)}.$$

Now if we choose s_0 so that

$$c(m) 2^{-8s_0(1-1/p)} \leqslant \theta < 1,$$

then according to the lemma

$$\Big\| \sum a_n (f_n^{(m)} - Q_n) \Big\|_{L_p} \leqslant \theta \Big\| \sum a_n f_n^{(m)} \Big\|_{L_p}.$$

But since $\{f_n^{(m)}\}$ is an unconditional basis of the space $L_p(0,1)$, by a theorem of the Wiener–Paley type the system of algebraic polynomials $\{Q_n\}$ is an unconditional basis in the same space and $\deg Q_n = \nu_n \leqslant n^{1+\varepsilon}$ for $n \geqslant 2^{s_0}$.

Thus for $p \in (1, 2)$ the proof is completed.

But the case of $p \in (2, \infty)$ can be reduced to the previous one, for if an orthonormal system consists of bounded functions and forms a basis for the space L_p $(1 < p < \infty)$, then it is also a basis in the conjugate space L_q, $1/p + 1/q = 1$ (see e.g. [9], p. 118), and this completes the proof of Theorem 1.

The following theorem may be proved in a similar way:

Theorem 2. *For any $p \in (1, \infty)$ and any $\varepsilon > 0$ in the space $L_p(0, 1)$ there exists an ONS of trigonometrical polynomials* $\{T_n(x) = \sum_{k=0}^{\nu_n} a_k \cos k\pi x\}_{\nu=0}^{\infty}$

which forms an unconditional basis in L_p *and satisfies the condition*

$$\deg T_n = \nu_n \leqslant n^{1+\varepsilon}$$

for $n > n_0(p, \varepsilon)$.

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DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF TIFLIS
TIFLIS, GEORGIA, SOVIET UNION

Received March 20, 1979 (1523)

STUDIA MATHEMATICA, T. LXXI. (1981)

Isomorphic embeddings of some generalized power series spaces

by

V. V. KASHIRIN (Rostov)

Abstract. The necessary and sufficient condition under which the generalized power series space $L_f(a_n, -1)$ contains a closed subspace of the class $(f)_1$ of all spaces $L_f(b_n, 1)$ is obtained in terms of (a_n). In particular, it is proved that every stable space $L_f(a_n, -1)$ contains a closed subspace isomorphic to the space $L_f(a_n, 1)$.

1. In the present paper we consider special classes of Köthe spaces, introduced by M. Dragilev (cf. [4]). For each fixed function $f(u)$, $u \in R$, which is odd, increasing and logarithmically convex for $u \geqslant 0$, and for $\lambda \in (-1, 0, 1, \infty)$, we consider the class $(f)_\lambda$ of Köthe spaces (called *generalized power series spaces*)

$$L_f(a_n, \lambda) = \lim\operatorname{pr} l_1\big(\exp f(\lambda_p a_n)\big),$$

where $a_n \uparrow \infty$, $\lambda_p \uparrow \lambda$. In the case of $f(u) = u$ we have $(f)_0 = R_0$, $f_\infty = R_\infty$, where R_0 and R_∞ are the classes of all Köthe spaces isomorphic, respectively, to the power series spaces of finite and infinite types.

We study the comparability of the linear dimensions of spaces belonging to different classes $(f)_\lambda$. In particular, we are interested in the necessary and sufficient conditions, in terms of (a_n), under which a space $L_f(a_n, \lambda)$ contains a closed subspace of class $(f)_\mu$. In the present article we consider the most interesting case, namely $\lambda = -1$, $\mu = 1$ only. For all the other pairs (λ, μ) the scheme of the proof remains the same, but the necessary and sufficient conditions, in terms of (a_n), obtained for different pairs (λ, μ) are different. All the cases were treated in the preprint [8], which contains complete proofs. The statement of the results (without proofs) can also be found in [10]. In Section 4 we state a unified necessary and sufficient condition in terms of the properties of the class of all continuous linear operators T: $L_f(a_n, \mu) \to L_f(a_n, \lambda)$.

Some special results, connected with the topic of this paper were obtained earlier in [11], [9], [7], [3].

Let

$$\tau(a) = \lim_{u\to\infty}\big(f(au)/f(u)\big) \qquad (1 < a < \infty).$$