which forms an unconditional basis in L_p and satisfies the condition

$$\deg T_n = \nu_n \leqslant n^{1+\varepsilon}$$

for $n > n_0(p, \varepsilon)$.

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DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY OF TIFLIS
TIFLIS, GEORGIA, SOVIET UNION

Received March 20, 1979 (1523)

STUDIA MATHEMATICA, T. LXXI. (1981)

Isomorphic embeddings of some generalized power series spaces

by

V. V. KASHIRIN (Rostov)

Abstract. The necessary and sufficient condition under which the generalized power series space $L_f(a_n, -1)$ contains a closed subspace of the class $(f)_1$ of all spaces $L_f(b_n, 1)$ is obtained in terms of (a_n). In particular, it is proved that every stable space $L_f(a_n, -1)$ contains a closed subspace isomorphic to the space $L_f(a_n, 1)$.

1. In the present paper we consider special classes of Köthe spaces, introduced by M. Dragilev (cf. [4]). For each fixed function $f(u)$, $u \in R$, which is odd, increasing and logarithmically convex for $u \geqslant 0$, and for $\lambda \in (-1, 0, 1, \infty)$, we consider the class $(f)_\lambda$ of Köthe spaces (called *generalized power series spaces*)

$$L_f(a_n, \lambda) = \operatorname{limpr} l_1\big(\exp f(\lambda_p a_n)\big),$$

where $a_n \uparrow \infty$, $\lambda_p \uparrow \lambda$. In the case of $f(u) = u$ we have $(f)_0 = R_0$, $f_\infty = R_\infty$, where R_0 and R_∞ are the classes of all Köthe spaces isomorphic, respectively, to the power series spaces of finite and infinite types.

We study the comparability of the linear dimensions of spaces belonging to different classes $(f)_\lambda$. In particular, we are interested in the necessary and sufficient conditions, in terms of (a_n), under which a space $L_f(a_n, \lambda)$ contains a closed subspace of class $(f)_\mu$. In the present article we consider the most interesting case, namely $\lambda = -1$, $\mu = 1$ only. For all the other pairs (λ, μ) the scheme of the proof remains the same, but the necessary and sufficient conditions, in terms of (a_n), obtained for different pairs (λ, μ) are different. All the cases were treated in the preprint [8], which contains complete proofs. The statement of the results (without proofs) can also be found in [10]. In Section 4 we state a unified necessary and sufficient condition in terms of the properties of the class of all continuous linear operators T: $L_f(a_n, \mu) \to L_f(a_n, \lambda)$.

Some special results, connected with the topic of this paper were obtained earlier in [11], [9], [7], [3].

Let

$$\tau(a) = \lim_{u\to\infty} \big(f(au)/f(u)\big) \quad (1 < a < \infty).$$

It is known [4] that the function f is always either rapidly increasing ($\tau(\alpha) \equiv \infty$) or slowly increasing ($\tau(\alpha) < \infty$ for all α).

From now on we shall assume that f is rapidly increasing. Otherwise we have $(f)_1 = (f)_{-1} = R_0$, but this means that $L_f(a_n, -1) \in (f)_1$ for all (a_n).

2. Definition 1 (cf. [12]). Assume that $X = L_f(a_n, \lambda)$, $Y = L_f(b_n, \mu)$. The pair (X, Y) is said to *satisfy the condition* R, written $(X, Y) \in R$, if every continuous linear operator $T\colon X \to Y$ is compact.

Let $(|\cdot|_p)$ and $(\|\cdot\|_p)$ be systems of norms determining the topology in the spaces X and Y, respectively. Denote by (e_n), $n = 1, 2, \ldots$, the coordinate sequence $(0, 0, \ldots, 0, 1, 0, \ldots)$, where the nth coordinate of e_n is equal to 1. Assume that $T\colon X \to Y$ is a linear operator and

$$T(e_k) = \sum_{i=1}^{\infty} t_{i,k} \cdot e_k.$$

The next lemma is well known.

Lemma 1. *An operator $T\colon X \to Y$ is continuous iff for every p there exists a $q = q(p)$ such that*

$$\sup_j \sum_{i=1}^{\infty} (|t_{i,j}| \cdot \|e_i\|_p / |e_j|_q) < \infty. \tag{1}$$

The operator T is compact iff there exists an index q such that for every p

$$\sup_j \sum_{i=1}^{\infty} (|t_{ij}| \cdot \|e_i\|_p / |e_j|_q) < \infty. \tag{2}$$

Lemma 2 (cf. [6]). *Suppose that X, Y are the generalized power series spaces whose systems of norms are $(|\cdot|_p)$, $(\|\cdot\|_p)$, respectively. If*

(3) $\exists p_1 \forall q_1 \exists q\, \forall p \exists p_2 \forall q_2 \forall k_n,\ k_n \to \infty$

$$\limsup_n \frac{\|e_{k_n}\|_p}{\|e_{k_n}\|_{p_2}} \cdot \frac{|e_n|_{q_2}}{|e_n|_q} \leqslant \limsup_n \frac{\|e_{k_n}\|_{p_1}}{\|e_{k_n}\|_p} \cdot \frac{|e_n|_q}{|e_n|_{q_1}},$$

then $(X, Y) \in R$.

Lemma 3. *If there is a constant $\delta > 0$ such that for every (k_n), $k_n \to \infty$, either*

$$\liminf_n (b_{k_n}/a_n) < 1 - \delta \tag{4}$$

or

$$\liminf_n (b_{k_n}/a_n) \geqslant 1, \tag{5}$$

then $(L_f(a_n, 1), L_f(b_n, -1)) \in R$.

Proof. Let us verify relation (3) of Lemma 2 for the matrices $|e_n|_p = \exp f(\lambda_p a_n)$ and $\|e_n\|_p = \exp f(\mu_p b_n)$, where $\lambda_p \uparrow 1$, $\mu_p \uparrow -1$. For a fixed index p_1 such that $|\mu_{p_1}| \cdot (1-\delta) < 1$ and for a fixed index q_1 we pick an index q with $q > q_1$, $\lambda_q > (1-\delta)|\mu_{p_1}|$ and for p, we let $p_2 > p$.

Assume that inequality (4) holds for a fixed sequence (k_n). Since f is rapidly increasing, we have

$$\begin{aligned}
\limsup_n \big((\|e_{k_n}\|_{p_1}/\|e_{k_n}\|_p) \cdot (|e_n|_q/|e_n|_{q_1})\big) &= \limsup_n \exp\big(f(\mu_{p_1} b_{k_n}) - f(\mu_p b_{k_n}) + f(\lambda_q a_n) - f(\lambda_{q_1} a_n)\big)\\
&\geqslant \limsup_n \exp\big(f(\lambda_q a_n) + f(\mu_{p_1} b_{k_n}) - f(\lambda_{q_1} a_n)\big)\\
&\geqslant \limsup_{n_j} \exp\big(f(\lambda_q a_{n_j}) + f(\mu_{p_1}(1-\delta) a_{n_j}) - f(\lambda_{q_1} a_{n_j})\big) = \infty,
\end{aligned}$$

where liminf in relation (4) is attained on the subsequence (n_j).

If (5) holds for a fixed sequence (k_n), then the inequality $|\mu_{p_2}| < |\mu_{p_1}|$ implies

$$\begin{aligned}
\lim_n \big((\|e_{k_n}\|_p/\|e_{k_n}\|_{p_2}) \cdot (|e_n|_{q_2}/|e_n|_q)\big) &= \lim_n \exp\big(f(\mu_p b_{k_n}) - f(\mu_{p_2} b_{k_n}) + f(\lambda_{q_2} a_n) - f(\lambda_q a_n)\big)\\
&\leqslant \lim_n \exp\big(f(\mu_p b_{k_n}) - f(\mu_{p_2} \cdot b_{k_n}) + f(\lambda_{q_2} \cdot b_{k_n})\big) = 0.
\end{aligned}$$

The relation $(L_f(a_n, 1), L_f(b_n, -1)) \in R$ follows from Lemma 2.

3. Lemma 4. *If there exist a sequence $(\varphi(r))_{r=1}^{\infty}$, $\varphi(r) \uparrow \infty$, and subsequences of indices $(k_r(s))_{s=1}^{\infty}$, $r = 1, 2, \ldots$, such that for fixed s, m ($s \neq m$) we have*

$$(k_r(s))_{r=1}^{s} \cap (k_r(m))_{r=1}^{m} = \emptyset$$

and such that for all $s \geqslant r$ the following estimates are fulfilled:

$$\forall \varepsilon,\ \varepsilon > 0 \quad \lim_s (1/s) \exp f(\varepsilon \cdot b_s) = \infty, \tag{6}$$

$$1 - 1/\varphi(r) < a_{k_r(s)}/b_s < 1 - 1/\varphi(r) + 1, \tag{7}$$

then the space $L_f(a_n, -1)$ contains a closed subspace isomorphic to the space $L_f(b_n, 1)$.

Proof. We shall consider the subspace X of $L_f(a_n, -1)$ defined by the block basic sequence (x_s), where

$$x_s = \sum_{r=1}^{s} (1/s)\big(\exp f((a_{k_r(s)}/2b_s + 1/2) b_s)\big) e_{k_r(s)}.$$

We shall prove that the space X is isomorphic to the space $L_f(b_s, -1)$. To this end, it is sufficient to show that the relations

$$\forall p \exists q \ |x_n|_p \leqslant c_p \|e_n\|_q, \tag{8}$$

$$\forall q \exists \theta \ \lim_n (|x_n|_\theta / \|e_n\|_q) = \infty \tag{9}$$

hold, where $(|\cdot|_p)$ and $(\|\cdot\|_p)$ are systems of norms for $L_f(a_n, -1)$, $L_f(b_n, -1)$, respectively. By definition we have

$$|x_s|_p = \sum_{r=1}^{s} (1/s) \exp f\big((a_{k_r(s)}/2b_s + 1/2) b_s\big) \exp f(\mu_p a_{k_r(s)}),$$

$$\|e_s\|_q = \exp f(\lambda_q a_{n(s)}),$$

where $\lambda_q \uparrow 1$, $\mu_p \uparrow -1$. From (7) we obtain

$$|x_s|_p \leqslant \sum_{r=1}^{s} (1/s) \exp\Big(f\Big(\big(1 - 1/2(\varphi(r)+1)\big) b_s\Big) + f\Big(\big(1 - 1/\varphi(r)\big) \mu_p b_s\Big)\Big).$$

For a fixed index p we can find an index m such that

$$|\mu_p| \cdot \big(1 - (1/\varphi(r))\big) > 1$$

for all $r \geqslant m$. Also we can find a q with

$$\lambda_q > 1 - \big(1/2\,(\varphi(m)+1)\big).$$

We check relation (8):

$$|x_s|_p \leqslant \sum_{r=1}^{m} (1/s) \exp f\Big(\big(1 - 1/2(\varphi(m)+1)\big) b_s\Big) + \sum_{r=1}^{s} (1/s)$$

$$\leqslant \exp\Big(f\Big(\big(1 - 1/2(\varphi(m)+1)\big) b_s\Big)\Big) + 1$$

$$\leqslant 2 \exp f\Big(\big(1 - 1/2\,(\varphi(m)+1)\big) b_s\Big) \leqslant 2\,\|e_s\|_q.$$

Now we pass to relation (9). For every fixed index q we find an index m such that

$$1 - \big(1/2\varphi(m)\big) > \lambda_q. \tag{10}$$

Since $\mu_p \uparrow -1$, we can find a θ such that

$$1 - \big(1/2\varphi(m)\big) > |\mu_\theta| \cdot \Big(1 - \big(1/(\varphi(m)+1)\big)\Big). \tag{11}$$

Write $I_s = |x_s|_\theta / \|e_s\|_q$. By (7) we have

$$I_s = \sum_{r=1}^{s} (1/s) \exp\Big(f\Big((1/2)\big((a_{k_r(s)}/b_s) + 1\big) b_s\Big) + f(\mu_\theta \cdot a_{k_r(s)}) - f(\lambda_q b_s)\Big)$$

$$\geqslant \sum_{r=1}^{s} (1/s) \exp\Big(f\Big(\big(1 - 1/2\varphi(r)\big) b_s\Big) + f\Big(\mu_\theta\big(1 - 1/(\varphi(r)+1)\big) a_{n(s)}\Big) - f(\lambda_q b_s)\Big)$$

$$\geqslant (1/s) \exp\Big(f\Big(\big(1 - 1/2\varphi(m)\big) b_s\Big) + f\Big(\mu_\theta\big(1 - 1/(\varphi(m)+1)\big) b_s\Big) - f(\lambda_q b_s)\Big).$$

By inequalities (10), (11), (6) and the fact that $f(u)$ is rapidly increasing, it is easy to show that

$$\lim_s (|x_s|_\theta / \|e_s\|_q) = \lim_s I_s = \infty.$$

This completes the proof.

COROLLARY. *Every stable space $L_f(a_n, -1)$ contains a closed subspace isomorphic to the space $L_f(a_n, 1)$.*

Proof. By definition $L_f(a_n, -1)$ is stable iff $L_f(a_n, -1)$ is isomorphic to the space $L_f(a_{2n}, -1)$ iff $\lim (a_{2n}/a_n) = 1$ (cf. [4]). This condition permits us (cf. [7]) to find subsequences $(k_r(n))_{n=1}^{\infty}$, $r = 1, 2, \ldots$, such that for fixed i, j $(i \neq j)$ we have

$$\{k_r(i)\colon r = 1, 2, \ldots, i\} \cap \{k_r(j)\colon r = 1, 2, \ldots, j\} = \emptyset,$$

but

$$1 - 1/\varphi(r) < a_{k_r(s)}/a_s < 1 - 1/\big(\varphi(r)+1\big), \qquad r = 1, 2, \ldots,$$

where $k_r(s) = \sigma(j_s^r, s)$, $\sigma(j, n) = 2^{j-1} \cdot (2n-1)$ and

$$a_{\sigma(j_n^r - 1, n)}/a_n \leqslant 1 - \big(1/\varphi(r)\big) < a_{\sigma(j_n^r, n)}/a_n$$

for $n \geqslant r$.

LEMMA 5. *If $\exists (n_j) \forall (n_{j(s)}) \forall r \exists \big(k_r(s)\big)$:*

$$1 - 1/r \leqslant \liminf_s (a_{k_r(s)}/b_{n(j(s))}) < 1 \qquad (r = 1, 2, \ldots), \tag{12}$$

then there are subsequences $\big(l_r(s)\big)$, $\big(m(s)\big)$ such that

$$1 - 1/r \leqslant \lim_s (a_{l_r(s)}/a_{m(s)}) < 1 \qquad (r = 1, 2, \ldots).$$

Proof. Using (12) for $r = 1$, we construct subsequences $\big(p_1(s)\big)$, $\big(m_1(s)\big) \subset (n_j)$ such that

$$0 < a_{p_1(s)}/b_{m_1(s)} \leqslant M(1) < 1.$$

For $r=2$ and for the already constructed subsequence $(m_1(s))$ we pick subsequences $(p_2(s))$ and $(m_2(s))\subset(m_1(s))$ satisfying the following inequality:

$$1-(1/2) < a_{p_2(s)}/b_{m_2(s)} \leqslant M(2) < 1 \quad \text{for} \quad s \geqslant 2.$$

Proceeding in the same way, we construct subsequences $(p_r(s))$ and $(m_r(s))\subset(m_{r-1}(s))$ with

$$1-1/r < a_{p_r(s)}/b_{m_r(s)} \leqslant M(r) < 1$$

for $s\geqslant r$. Using the diagonal method, we find subsequences $(m_s(s))$ and $(l_r(s))\subset(p_r(s))$ such that

$$1-1/r < \lim_s (a_{l_r(s)}/b_{m_s(n_s)}) \leqslant M(r) < 1.$$

Finally, letting $m(s)=l_s(s)$, we have

$$1-1/r < \lim_s (a_{l_r(s)}/a_{l_s(s)}) = \lim_s\big((a_{l_r(s)}/b_{m_s(n_s)})\cdot(b_{m_s(n_s)}/a_{l_s(s)})\big) \leqslant M(r) < 1.$$

THEOREM 1. *The space $L_f(a_n, -1)$ contains a closed subspace of class $(f)_1$ iff there exist subsequences $(n(s))$ and $(k_r(s))$, $r=1,2,\ldots$, such that for all $s\geqslant r$ the following estimate is fulfilled*:

$$(13) \qquad 1-1/r < a_{k_r(s)}/a_{n(s)} \leqslant c(r) < 1 \quad (r=1,2,\ldots).$$

Proof. *Sufficiency*. Without loss of generality we may assume that the sequences (b_s) $(b_s=a_{n(s)})$ and $(a_{k_r(s)})$ have properties (6), (7) and for fixed s, m $(s\neq m)$

$$(k_r(s))_{r=1}^{s}\cap(k_r(m))_{r=1}^{m}=\emptyset.$$

By Lemma 4 we assume that the space $L_f(a_n, -1)$ contains a closed subspace isomorphic to $L_f(a_{n(s)}, 1)$.

Necessity. Assume that condition (13) does not hold, but there is a subspace X of $L_f(a_n, -1)$ which is isomorphic to a space $L_f(b_n, +1)$. From Lemma 5 it follows that, for every subsequence (n_j), we can find a subsequence $(n_{j(s)})$ and a constant $\delta>0$ such that, for every sequence (k_s), $k_s\to\infty$, we have either

$$\liminf_s (a_{k_s}/b_{n_{j(s)}}) < 1-\delta \quad \text{or} \quad \liminf_s (a_{k_s}/b_{n_{j(s)}}) \geqslant 1.$$

Using Lemma 3, we have

$$\big(L_f(b_{n(j(s))}, 1),\ L_f(a_n, -1)\big)\in R.$$

Now it is easy to show that $L_f(a_n, -1)$ has no subspace isomorphic to $L_f(b_{n(j(s))}, 1)$, and we get a contradiction.

COROLLARY 1. *If the space $L_f(a_n, -1)$ is isomorphic to its subspace of codimension one, then $L_f(a_n, -1)$ contains a closed subspace of class $(f)_1$.*

Proof. It is known (cf. [4]) that $L_f(a_n, -1)$ is isomorphic to its closed hyperspace iff $a_{n+1}/a_n\to 1$. We find subsequences $(l_r(n))$ such that

$$1-1/r < a_{l_r(n)}/a_n, \text{ but } a_{l_r(n)-1}/a_n \leqslant 1-1/r.$$

Hence we have

$$1-1/r < a_{l_r(n)}/a_n = (a_{l_r(n)-1}/a_n)(a_{l_r(n)}/a_{l_r(n)-1}) \leqslant M(r) < 1$$

for all $n\geqslant N(r, M)$. It is not difficult to find sequences $(n(s))$ and $(k_r(s))$ satisfying relation (13).

COROLLARY 2. *If*

$$\liminf_n (a_{n+1}/a_n) = 1/(1-\delta) > 1, \qquad 0<\delta<1,$$

then the space $L_f(a_n, -1)$ has no subspace of the class $(f)_1$.

Proof. For arbitrary fixed subsequences $(k_r(s))$ and $(n(s))$ we have

(1) if $k_r(s_j) < n(s_j)$, then $\liminf_j (a_{k_r(s_j)}/a_{n(s_j)}) \leqslant 1-\delta$,

(2) if $k_r(s)\geqslant n(s)$ for all $s>s_0$, then

$$\liminf_s (a_{k_r(s)}/a_{n(s)}) \geqslant 1.$$

Hence condition (13) is not satisfied.

4. Obviously, if X, Y are infinite-dimensional Köthe spaces and $(X, Y)\in R$, then Y has no subspaces isomorphic to X. On the other hand, it may happen that X has no subspace isomorphic to Y, but $(Y, X)\notin R$ (cf. [9]). We shall introduce relation R_1, weaker than R and with its help we shall state a unified condition under which the space $L_f(a_n, \lambda)$ has a subspace of the class $(f)_\mu$.

Let us assume that $X=L_f(a_n, \lambda)$, $Y=L_f(b_n, \mu)$ be generalized power series spaces.

DEFINITION 2. We write $(X, Y)\in R_1$ if for every absolute basis (x_n) for X every subsequence $(x_{n(j)})$ contains a subsequence $(x_{n(j(s))})$ such that for every continuous linear operator $T\colon X\to Y$ the restriction of T to the subspace of X generated by $(x_{n(j(s))})$ is compact.

The next three lemmas are obvious.

LEMMA 6. *If $X\sim X_1$, $Y\sim Y_1$ and $(X, Y)\in R_1$, then $(X_1, Y_1)\in R_1$.*

LEMMA 7. *If $(X, Y)\in R_1$, then Y has no subspace isomorphic to X.*

LEMMA 8. *If $(X, Y)\in R$, then $(X, Y)\in R_1$.*

LEMMA 9. *$(X, Y)\in R_1$ if there exists an absolute basis (x_n) for X such that every subsequence $(x_{n(j)})$ contains a subsequence $(x_{n(j(s))})$ such that,* *for every continuous linear operator $T: X \to Y$, the restriction of T to the subspace of X generated by $(x_{n(j(s))})$ is compact.*

Proof. We shall show that every absolute basis (z_n) for X has the property mentioned above. First we find positive numbers (λ_n), a subsequence (k_n), $k_n \to \infty$, and fundamental systems of norms $(|\cdot|_p)$ and $(\|\cdot\|_p)$ for X such that (cf. [1]):

$$\forall p \exists q \quad \lambda_n \cdot |z_n|_p / \|x_{k(n)}\|_q \leqslant C(p) < \infty, \tag{14}$$

$$\forall s \exists r \quad \|x_{k(n)}\|_s / \lambda_n \cdot |z_n|_r \leqslant C(s) < \infty. \tag{15}$$

Further, for every subsequence (n_j) we find a subsequence $(n_{j(s)})$ such that $(k_{n(j(s))})$ is strictly increasing.

An arbitrary continuous linear operator T acting from the space X with the basis (z_n) to the space Y with the basis (y_n) can be defined by the matrix (t_{in}), where $t_{in} = y_i'(T(z_n))$ and y_i' is the sequence of functionals for Y, biorthogonal to the basis (y_n). By Lemma 1,

$$\forall p \exists q \sup_n \sum_{i=1}^{\infty} (|t_{in}| \cdot |y_i|_p^* / |z_n|_q) < c(p) < \infty,$$

where $(|\cdot|_p^*)$ is the system of norms determining the topology in the space Y. We shall consider the operator $T_1: X \to Y$ defined by the matrix (l_{ir}), where

$$l_{i,n(j(s))} = \lambda_{n(j(s))} t_{i,n(j(s))} = y_i'\big(T_1(x_{k(n(j(s)))})\big)$$

and $l_{i,r} = 0$ for the remaining pairs i, r. T is continuous since by (14) for q we can find a q_1 such that

$$\forall p \exists q_1 \sup_n \sum_{i=1}^{\infty} (|t_{in}| \cdot \lambda_n \cdot |y_i|_p^* / \|x_{k_n}\|_{q_1})$$
$$\leqslant \sup_n \sum_{i=1}^{\infty} (|t_{in}| \cdot |y_i|_p^* / \|z_n\|_q) \leqslant c(p) \cdot C(p) < \infty.$$

By the assumption of Lemma 9, we can find a subsequence $(k_{m(s)})$ $(k_{n(j(s))})$ for which the restriction of the continuous linear operator $T_1: X \to Y$ to the subspace of X generated by $(x_{k(m(s))})$ is compact. This means that

$$\exists q_2 \forall p \sup_s \sum_{i=1}^{\infty} (|t_{i,m(s)}| \cdot \lambda_{m(s)} \cdot |y_i|_p^* / \|x_{k_{m(s)}}\|_{q_2}) \leqslant C(p) < \infty.$$

Applying inequality (15) for the fixed index q_2, we find q_3 such that

$$\exists q_3 \forall p \sup_s \sum_{i=1}^{\infty} (|t_{i,m_s}| \cdot |y_i|_p^* / |z_{m_s}|_q)$$
$$\leqslant C(p) \cdot \sup_s \sum_{i=1}^{\infty} (|t_{i,m_s}| \cdot \lambda_{m_s} \cdot |y_i|_p^* / \|x_{k(m(s))}\|_q) < \infty.$$

By Lemma 1 the last inequality means that the restriction of T to the subspace of X generated by $(z_{m(s)})$ is compact. That is, $(X, Y) \in R_1$.

Remark. All the results mentioned above and connected with the relation R_1, can be proved for nuclear Köthe spaces (cf. [2], [5]).

THEOREM 2. *The space $L_f(a_n, -1)$ has no subspace of class $(f)_1$ iff*

$$\big(L_f(a_n, 1), L_f(a_n, -1)\big) \in R_1. \tag{16}$$

Proof. *Sufficiency.* Let us assume that (16) holds. If $L_f(a_n, -1)$ contains a subspace of class $(f)_1$, then condition (13) takes place. This means that there exist a subsequence $(n(s))$ and a subspace Z of $L_f(a_n, -1)$ such that Z is isomorphic to $L_f(a_{n(s)}, 1)$. Hence (see Lemma 7) $\big(L_f(a_{n(s)}, 1), L_f(a_n, -1)\big) \in R_1$. Therefore relation (16) does not hold either. This contradiction completes the proof of the sufficiency.

Necessity. If $L_f(a_n, -1)$ has no subspace of class $(f)_1$ then, by Theorem 1, relation (13) does not hold. Moreover, by Lemma 5, every subsequence (n_j) has a subsequence $(n_{j(s)})$ for which relation (12) is not true for the ordered pair $(a_{n(j(s))}, a_s)$. By Lemma 3

$$\big(L_f(a_{n(j(s))}, 1), L_f(a_n, -1)\big) \in R.$$

The space Z generated by the basis subsequence $(e_{n(j(s))})$ for $L_f(a_n, 1)$ is isomorphic to $L_f(a_{n(j(s))}, 1)$. Therefore $\big(Z, L_f(a_n, -1)\big) \in R$. Hence, by Lemma 9 we have (16).

Acknowledgement. The author thanks Professor C. Bessaga for his interest and for his help in the preparation of the English version of this text.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WARSAW
INSTITUTE OF TECHNOLOGY
344022 ROSTOV, SOVIET UNION

Received March 28, 1979 (1531)

STUDIA MATHEMATICA, T. LXXI. (1981)

On domination and separation of ideals in commutative Banach algebras

by

W. Żelazko* (Warszawa)

Abstract. We introduce and study the concept of domination property and approximate domination property of ideals in commutative Banach algebras, and connections of these concepts with separation of ideals by means of annihilating nets. Among other results we show that domination and separation properties coincide, as well as the approximate domination and bounded separation properties.

1. Separation of ideals. All algebras in this paper are assumed to be commutative, complex Banach algebras with unit, unless otherwise stated. The unit element will be designated by e.

The maximal ideal space of an algebra A will be designated by $\mathfrak{M}(A)$, and its Shilov boundary by $\Gamma(A)$. We shall treat elements of $\mathfrak{M}(A)$ both as ideals and as multiplicative-linear functionals. We say that an ideal $I \subset A$ *consists of joint topological divisors of zero* if there is a net $(z_a) \subset A$, $\|z_a\| \geqslant \varepsilon > 0$, such that $\lim_a z_a x = 0$ for all $x \in I$. In this case we say that the net (z_a) *annihilates* I and write $(z_a) \perp I$, or $(z_a) \in I^{\perp}$. The symbol $l(A)$ will designate the set of all (not necessarily closed) proper ideals of A, including the zero ideal, consisting of joint topological divisors of zero. The members of $l(A)$ will be called shortly *l-ideals*. We put also $\pounds(A) = l(A) \cap \mathfrak{M}(A)$. It is known that the closure of an l-ideal is again an l-ideal, $\pounds(A)$ is a closed subset of $\mathfrak{M}(A)$ containing $\Gamma(A)$ (cf. [9]), and every l-ideal is contained in a maximal ideal M belonging to $\pounds(A)$ (cf. [6]).

1.1. Definition. We say that an ideal $I \subset A$ can be *separated from an element* $x_0 \notin I$ if there is a net $(z_a) \perp I$ and $z_a x_0 \nrightarrow 0$. We say that two ideals I_1 and I_2 can be *separated* if one of them can be separated from an element of the other. We say that an ideal $I \subset A$ has the *separation property* if it can be separated from each element $x \notin I$. If in above definition we replace nets by bounded nets, we obtain the concepts of bounded separation of an ideal I from an element x, bounded separation of two

* This paper was written during the authors stay at the University of Kansas in second semester of 1978/79.