[4] M. M. Dragilev, *On regular bases in nuclear spaces*, Amer. Math. Soc. Transl (2) 93 (1970), 61–82.
[5] — *On special dimensionalities defined on some classes of Köthe spaces* (in Russian), Mat. Sb. 80, 2 (1969), 225–240.
[6] — *Riesz classes and multiple regular bases* (in Russian), Teor. Funkciĭ Funkcional. Anal. i Priložen. 15 (1972), 55–78.
[7] Ed Dubinsky, *Infinite type power series subspaces of finite type power series spaces*, Israel J. Math. 15 (1973), 257–281.
[8] V. V. Kashirin, *Isomorphic embedding of some generalized power series spaces*, preprint.
[9] — *On subspaces with finite center of the absolute Riesz scale* (in Russian), Sibirsk. Mat. Ž. 16, 4 (1975), 863–865.
[10] — *Comparison of linear dimensions of some generalized power series spaces*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 27 (1979).
[11] S. Rolewicz, *On spaces of holomorphic functions*, Studia Math. 21 (1961), 135–160.
[12] V. P. Zahariuta, *On the isomorphism of Cartesian products of locally convex spaces*, Studia Math. 46 (1973), 201–221.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WARSAW
INSTITUTE OF TECHNOLOGY
344022 ROSTOV, SOVIET UNION

Received March 28, 1979 (1531)

STUDIA MATHEMATICA, T. LXXI. (1981)

On domination and separation of ideals in commutative Banach algebras

by

W. Żelazko* (Warszawa)

Abstract. We introduce and study the concept of domination property and approximate domination property of ideals in commutative Banach algebras, and connections of these concepts with separation of ideals by means of annihilating nets. Among other results we show that domination and separation properties coincide, as well as the approximate domination and bounded separation properties.

1. Separation of ideals. All algebras in this paper are assumed to be commutative, complex Banach algebras with unit, unless otherwise stated. The unit element will be designated by e.

The maximal ideal space of an algebra A will be designated by $\mathfrak{M}(A)$, and its Shilov boundary by $\Gamma(A)$. We shall treat elements of $\mathfrak{M}(A)$ both as ideals and as multiplicative-linear functionals. We say that an ideal $I \subset A$ *consists of joint topological divisors of zero* if there is a net $(z_a) \subset A$, $\|z_a\| \geqslant \varepsilon > 0$, such that $\lim_a z_a x = 0$ for all $x \in I$. In this case we say that the net (z_a) *annihilates* I and write $(z_a) \perp I$, or $(z_a) \in I^{\perp}$. The symbol $l(A)$ will designate the set of all (not necessarily closed) proper ideals of A, including the zero ideal, consisting of joint topological divisors of zero. The members of $l(A)$ will be called shortly *l-ideals*. We put also $\pounds(A) = l(A) \cap \mathfrak{M}(A)$. It is known that the closure of an l-ideal is again an l-ideal, $\pounds(A)$ is a closed subset of $\mathfrak{M}(A)$ containing $\Gamma(A)$ (cf. [9]), and every l-ideal is contained in a maximal ideal M belonging to $\pounds(A)$ (cf. [6]).

1.1. Definition. We say that an ideal $I \subset A$ can be *separated from an element* $x_0 \notin I$ if there is a net $(z_a) \perp I$ and $z_a x_0 \nrightarrow 0$. We say that two ideals I_1 and I_2 can be *separated* if one of them can be separated from an element of the other. We say that an ideal $I \subset A$ has the *separation property* if it can be separated from each element $x \notin I$. If in above definition we replace nets by bounded nets, we obtain the concepts of bounded separation of an ideal I from an element x, bounded separation of two

* This paper was written during the authors stay at the University of Kansas in second semester of 1978/79.

ideals and bounded separation property. One can easily see that instead of using all bounded nets, we can limit ourselves to nets (z_α) with $\|z_\alpha\| = 1$ for all α.

The above concepts make sense only for l-ideals, and particularly when all ideals in A are l-ideals. The latter holds when $\mathfrak{M}(A) = \pounds(A)$ and this condition is satisfied for regular algebras, i.e. semi-simple algebras such that for every closed subset $F \subset \mathfrak{M}(A)$ and $M_0 \notin F$ there is an element $x \in A$ whose Gelfand transform $x\hat{}$ is zero on F and $x\hat{}\,(M_0) \neq 0$.

The *hull* of an ideal $I \subset A$ is the set $h(I) = \{M \in \mathfrak{M}(A):\ I \subset M\}$ and we say that two ideals I_1 and I_2 in A can be *separated by means of spectral synthesis* if $h(I_1) \neq h(I_2)$. An important example of regular algebras are the group algebras $L_1(G)$ for LCA groups G, or rather these algebras when a unit is adjoined. An important theorem of Malliavin [3], [5] states that for a non-compact LCA group G the algebra $L_1(G)$ always contains closed ideals $I_1 \neq I_2$ with $h(I_1) = h(I_2)$.

The motivation for the study of the concept of separation by means of nets is to provide a tool which works better than spectral synthesis.

The following results are proved in [10]:

1.2. PROPOSITION. *If two ideals I_1 and I_2 of a regular Banach algebra can be separated by means of spectral synthesis, then they can also be boundedly separated.*

For a regular algebra A and for any closed set $F \subset \mathfrak{M}(A)$ there always exists a unique smallest ideal I_F^0 with $h(I_F^0) = F$. It is

$$I_F^0 = \{x \in A:\ x\hat{}\,(M) = 0 \text{ for all } M \text{ in some open neighborhood of } F\}.$$

1.3. PROPOSITION. *If A is regular, then for any closed $F \subset \mathfrak{M}(A)$ the ideal I_F^0 has the separation property.*

Since usually the ideal I_F^0 is non-closed, the above proposition shows that it is possible to separate an ideal from its closure.

1.4. PROPOSITION. *There exists a regular algebra A and two ideals I_1, I_2 in A which can be separated, but cannot be boundedly separated.*

2. Domination property.

2.1. DEFINITION. We say that an element $x \in A$ is *dominated by elements* $x_1, x_2, \ldots, x_n \in A$ if there exists a constant $C > 0$ such that for all $z \in A$ the following holds true:

$$\|xz\| \leqslant C \sum_{i=1}^{n} \|x_i z\|. \tag{1}$$

In this case we write $x < (x_1, \ldots, x_n)$. We say that an element $x \in A$ is *dominated by an ideal* $I \subset A$ if $x < (x_1, \ldots, x_n)$ for some n-tuple $(x_1, \ldots, x_n)$ of elements of I. In this case we write $x < I$. We say that an ideal $I \subset A$

has the *domination property* if the relation $x < I$ implies $x \in I$. The family of all (not necessarily closed) ideals in A with domination property will be designated by $D(A)$.

We say that a family $F = (I_\alpha)$ of ideals of A is *directed*, if for any $I_\alpha, I_\beta \in F$ there exists $I_\gamma \in F$ such that $I_\alpha \cup I_\beta \subset I_\gamma$.

2.2. PROPOSITION. *Let A be a commutative Banach algebra with unit. Then*

(i) $\pounds(A) \subset D(A) \subset l(A)$,

(ii) *If $I_\alpha \in D(A)$, then $\bigcap_\alpha I_\alpha \in D(A)$,*

(iii) *If (I_α) is a directed family of ideals and $I_\alpha \in D(A)$ for all α, then $\bigcup_\alpha I_\alpha \in D(A)$.*

Proof. (i): Let $M \in \pounds(A)$ and $x_1, x_2, \ldots, x_n \in M$. If $x < (x_1, \ldots, x_n)$, then any net annihilating M annihilates x also. Since M is maximal, we have $x \in M$ and so $M \in D(A)$. If $I \notin l(A)$, then the unit element e is dominated by some elements in I and $e \notin I$. Thus $I \notin D(A)$.

(ii): If $I = \bigcap I_\alpha$ and $x_1, \ldots, x_n \in I$, then any element x dominated by $(x_1, \ldots, x_n)$ is in I_α for all α and so $x \in I$. Thus $I \in D(A)$.

(iii): The proof follows immediately from Definition 2.1.

Remark. Since the ideal I_F^0 can be expressed as a union of a directed family of ideals which are intersections of maximal ideals, then Proposition 1.3 is a consequence of above proposition and Theorem 2.8 below.

2.3. COROLLARY. $D(A) \cap \mathfrak{M}(A) = \pounds(A)$.

2.4. COROLLARY. *Every ideal in $D(A)$ is contained in a maximal ideal which also belongs to $D(A)$.*

PROBLEM 1. Suppose that $I \in D(A)$. Does the closure $\bar{I}$ belong to $D(A)$, too?

2.5. COROLLARY. *If Ω is a compact Hausdorff space, then every closed ideal in $A = C(\Omega)$ is in $D(A)$.*

The proof follows from the fact that A is a regular algebra and every closed ideal in A is an intersection of maximal ideals.

There are, however, non-closed ideals in $A = C[0,1]$ which are not in $D(A)$. If $x_0(t) = t$ and I is the principal ideal $x_0 A$, then the element $x(t) = t\sin t^{-1}$ is dominated by I and does not belong to I.

2.6. LEMMA. *Let R be a set of indices and let $\{(z_\gamma^\varrho)_{\gamma \in S}\}$, $\varrho \in R$ be a family of nets of elements of A indexed by the same directed system S of indices γ. Then there exists a single net $(z_\alpha) \subset A$ such that*

$$\{x \in A:\ z_\alpha x \to 0\} = \bigcap_{\varrho \in R} \{x \in A:\ z_\gamma^\varrho x \xrightarrow[\gamma]{} 0\}.$$

If, moreover, $\|z_\gamma^\varrho\| = 1$ for all γ in S and all $\varrho \in R$, then $\|z_\alpha\| = 1$ for all α.

Proof. Put

$$J = \bigcap_{\varrho\in R}\{x \in A:\ z_\gamma^\varrho x \underset{\gamma}{\to} 0\}.$$

If $J = A$, then we can assume $\alpha = \gamma$ and put $z_\alpha = z_\gamma^\varrho$ for an arbitrarily choosen fixed $\varrho \in R$. Suppose then that $J \neq A$ and form a new directed system of indices β in the following way. We put $\beta = \{F_1, F_2, k\}$, where $F_1 = F_1(\beta)$ is a finite non-void subset of $A\setminus J$, $F_2 = F_2(\beta)$ is a finite non-void subset of J and $k = k(\beta)$ is a positive integer. Write $\beta_1 \geqslant \beta_2$ if for the corresponding sets F_1, F_2 and integers k we have $F_1(\beta_1) \supset F_1(\beta_2)$, $F_2(\beta_1) \supset F_2(\beta_2)$, and $k(\beta_1) \geqslant k(\beta_2)$.

By assumption, to any $x \notin J$ there correspond an index $\varrho = \varrho(x)$ and a positive constant $c(x)$ such that

$$\limsup_\gamma \|z_\gamma^{\varrho(x)} x\| = c(x) > 0.$$

Now, to each index β we associate n indices $\alpha_1(\beta), \ldots, \alpha_n(\beta)$, where n is the cardinality of $F_1(\beta)$, and order them in the following way. If $\alpha = \alpha_i(\beta)$, $\alpha' = \alpha_i'(\beta')$, then $\alpha > \alpha'$ if $\beta > \beta'$, or in the case when $\beta = \beta'$ if $i > i'$. In this way we obtain a directed system of indices α. Define now the net (z_α) in the following way. For $\alpha = \alpha_i(\beta)$ write $F_1(\beta) = (x_1, \ldots, x_n)$, $F_2(\beta) = (y_1, \ldots, y_m)$ and put $z_\alpha = z_\gamma^{\varrho(x_i)}$ where $\gamma \in S$ is choosen in such a way that

$$\|z_\gamma^{\varrho(x_i)} y_j\| < 1/k(\beta) \quad \text{for} \quad j = 1, 2, \ldots, m$$

and

$$\|z_\gamma^{\varrho(x_i)} x_i\| > \tfrac{1}{2}c(x_i).$$

Such an index γ exists because of definition of $c(x_i)$ and because the net $(z_\gamma^{\varrho(x_i)})_{\gamma\in S}$ annihilates J. This is clear that the net (z_α) annihilates J and does not annihilate $A\setminus J$ since for any $x \in A\setminus J$ we have

$$\limsup_\alpha \|z_\alpha x\| \geqslant \tfrac{1}{2}c(x) > 0.$$

2.7. Theorem. *Let A be a commutative Banach algebra with unit e. Then an ideal $I \subset A$ has the domination property if and only if it is of the form*

$$I = \{x \in A:\ z_\alpha x \to 0\}, \tag{2}$$

where (z_α) is a net of elements of A which does not tend to zero.

Proof. Suppose that I is a subset of A of the form (2). Since $z_\alpha \not\to 0$, it is a proper ideal in A. If $x < I$, then relation (1) shows that $z_\alpha x \to 0$ and so $x \in I$. Thus $I \in D(A)$. Suppose now that $I \in D(A)$. We shall construct a directed system S of indices γ such that for each $x \notin I$ there is a net $(z_\gamma(x))_{\gamma\in S}$ in $I^\perp$ with $z_\gamma(x)x \not\to 0$. Once we have the family of nets $(z_\gamma(x))$, $x \in A\setminus I$, then Lemma 2.6, with $R = A\setminus I$ and $z_\gamma^\varrho = z_\gamma(x)$, provides us with a net (z_α) which realizes formula (2). The directed systems of indices γ will consist of pairs $\gamma = (F, k)$, where F is a finite subset of I and k is a positive integer. For $\gamma = (F, k)$ and $\gamma' = (F', k')$ we write $\gamma \geqslant \gamma'$ if $F \supset F'$ and $k \geqslant k'$. Fix an element $x \notin I$. Since I has the domination property and $x \notin I$, for each $\gamma = (F, k)$, $F = (x_1, \ldots, x_n) \subset I$, there exists by formula (1) an element $z_\gamma(x)$ such that

$$1 = \|z_\gamma(x)\cdot x\| > k\sum_{i=1}^{n}\|x_i z_\gamma(x)\|.$$

It follows that $(z_\alpha(x)) \perp I$ and $z_\alpha(x)\cdot x \not\to 0$. The conclusion follows.

2.8. Theorem. *An ideal $I \subset A$ has the domination property if and only if it has the separation property.*

Proof. If $I \in D(A)$, then by formula (2) the ideal I has the separation property. If $I \notin D(A)$, then there is an element $x \notin I$ which is dominated by an n-tuple $(x_1, \ldots, x_n) \subset I$. Formula (1) implies that every net annihilating I must also annihilate x, and so I cannot be separated from x.

For an arbitrary ideal $I \in \mathfrak{l}(A)$ there always exists a unique smallest ideal in $D(A)$ which contains I. It will be denoted by I^D. The proof of the following proposition is left to the reader.

2.9. Proposition. *Let $I \in \mathfrak{l}(A)$. Then*

$$I^D = I^{\perp\perp} = \bigcap\{J \in D(A):\ I \subset J\} = \{x \in A:\ x < I\}.$$

Here for a family N of nets of elements of A we put $N^\perp = \{x \in A:\ z_\alpha x \to 0$ for all $(z_\alpha) \in N\}$. The symbol $I^{\perp\perp}$ denotes $(I^\perp)^\perp$. We have also $N^\perp = N^{\perp\perp\perp}$ and $I^\perp = I^{\perp\perp\perp}$ for any family of nets N, containing at least one net which does not tend to zero, and for any ideal in $\mathfrak{l}(A)$.

For two ideals $I, J \in \mathfrak{l}(A)$ write $I < J$ if for each $x \in I$ we have $x < J$. One easily sees that $I < J$ is equivalent to each of the following relations $I^D < J$, $I < J^D$ or $I^D < J^D$, also $I_1 < I_2$, $I_2 < I_3$ imply $I_1 < I_3$. The relations $I < J$ and $J < I$ imply $I^D = J^D$, so they imply $I = J$ in the case where $I, J \in D(A)$. One has also $I^\perp = I^{D\perp}$ for all $I \in \mathfrak{l}(A)$. Since two ideals $I, J \in \mathfrak{l}(A)$ can be separated if and only if $I^\perp \neq J^\perp$, we have the following

2.10. Proposition. *Two ideals $I, J \in \mathfrak{l}(A)$ can be separated if and only if $I^D \neq J^D$.*

Another characterization of $D(A)$ will be obtained by means of the concept of extension of an algebra.

An *extension* or *superalgebra* of A is an algebra B with unit which contains A under a unital topological isomorphism. In this situation we write $A \subset B$. By a theorem of Lindberg [2] we can always replace the norm in B by an equivalent norm, so that the embedding in question becomes an isometry. Thus without loss of generality we can consider only isometric extensions of A. If I is an ideal in A, then for any extension $B \supset A$ we denote by I_B the smallest ideal in B containing I. $\bar{I}_B$ will designate the closure of I_B in B. We have $I_B = \{\sum_1^n x_i b_i:\ x_i \in I,\ b_i \in B\}$. If $I \in \mathfrak{l}(A)$, then $\bar{I}_B \in \mathfrak{l}(B)$, so it is a proper ideal in B. We put $I_B^D = (I_B)^D$ for any $I \in \mathfrak{l}(A)$, $A \subset B$. Under this notation we have

2.11. Proposition. *An ideal $I \in \mathfrak{l}(A)$ is in $D(A)$ if and only if for each extension $B \supset A$ we have*

$$I = I_B^D \cap A.$$

Proof. Taking $B = A$ we see that the above formula implies $I = I^D \in D(A)$. Suppose now that $I \in D(A)$. Since obviously $I \subset I_B^D \cap A$, we have to show that $I_B^D \cap A \subset I$. So let $x \in I_B^D \cap A$. If $(z_a) \perp I$, then $(z_a) \perp I_B$. But $I_B^D = I_B^{\perp\perp}$ and $I_B^{\perp} = I_B^{\perp\perp\perp}$, thus $(z_a) \perp I_B^D$ and so $z_a x \to 0$. Thus $x \in I^{\perp\perp} = I$ and $I_B^D \cap A \subset I$.

In [10] we posed the following domination conjecture: If $x < (x_1, \ldots, x_n) \subset A$, then there is an extension $B \supset A$ and elements $b_1, \ldots, b_n \in B$ such that

$$x = \sum_1^n x_i b_i.$$

(Clearly every element in A of this from is dominated by $(x_1, \ldots, x_n)$.) However, as was recently shown by Vladimir Müller [4], this conjecture fails in general. Nevertheless, in some instances the conjecture is true and we have the following domination theorems.

2.12. Theorem (Arens [1]). *If $x, y \in A$, $x < y$, then there is extension $B \supset A$ and element $b \in B$ such that*

$$x = yb.$$

2.13. Theorem ([11]). *If A is a uniform algebra and $x < (x_1, \ldots, x_n) \subset A$, then there is an extension $B \supset A$, which is also a uniform algebra, and elements $b_1, \ldots, b_n \in B$ such that*

$$x = \sum x_i b_i.$$

In order to obtain results on separation of ideals in some algebras without unit we need an extension of Arens theorem. An approximate identity for A is a net $(\delta_a) \subset A$ such that $\delta_a x \to x$ for all $x \in A$. We do not assume that the net (δ_a) is bounded.

2.14. Proposition. *Let A be a commutative Banach algebra with an approximate identity (δ_a). Let $x, y \in A$ and suppose that there is a positive C such that for all $z \in A$*

$$\|xz\| \leqslant C\|yz\|. \tag{3}$$

Then there exists a superalgebra $B \supset A$ and an element $b \in B$ such that $x = yb$.

Proof. Let $A_1 = A \oplus Ce$ be A with an adjoined identity e and with the norm given by $\|x + \lambda e\| = \|x\| + |\lambda|$ for $x \in A$, $\lambda \in C$. Relation (3) implies $\|x(z + \lambda\delta_a)\| \leqslant C\|y(z + \lambda\delta_a)\|$ for all $z \in A$, $\lambda \in C$. By passing to the limit we obtain $\|x(z + \lambda e)\| \leqslant \|y(z + \lambda e)\|$, so $x < y$ in A_1 and we apply Arens domination theorem.

For a not necessarily unital commutative Banach algebra A and $x_0 \in A$ denote by $I_n(x_0)$ the smallest ideal in A containing x_0^n, i.e. $I_n(x_0) = \{x_0^n y + \lambda x_0^n:\ y \in A,\ \lambda \in C\}$.

We shall use Proposition 2.14 in the proof of the following result.

2.15. Proposition. *Let A be a commutative regular Banach algebra either with unit element or with an approximate identiy. If $x_0 \in A$, $0 \in \sigma(x_0)$, and 0 is not an isolated point of the spectrum $\sigma(x_0)$, then all ideals $I_n(x_0)$, $n = 1, 2, \ldots$ can be mutually separated.*

Proof. It is clear that an element x is dominated by $I_n(x_0)$ if and only if $x < x_0^n$, so by Proposition 2.10 it is sufficient to show that for $n > m$ the element x_0^m cannot be dominated by x_0^n. Suppose then that $x_0^m < x_0^n$. By Theorem 2.11 we have $x_0^m = bx_0^n$ for an element b in some extension B of A. We have

$$x_0^m(e - bx_0^{n-m}) = 0 \tag{4}$$

and there is a sequence $f_k \in \mathfrak{M}(A)$ with $0 \neq f_k(x_0) \to 0$. Since A is regular, we have $\mathfrak{M}(A) = \Gamma(A)$ and all functionals f_k can be extended to functionals F_k in $\mathfrak{M}(B)$. Since $F_k(x_0) \neq 0$, relation (4) implies $F_k(b)F_k(x_0)^{n-m} = 1$ for all k, and so $F_k(b) \to \infty$. A contradiction, since $|F_k(b)| \leqslant \|b\|$, and the conclusion follows.

We close this section with the following questions.

Problem 2. For which Banach algebras is the domination conjecture true?

Problem 3. Let A be a regular Banach algebra with unit element. Do all closed ideals in A belong to $D(A)$?

Problem 4.(²) Suppose that $x < (x_1, \ldots, x_n) \subset A$. Does there exist an extension $B \supset A$ and elements $b_1, \ldots, b_n \in B$ such that $x = \sum_1^n x_i b_i$? Here we do not assume that B is a Banach algebra, but any topological algebra containing A under a homeomorphic imbedding preserving the unit element.

(²) A negative answer to this problem has been given in [12] [added in proof].

3. Approximate domination.

3.1. Definition. Let A be a commutative Banach algebra with unit element e and let I be a (proper) ideal in A. An element $x \in A$ is said to be *approximately dominated by* I if for each $\varepsilon > 0$ there exist elements $x_1, \ldots, x_n \in I$ such that for all $z \in A$ we have

$$\|xz\| \leqslant \sum_{i=1}^{n} \|x_i z\| + \varepsilon \|z\|. \tag{5}$$

In this case we write $x <_a I$. We say that an ideal $I \subset A$ possesses the *approximate domination property* if the relation $x <_a I$ implies $x \in I$. The family of all ideals in A possessing the approximate domination property will be designated by $D_0(A)$.

3.2. Proposition. *If A is as above, then $D_0(A) \subset D(A)$.*

Proof. The relation $x < I$ clearly implies $x <_a I$. Thus if $I \in D_0(A)$ and $x < I$, then $x <_a I$. So $x \in I$ and $I \in D(A)$.

3.3. Corollary. $D_0(A) \subset l(A)$.

3.4. Proposition. *Every ideal in $D_0(A)$ is closed.*

Proof. Let $I \in D_0(A)$ and $y \in \bar{I}$. For any given $\varepsilon > 0$ we find an element $x \in I$ with $\|x - y\| < \varepsilon$. Since $\|yz\| = \|(y-x)z + xz\| \leqslant \|xz\| + \varepsilon\|z\|$ for all $z \in A$, then $y <_a I$ and so $y \in I$. Thus $I = \bar{I}$.

3.5. Proposition. *Let A be as above. Then*

(i) $\mathfrak{L}(A) \subset D_0(A)$,

(ii) *If $I_a \in D_0(A)$, then $I = \bigcap_a I_a \in D_0(A)$.*

Proof. (i): Let $M \in \mathfrak{L}(A)$ and choose a net $(z_a) \perp M$, $\|z_a\| = 1$. If $x <_a M$, then relation (5) implies that for each $\varepsilon > 0$ we have $\limsup_a \|xz_a\| \leqslant \varepsilon$, so $\lim_a \|xz_a\| = 0$ and by the maximality of M we have $x \in M$. So $M \in D_0(A)$.

(ii): If $x <_a I$, then $x <_a I_a$ for all a, so $x \in I_a$ for all a and $x \in I$.

3.6. Theorem. *Let A be a commutative Banach algebra with unit e. Then an ideal $I \subset A$ has the approximate domination property if and only if it is of the form*

$$I = \{x \in A\colon z_a x \to 0\}, \tag{6}$$

where (z_a) is a net of elements of A such that $\|z_a\| = 1$ for all a.

Proof. Suppose that an ideal $I \subset A$ is of the form (6). If $x <_a I$, then relation (5) implies that for each $\varepsilon > 0$ we have $\limsup_a \|xz_a\| \leqslant \varepsilon$ and so $x \in I$. Thus $I \in D_0(A)$. Suppose now that $I \in D_0(A)$. We should have a directed system S of indices γ such that for each $x \notin I$ there is a net $(z_\gamma(x)) \perp I$, $\|z_\gamma(x)\| = 1$ for all $\gamma \in S$, and $z_\gamma(x) \cdot x \nrightarrow 0$. Once we have the family of nets $(z_a(x)) \in I^\perp$, $x \in A \setminus I$, with $\|z_\gamma(x)\| = 1$ for all $\gamma \in S$, then, by Lemma 2.6 with $R = A \setminus I$ and $z_\gamma^x = z_\gamma(x)$, we obtain the desired net z_a. The system S is exactly the same as in the proof of Theorem 2.7. Fix an element $x \notin I$ and $\varepsilon > 0$. Since I has the approximate domination property, then for each $\gamma = (F_\gamma, k_\gamma)$, $F_\gamma = (x_1, \ldots, x_n) \subset I$, there exists by formula (5) an element $z_\gamma(x)$ with $\|z_\gamma(x)\| = 1$ such that

$$\|xz_\gamma(x)\| \geqslant k_\gamma \sum_{i=1}^{n} \|x_i z_\gamma(x)\| + \varepsilon \geqslant \varepsilon \tag{7}$$

(we use in the negation of (5), $k_\gamma x_i$ instead of x_i). Relation (7) shows that $\|x_i z_\gamma(x)\| \leqslant \|x\|/k_\gamma$ for all $\gamma > \gamma_0$, where $\gamma_0 = (F_0, k_0)$ and $x_i \in F_0$. Since $k_\gamma \to \infty$, this implies that $z_\gamma(x) \cdot x_i \xrightarrow[\gamma]{} 0$. Since we can take for x_i an arbitrary element of I, this implies $(z_\gamma(x)) \perp I$. On the other hand, formula (7) shows that $xz_\gamma(x) \nrightarrow 0$. The conclusion follows.

3.7. Theorem. *An ideal $I \subset A$ has the approximate domination property if and only if it has the bounded separation property.*

Proof. If $I \in D_0(A)$, then by formula (6) the ideal I possesses the bounded separation property. If $I \notin D_0(A)$, then there is an element $x \notin I$ such that for each $\varepsilon > 0$ there are elements $x_1, x_2, \ldots, x_n \in I$ with

$$\|xz\| \leqslant \sum \|x_i z\| + \varepsilon \|z\| \tag{8}$$

for all $z \in A$. Let $(z_a) \perp I$ and $\|z_a\| = 1$ for all a. Formula (8) implies that $\limsup_a \|z_a x\| \leqslant \varepsilon$, and since it holds for all $\varepsilon > 0$ we have $\lim_a z_a x = 0$. Thus every bounded net (z_a) annihilating I annihilates x also and so I does not possess the bounded separation property.

For an ideal $I \in \mathfrak{L}(A)$ denote by $I^\#$ the set of all nets (z_a) in $I^\perp$ with $\|z_a\| = 1$ for all a, and denote by I^{D_0} the smallest ideal in $D_0(A)$ which contains I. The proof of the following proposition is left to the reader.

3.8. Proposition. *Let $I \in l(A)$. Then*

$$I^{D_0} = \bar{I}^{D_0} = I^{\#\perp} = \bigcap \{J \in D_0(A)\colon I \subset J\} = \{x \in A\colon x <_a I\}.$$

If in the relations following Proposition 2.9 we replace D by D_0, $<$ by $<_a$ and $\perp$ by $\#$ we obtain again valid relations. In particular we obtain

3.9. Proposition. *If $I, J \in l(A)$, then I can be boundedly separated from J if and only if $I^{D_0} \neq J^{D_0}$.*

If for $I \in l(A)$ and for an extension $B \supset A$ we put $I_B^{D_0} = (I_B)^{D_0}$ and $\bar{I}_B^{D_0} = (\bar{I}_B)^{D_0}$ we obtain the following proposition whose proof is similar to that of Proposition 2.11.

3.10. Proposition. *Let $I \in l(A)$. Then $I \in D_0(A)$ if and only if for each extension $B \supset A$ we have*

$$I = I_B^{D_0} \cap A \ (= \bar{I}_B^{D_0} \cap A).$$

We do not know any example of an ideal $I \in D_0(A)$ for a semi-simple Banach algebra A which is not of the form

$$I = \bigcap \{M \in \mathfrak{L}(A): I \subset M\}. \tag{9}$$

By Proposition 3.5 every ideal of this form is in $D_0(A)$. We shall now show that for uniform algebras $D_0(A)$ coincides with the ideals of the form (9).

3.11. PROPOSITION. *Let A be a uniform algebra with unit e. Then every ideal in $D_0(A)$ is of the form* (9).

Proof. It is shown in [8] that for a uniform algebra A we have $\Gamma(A) = \mathfrak{L}(A)$. Let $I \in D_0(A)$ and consider the imbedding $A \subset B = C(\Gamma(A))$. Since every closed ideal of B is an intersection of maximal ideals (cf. e.g. [7]), we have

$$\bar{I}_B = \bigcap \{\tilde{M} \in \mathfrak{M}(B): I_B \subset \tilde{M}\} = \bigcap \{\tilde{M} \in \mathfrak{M}(B): I \subset \tilde{M}\}. \tag{10}$$

Thus

$$\bar{I}_B \cap A = \bigcap \{\tilde{M} \cap A: \tilde{M} \in \mathfrak{M}(B), I \subset \tilde{M} \cap A\}.$$

But the intersections $\tilde{M} \cap A$, $\tilde{M} \in \mathfrak{M}(B)$ are precisely the elements of $\Gamma(A) = \mathfrak{L}(A)$. So

$$\bar{I}_B \cap A = \bigcap \{M \in \mathfrak{L}(A): I \subset M\}.$$

The conclusion will follow if we show $I = \bar{I}_B \cap A$. But this follows immediately from Proposition 3.10, since by formula (10) and Propositions 3.4, 3.5 and 3.8 we have $I_B^{D_0} = \bar{I}_B$.

3.12. COROLLARY. *For uniform algebras the spectral synthesis separates better than bounded nets. So by Proposition* 1.2 *separation by spectral synthesis is for uniform regular algebras equivalent to separation by bounded nets.*

Another type of ideal in $D_0(A)$ which must be of the form (9) is given by the following result.

3.13. PROPOSITION. *If A is a semi-simple algebra and $a \in A$, then the ideal $I = a^{\perp}$ is of the form* (9).

Proof. Let $S = \{M \in \mathfrak{L}(A): a\hat{\ }(M) \neq 0\}$. If $x \in a^{\perp}$, then $ax = 0$ and so $x\hat{\ }(M) = 0$ for all $M \in S$. Thus $a^{\perp} \subset M$ for all $M \in S$. Put

$$J = \bigcap \{M: M \in S\}.$$

So $a^{\perp} \subset J$. If $x \in J$, then $x\hat{\ }(M) = 0$ for all $M \in S$ and so $x\hat{\ }(M)a\hat{\ }(M) = 0$ for all $M \in \mathfrak{L}(A)$. Since $\Gamma(A) \subset \mathfrak{L}(A)$ and A is semi-simple it follows that $xa = 0$ or $x \in a^{\perp}$. So $a^{\perp} = J$ and the conclusion follows.

If A is not a semi-simple algebra, then there are ideals in $D_0(A)$ which are not of the form (9). For example, if $x_0 \in A$ is an element such that $x_0^2 = 0$ and $x_0 \operatorname{rad} A \neq (0)$, then $x_0^{\perp} \in D_0(A)$ and $x_0^{\perp}$ is not of the form (9).

Such an element x_0 can be found in the convolution algebra $L_1(0, 1)$ with unit adjoined.

PROBLEM 5. Is every ideal in $D_0(A)$ of the form (9), if A is a regular algebra?

The following problem is an analog of the domination conjecture, it is not disproved by the Müller example.

PROBLEM 6. Let $x <_a I \subset A$. Does there exist an extension $B \supset A$ such that $x \in \bar{I}_B \cap A$?

We give a proof of this conjecture for uniform algebras.

3.14. PROPOSITION. *If A is a uniform algebra and $x <_a I$, where I is an ideal in A, then there is an extension $B \supset A$ such that $x \in \bar{I}_B \cap A$.*

Proof. Let $B = C(\Gamma(A))$. If I is not contained in an ideal $M \in \Gamma(A)$, then $I_B = B$ and so $x \in \bar{I}_B \cap A$. Otherwise $\bar{I}_B = \bigcap \{\tilde{M} \in \mathfrak{M}(B): I \subset \tilde{M}\}$ and $\bar{I}_B \cap A = \bigcap \{M \in \mathfrak{L}(A): I \subset M\}$. By Proposition 3.5 we have $\bar{I}_B \cap A \in D_0(A)$. Since $I \subset \bar{I}_B \cap A$, the relation $x <_a I$ implies $x <_a \bar{I}_B \cap A$ and so $x \in \bar{I}_B \cap A$.

References

[1] R. Arens, *Extensions of Banach algebras*, Pacific J. Math. 10 (1960), 1–16.

[2] J. A. Lindberg, *Extension of algebra norms and applications*, Studia Math. 40 (1971), 35–39.

[3] P. Malliavin, *Impossibilité de la synthèse spectrale sur des groupes abeliens, non compacts*, Publications Mathematiques de L'Institut des Hautes Etudes Scientifiques Paris 2 (1959), 61–68.

[4] V. Müller, *On domination problem in Banach algebras*, Comm. Math. Univ. Carolinae (in print).

[5] W. Rudin, *Fourier analysis on groups*, New York 1962.

[6] Z. Słodkowski, *On ideals consisting of joint topological divisors of zero*, Studia Math. 48 (1973), 83–88.

[7] W. Żelazko, *Banach algebras*, Elsevier-PWN, 1973.

[8] — *A characterization of Shilov boundary in function algebras*, Math. Comm. 14 (1970), 59–64.

[9] — *On a certain class of non-removable ideals in Banach algebras*, Studia Math. 44 (1972), 87–92.

[10] — *On some classes of ideals in commutative Banach algebras*, Rend. Mat. Fis. Milano 48 (in print).

[11] — *A domination theorem in function algebras*, Collect. Math. (submitted).

[12] V. Müller, *On domination and extensions of Banach algebras*, to appear in Studia Math. 73.

MATHEMATICAL INSTITUTE
POLISH ACADEMY OF SCIENCES

Received May 21, 1979 (1550)
Revised version September 5, 1979