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Consistency theory in semiconservative spaces

by

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Abstract. Useful sufficient conditions are provided on a sequence space V such that every matrix summing V must be conull. This results in generalizations of Copping's conullity theorem and part of the classical consistency theory of matrices to a semiconservative setting. For certain sequence spaces K_0 and V it is shown that K_0^β is $\sigma(K_0^\beta, W_E \cap V)$ sequentially complete for all FK spaces E containing K_0 whenever the multiplier algebra of V contains sufficiently many slowly oscillating sequences. Known completeness and consistency theorems for almost convergence follow as corollaries.

§ 1. Introduction. A theorem of J. Copping states that every matrix which sums the bounded convergence domain of a conull matrix must be conull. This property of conullity is known to be equivalent to the Mazur–Orlicz Bounded Consistency Theorem. Furthermore, G. Bennett and N. Kalton have observed that portions of the classical consistency theory of matrices can be reduced to problems involving the sequential completeness of l^1, the space of absolutely summable sequences, under appropriate weak topologies. The above work is placed in the context of FK spaces containing c_0, the space of null sequences.

We shall develop a consistency theory for semiconservative spaces which extends the above results. An essential feature of this extension is the provision of useful sufficient conditions on a sequence space V such that every matrix summing V must be conull. Copping's conullity theorem is thereby generalized to an appropriate semiconservative setting. For certain sequence spaces K_0 and V we show that K_0^β is $\sigma(K_0^\beta, W_E \cap V)$ sequentially complete for all FK spaces E containing K_0 whenever the multiplier algebra of V contains sufficiently many slowly oscillating sequences.

As an application we observe that the multiplier algebra of ac_0 contains appropriate oscillating sequences, where ac_0 is the space of sequences which are almost convergent to zero. Known completeness and consistency theorems for almost convergence follow from this one essential property of ac_0.

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The following notation will be used:

e, e^n are the sequences given by $e_k = 1$ for all k and $e^n_k = 0$ for $k \neq n$, $e^n_n = 1$;

φ is the linear span of $\{e^n\colon n = 1, 2, \ldots\}$;

ω is the space of all complex sequences;

$c_0 = \{x \in \omega\colon \lim x = 0\}$;

$c = \{x \in \omega\colon \lim x \text{ exists}\}$;

$m = \{x \in \omega\colon \|x\|_\infty = \sup\limits_k |x_k| < \infty\}$;

$l^p = \{x \in \omega\colon \|x\|_p = \big(\sum\limits_k |x_k|^p\big)^{1/p} < \infty\}$ for $1 \leqslant p < \infty$;

$bv = \{x \in \omega\colon \|x\|_{bv} = \lim |x_k| + \sum\limits_k |x_k - x_{k+1}| < \infty\}$;

$bs = \{x \in \omega\colon \|x\|_{bs} = \sup\limits_n \big|\sum\limits_{k=1}^{n} x_k\big| < \infty\}$;

$cs = \{x \in \omega\colon \sum\limits_k x_k \text{ is convergent}\}$ with $\|x\|_{cs} = \|x\|_{bs}$.

Note for $x \in bs$ and $y \in \varphi$ that $\big|\sum\limits_k x_k y_k\big| \leqslant \|x\|_{bs}\|y\|_{bv}$.

A *sequence space* is a linear subspace of ω. A sequence space E is an FK *space* if E is a locally convex Fréchet space on which the coordinate functionals $x \to x_n$ are continuous for each n. An FK space whose topology is normable is a BK *space*. The spaces c_0, c, and m are BK spaces under the norm of m. The spaces l^p, bv, bs, and cs are BK spaces under the indicated norms.

We shall consider only sequence spaces containing φ.

For $x \in \omega$ let $P_n x = \sum\limits_{k=1}^{n} x_k e^k$. If E is an FK space, then

$$W_E = \{x \in E\colon P_n x \to x \text{ weakly in } E\}.$$

The space E is an AK *space* if $P_n x \to x$ for all $x \in E$.

Let $\langle E, F\rangle$ be paired linear spaces under a bilinear form $\langle\cdot,\cdot\rangle$. A sequence $\{x^n\}$ in E is $\sigma(E, F)$ *Cauchy* if the map $y \to \{\langle x^n, y\rangle\}$ takes F into c. An FK space E is called *semiconservative* if $\{P_n e\}$ is $\sigma(E, E')$ Cauchy, where E' is the space of continuous linear functionals on E. E is called *conull* if $e \in W_E$, i.e. if $\{P_n e\}$ is $\sigma(E, E')$ convergent in E. Hence, conull spaces are semiconservative.

If V and W are sequence spaces, let $M(V, W)$ denote the sequence space of multipliers of V into W. Thus, $u \in M(V, W)$ if and only if $uv = \{u_k v_k\} \in W$ for all $v \in V$. For any $u \in \omega$ let $uV = \{uv\colon v \in V\}$. If E and F are BK spaces, then $M(E, F)$ may be identified as a space of bounded maps from E into F, and $M(E, F)$ is a BK space under the operator norm. Let $M(V)$ denote $M(V, V)$.

If V is a sequence space, let $V^\beta = M(V, cs)$. If V and W are sequence spaces such that $V \subset W^\beta$, then the bilinear form $\langle v, w\rangle = \sum\limits_k v_k w_k$ is defined on $V \times W$. We shall be concerned with the weak topology $\sigma(V, W)$ on V relative to the pairing $\langle V, W\rangle$.

Finally, we shall consider matrix maps on sequence spaces. Let $A = (a_{nk})$, $n, k = 1, 2, \ldots$, be an infinite matrix of scalars. If $x \in \omega$ and $\sum\limits_k a_{nk} x_k$ converges for all n, let Ax be the sequence given by

$$(Ax)_n = \sum_k a_{nk} x_k.$$

If V is a sequence space, let $V_A = \{x \in \omega\colon Ax \in V\}$. The space c_A is the *convergence domain* of A. It is known that c_A is an FK space. The matrix A is called *semiconservative* if c_A is semiconservative. The functional $\lim\limits_A \in c'_A$ is defined by $\lim\limits_A x = \lim(Ax)$. Whenever $e \in c_A$ and $\{\lim\limits_A e^k\} \in cs$, we let

$$\chi(A) = \lim_n \sum_k a_{nk} - \sum_k \lim_n a_{nk} = \lim_A e - \sum_k \lim_A e^k.$$

Let A be a semiconservative matrix with $e \in c_A$. A is called *conull* if $\chi(A) = 0$. It is known that c_A is conull if and only if A is conull. When dealing with the FK space c_A we shall abbreviate W_{c_A} by W_A. A matrix A is *row finite* if the rows of A are in φ.

The required properties of FK spaces and convergence domains may be found in [7] or [8]. See [5] for a discussion of semiconservative spaces.

§ 2. Pseudoconull spaces. A sequence space V will be called *pseudoconull* if every convergence domain containing V is conull. Then every conull FK space is pseudoconull. The standard example of a pseudoconull space which is not conull is m. In this section we provide a useful sufficient condition for a sequence space to be pseudoconull.

Let m and n be positive integers, $m < n$. Then $[m, n]$ will denote the interval of positive integers $\{k\colon m \leqslant k \leqslant n\}$.

A sequence space V will be said to have the (*strong*) *oscillating sequence property* if there exists a constant M with the property that for each disjoint increasing sequence of intervals of positive integers there exists a subsequence $\{I_k\}$, $I_k = [m_k, n_k]$, and a sequence $\{v^k\} \subset \omega$ such that for each k,

(i) $v_i^k = 0$ for all $i \notin [n_{2k-1}+1, m_{2k+1}-1]$;
(ii) $v_i^k = 1$ for all $i \in I_{2k}$;
(iii) $\|v^k\|_{bv} \leqslant M$; and
(iv) the pointwise sum $\sum_k v^k \in V$ ($\sum_k v^{q_k} \in V$ for every subsequence $\{v^{q_k}\}$ of $\{v^k\}$).

Of course, if V has the (strong) oscillating sequence property, then so does $V \cap m$.

LEMMA 1. *Let the matrix A be row finite with $e \in c_A$ and $\{\lim_A e^k\} \in cs$. If c_A has the oscillating sequence property, then $\chi(A) = 0$.*

Proof. Let M be a constant given by the oscillating sequence property. Assume that $\chi(A) \neq 0$. Let a be the sequence $\{\lim_A e^k\}$. Since cs is an AK space, there exists an increasing sequence $\{l_k\}$ of positive integers such that

$$\sum_k \|a - P_{l_k-1}a\|_{cs} < \frac{1}{3M} |\chi(A)|.$$

Choose sequences $\{p_k\}$, $\{m_k\}$, $\{n_k\}$ of positive integers as follows: Let $p_1 = 1$, $m_1 = l_1$. Choose $n_1 > m_1$ so that $a_{p_1,i} = 0$ for $i \geqslant n_1$. Assume p_k, m_k, n_k have been chosen. Choose $m_{k+1} > \max\{n_k, l_{k+1}\}$. Choose $p_{k+1} > p_k$ so that

$$\sum_{i=1}^{m_{k+1}} |a_i - a_{p_{k+1},i}| < 1/k.$$

Choose $n_{k+1} > m_{k+1}$ so that $a_{p_{k+1},i} = 0$ for $i \geqslant n_{k+1}$.

By hypothesis there exists a subsequence $\{[r_k, s_k]\}$ of the intervals $\{[m_k, n_k]\}$ and a sequence $u \in c_A$ so that

$$u_i = 0 \quad \text{for} \quad i \leqslant r_1;$$
$$u_i = 0 \quad \text{for} \quad r_{2k-1} \leqslant i \leqslant s_{2k-1};$$
$$u_i = 1 \quad \text{for} \quad r_{2k} \leqslant i \leqslant s_{2k};$$
$$\Big\|\sum_{i=r_k}^{r_{k+1}-1} u_i e^i\Big\|_{bv} \leqslant M.$$

By deleting rows of A we may assume that A has the following form:

$$\sum_{i=1}^{r_k} |a_i - a_{ki}| \to 0 \quad \text{as} \quad k \to \infty;$$
$$a_{ki} = 0 \quad \text{for} \quad i \geqslant s_k.$$

Finally, we may assume that $a_{ki} = a_i$ for $1 \leqslant i \leqslant r_k$, since $u \in m$. We are adjusting A by adding a matrix B with $Bu \in c$ and $\chi(B) = 0$.

Note next that

$$(Au)_{2k} = \sum_i a_{2k,i} u_i = \sum_{j=1}^{2k-1} \Big(\sum_{i=r_j}^{r_{j+1}-1} a_i u_i\Big) + \sum_{i=r_{2k}}^{s_{2k}} a_{2k,i}.$$

But

$$\sum_{j=1}^{2k-1} \Big|\sum_{i=r_j}^{r_{j+1}-1} a_i u_i\Big| \leqslant \sum_{j=1}^{2k-1} \|a - P_{l_j-1}a\|_{cs} \Big\|\sum_{i=r_j}^{r_{j+1}-1} u_i e^i\Big\|_{bv} \leqslant \tfrac{1}{3}|\chi(A)|.$$

Also,

$$\sum_{i=r_{2k}}^{s_{2k}} a_{2k,i} = \sum_{i=1}^{\infty} a_{2k,i} - \sum_{i=1}^{r_{2k}-1} a_i \to \chi(A).$$

Therefore,

$$\limsup_k |(Au)_{2k} - \chi(A)| \leqslant \tfrac{1}{3}|\chi(A)|.$$

Similarly,

$$(Au)_{2k+1} = \sum_i a_{2k+1,i} u_i = \sum_{i=1}^{r_{2k+1}-1} a_i u_i = \sum_{j=1}^{2k} \Big(\sum_{i=r_j}^{r_{j+1}-1} a_i u_i\Big).$$

As above,

$$|(Au)_{2k+1}| < \tfrac{1}{3}|\chi(A)|.$$

The contradiction $Au \notin c$ follows easily. ■

LEMMA 2. *Let K_0 be a semiconservative FK space, and let V be a sequence space satisfying $K_0 \subset V \subset M(K_0^\beta)$. Then $K_0^\beta = V^\beta$.*

Proof. The inclusion $V^\beta \subset K_0^\beta$ is obvious.

Let $u \in K_0^\beta$. If $v \in V$, then $uv \in K_0^\beta$, since $V \subset M(K_0^\beta)$. Also, $K_0^\beta \subset cs$, since K_0^β may be identified as a subspace of the dual of the semiconservative space K_0. It follows that $uv \in cs$, so $u \in V^\beta$. ■

Recall that if K_0 is a BK space, then K_0^β is a natural BK space under the norm of $M(K_0, cs)$.

THEOREM 1. *Let K_0 be a semiconservative BK space such that φ is dense in K_0^β and $K_0 \subset M(K_0^\beta)$. Let V be a sequence space such that $e \in V$ and $K_0 \subset V \subset M(K_0^\beta)$. If V has the oscillating sequence property, then V is pseudoconull.*

Proof. Assume that $V \subset c_A$. Let a^n be the n^{th} row of the matrix A. Since φ is dense in K_0^β, there exists $b^n \in \varphi$ for each n such that $a^n - b^n \to 0$ in K_0^β. Let B be the matrix whose n^{th} row is b^n.

Note next that $V \subset c_B$, for suppose $v \in V$. By Lemma 2, $K_0^\beta = M(K_0^\beta)^\beta$. Furthermore, $a^n - b^n \to 0$ in $K_0^\beta = M(K_0^\beta)^\beta = M\big(M(K_0^\beta), cs\big)$ and $v \in M(K_0^\beta)$. It follows that $(a^n - b^n)v \to 0$ in cs, so

$$\sum_i a_i^n v_i - \sum_i b_i^n v_i \to 0. \tag{1}$$

Thus, $V \subset c_B$ since $V \subset c_A$.

But B is row finite. Also, $e \in V \subset c_B$. Finally, K_0 is semiconservative and $K_0 \subset V \subset c_B$, so c_B must be semiconservative. Hence $\{\lim_B e^k\} \in cs$. By Theorem 1, $\chi(B) = 0$, so B is conull.

Now $A-B$ maps φ and e into c_0 according to (1), so $\chi(A-B) = 0$. Since B and $A-B$ are conull, it follows that A is conull. ∎

We next provide a large class of sequence spaces with the oscillating sequence property.

LEMMA 3. *Let E be a semiconservative* FK *space, and let $\{p_n\}$ be an increasing sequence of positive integers. For $x \in E$ let $x^n = \sum_{k=p_n+1}^{p_{n+1}} x_k e^k$. If the series $\sum_n x^n$ converges weakly in E and $\{x^n\}$ is bounded in bv, then $x \in W_E$.*

Proof. If $f \in E'$, then $\{f(e^k)\} \in cs$, since E is semiconservative. Also,

$$\Big\| \sum_{k=p_n+1}^{p_{n+1}} f(e^k) e^k \Big\|_{cs} \to 0,$$

since cs is an AK space.

For any integer m suppose $p_n < m \leqslant p_{n+1}$. Now

$$f\Big(\sum_{k=n+1}^{\infty} x^k\Big) \to 0 \quad \text{as} \quad n \to \infty,$$

and

$$\Big|\sum_{k=m+1}^{p_{n+1}} x_k f(e^k)\Big| \leqslant \Big\|\sum_{k=p_n+1}^{p_{n+1}} f(e^k)e^k\Big\|_{cs} \Big\|\sum_{k=m}^{p_{n+1}} x_k e^k\Big\|_{bv} \leqslant 2 \Big\|\sum_{k=p_n+1}^{p_{n+1}} f(e^k)e^k\Big\|_{cs} \|x^n\|_{bv} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, since

$$f(x) = \sum_{k=1}^{m} x_k f(e^k) + \sum_{k=m+1}^{p_{n+1}} x_k f(e^k) + f\Big(\sum_{k=n+1}^{\infty} x^k\Big),$$

it follows that

$$f(x) = \sum_{k=1}^{\infty} x_k f(e^k).$$

Therefore, $x \in W_E$. ∎

LEMMA 4. *If E is a conull* FK *space, then W_E has the strong oscillating sequence property.*

Proof. By hypothesis, $e - P_n e \to 0$ weakly in E, so appropriate convex combinations of the sequence $\{e-P_n e\}$ will converge to 0 in E. Hence, there exists $\{x^k\} \subset E$ such that for each k

(a) $x_i^k = 1$ for i large;

(b) $0 \leqslant x_i^k \leqslant 1$ for all i;

(c) x_i^k is nondecreasing;

(d) $!x^k! < 2^{-k}$

where $! \cdot !$ is a paranorm providing the topology of E. Now $x^k \to 0$ pointwise. Thus, we may assume without loss of generality that $x_i^k = 0$ for $1 \leqslant i \leqslant k$.

Let $p_1 = 1$. Choose an interval $I_1 = [m_1, n_1]$ from the given sequence of intervals such that $x_i^{p_1} = 1$ for all $i \geqslant m_1$. Assume that p_k and $I_k = [m_k, n_k]$ have been chosen. Choose $p_{k+1} > \max\{p_k, n_k\}$. Then choose $I_{k+1} = [m_{k+1}, n_{k+1}]$ from the given sequence of intervals with $m_{k+1} > n_k$ so that $x_i^{p_{k+1}} = 1$ for all $i \geqslant m_{k+1}$.

By this process one obtains a sequence $\{u^k\} \subset E$ and a subsequence of intervals $I_k = [m_k, n_k]$ such that for each k

(a)′ $u_i^k = 1$ for $i \geqslant m_k$ and $u_i^k = 0$ for $i \leqslant n_{k-1}$;

(b)′ $0 \leqslant u_i^k \leqslant 1$ for all i;

(c)′ u_i^k is nondecreasing;

(d)′ $!u^k! < 2^{-k}$.

Let $v^k = u^{2k} - u^{2k+1}$ for each k. The conditions (i) through (iii) of the strong oscillating sequence property are satisfied for the constant $M = 2$.

Let $\{q_k\}$ be an increasing sequence of positive integers. The series $\sum_k v^{q_k}$ converges absolutely in E, hence weakly. By Lemma 3, $\sum_k v^{q_k} \in W_E$, so condition (iv) is satisfied. ∎

Let V be a sequence space. Assume that for each increasing sequence $\{p_n\}$ of positive integers and for each sequence $\{x^n\} \subset \omega$ satisfying $x_i^n = 0$ for $i \notin [p_n, p_{n+1}]$ and $\{x^n\}$ bounded in bv, there exists a subsequence $\{x^{q_n}\}$ such that the pointwise sum $\sum_n x^{q_n} \in V$. Then V will be said to have the *gliding humps property.*

LEMMA 5. *Let V and W be sequence spaces. If V has the strong oscillating sequence property and W has the gliding humps property, then $V \cap W$ has the oscillating sequence property.*

Proof. Given a disjoint increasing sequence of intervals of positive integers, choose a subsequence $\{I_k\}$ and a sequence $\{v^k\} \subset \varphi$ satisfying conditions (i) through (iv) of the definition for the strong oscillating sequence property of V. Since W has the gliding humps property, there is a subsequence $\{v^{q_k}\}$ so that $\sum_k v^{q_k} \in W$. But $\sum_k v^{q_k} \in V$ as well. ∎

THEOREM 2. *Let K_0 be a semiconservative* BK, AK *space such that φ is dense in K_0^β and $K_0 \subset M(K_0)$. Let V be a sequence space such that $e \in V$ and $K_0 \subset V \subset M(K_0^\beta)$. If E is a conull* FK *space, $K_0 \subset E$, and if V has the gliding humps property, then $W_E \cap V$ is pseudoconull.*

Proof. Note that $K_0 \subset W_E$ since K_0 is an AK space. By Lemma 4 and Lemma 5, $W_E \cap V$ has the oscillating sequence property. Therefore, Theorem 1 applies. ∎

Theorem 2 generalizes Copping's Theorem [3], Theorem 3, to a semiconservative setting. If $K_0 = c_0$, then $M(K_0^\beta) = m$. Clearly, $V = m$ has the gliding humps property.

Other proofs of Copping's Theorem have been given by Wilansky in [6] and by Bennett and Kalton in [1].

Using matrices in the setting of spaces containing c_0, one can provide spaces with the oscillating sequence property without resorting to conullity.

THEOREM 3. *Let V be a sequence space satisfying $c_0 \subset V \subset m$, and let A be a matrix with null columns such that $c \subset V_A$. If $M(V)$ has the gliding humps property, then V_A has the oscillating sequence property. Hence $V_A \cap m$ is pseudoconull.*

Proof. Since $A(c) \subset V \subset m$, it is known that $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$. Without loss of generality we may assume that the rows and columns of A are in φ.

Choose increasing sequences $\{m_n\}$, $\{k_n\}$ of positive integers as follows: Choose m_1 so that $a_{1i} = 0$ for all $i \geqslant m_1$. Choose k_1 so that $a_{pi} = 0$ for $p \geqslant k_1$ and $1 \leqslant i \leqslant m_1$. Assume m_n and k_n have been chosen. Choose $m_{n+1} > m_n$ so that $a_{pi} = 0$ for $i \geqslant m_{n+1}$ and $1 \leqslant p \leqslant k_n$. Then choose $k_{n+1} > k_n$ so that $a_{pi} = 0$ for $p \geqslant k_{n+1}$ and $1 \leqslant i \leqslant m_{n+1}$.

If $M(V)$ has the gliding humps property, the following construction is possible: Given a disjoint increasing sequence of intervals of positive integers, there exists a subsequence $\{I_n\}$ and a scalar sequence $\{t_i\}$ satisfying

(i) $0 \leqslant t_i \leqslant 1$ for all i;

(ii) $t_{i+1} - t_i \to 0$ as $i \to \infty$;

(iii) Let u be given by $u_j = t_i$ for $m_{2i-2} \leqslant j < m_{2i}$. Then for each n, $u_j = 0$ on I_{2n-1}, $u_j = 1$ on I_{2n}, u_j is nondecreasing between I_{2n-1}, I_{2n} and u_j is nonincreasing between I_{2n}, I_{2n+1}.

(iv) Let v be given by $v_j = t_i$ for $k_{2i-2} \leqslant j < k_{2i}$. Then $v \in M(V)$.

Observe next that $v(Ae) - Au \in c_0$, for suppose n is a given positive integer. Choose i so that $k_{2i-1} \leqslant n < k_{2i+1}$. If $k_{2i-1} \leqslant n < k_{2i}$, then

$$\begin{aligned} v_n(Ae)_n - (Au)_n &= v_n \sum_j a_{nj} - \sum_j a_{nj} u_j \\ &= t_i \sum_{j=m_{2i-1}}^{m_{2i+1}} a_{nj} - \Big(t_i \sum_{j=m_{2i-1}}^{m_{2i}-1} a_{nj} + t_{i+1} \sum_{j=m_{2i}}^{m_{2i+1}} a_{nj}\Big) \\ &= (t_i - t_{i+1}) \sum_{j=m_{2i}}^{m_{2i+1}} a_{nj}. \end{aligned}$$

Similarly, if $k_{2i} \leqslant n < k_{2i+1}$, then

$$v_n(Ae)_n = t_{i+1} \sum_{j=m_{2i}}^{m_{2i+2}} a_{nj} = (Au)_n.$$

Now $Ae \in V$ so $v(Ae) \in V$, since $v \in M(V)$. Therefore, $Au \in V$. We have shown that V_A has the oscillating sequence property. According to Theorem 1, $V_A \cap m$ must be pseudoconull. ∎

§ 3. A semiconservative consistency theory. In this section we develop for appropriate semiconservative spaces a consistency theory based on the pseudoconull property.

Let $u \in \omega$ with $u_i \neq 0$ for all i, let E be an FK space, and let $A = (a_{nk})$ be a matrix. Let $\frac{1}{u}$ denote the sequence $\left\{\frac{1}{u_n}\right\}$. Then $\frac{1}{u} W_E = W_{\frac{1}{u}E}$, so $u \in W_E$ if and only if $\frac{1}{u} E$ is conull. Also, note that for any sequence space V, $\frac{1}{u} V_A = V_{Au}$ where Au is the matrix $(a_{nk}u_k)$. In what follows it will be clear from the context whether the symbol Au represents a matrix or a sequence.

LEMMA 6. *Let K_0 be an* FK, AK *space such that $K_0 \subset M(K_0)$. Let V be a sequence space such that $K_0 \overset{\bullet}{\subset} V \subset M(K_0)$. Assume that $W_E \cap M(V)$ is pseudoconull for all conull* FK *spaces $E \supset K_0$. Then for all* FK *spaces $E \supset K_0$, $W_E \cap V \subset W_A$ whenever $W_E \cap V \subset c_A$.*

Proof. Assume that $W_E \cap V \subset c_A$ and $u \in W_E \cap V$. We may assume without loss of generality that $u_i \neq 0$ for all i, because there exists $w \in K_0$ such that $u_i + w_i \neq 0$ for all i. Then $w \in W_A$ since K_0 is an AK space.

Now $\frac{1}{u} W_E \cap \frac{1}{u} V \subset \frac{1}{u} c_A$, and $\frac{1}{u} E$ is conull. Also, $K_0 \subset \frac{1}{u} E$ since $V \subset M(K_0)$ and $K_0 \subset E$. Finally, $M(V) \subset \frac{1}{u} V$ since $u \in V$. Therefore, $W_{\frac{1}{u}E} \cap M(V) \subset c_{Au}$. By hypothesis, $c_{Au} = \frac{1}{u} c_A$ is conull, so $u \in W_A$. ∎

Note that if a matrix A is semiconservative and $t \in l^1$, then $\sum_k \sum_i t_k a_{ki} = \sum_i \sum_k t_k a_{ki}$. See the proof of Theorem 6 of [5].

LEMMA 7. *Let K_0 be a semiconservative* FK *space such that $K_0 \subset M(K_0)$. Let V be a sequence space with $K_0 \subset V \subset M(K_0)$. Then $V \subset W_A$ whenever $V \subset c_A$ if and only if K_0^β is $\sigma(K_0^\beta, V)$ sequentially complete.*

Proof. Assume that $V \subset W_A$ whenever $V \subset c_A$. Now $M(K_0) \subset M(K_0^\beta)$ so by Lemma 2, $K_0^\beta = V^\beta$ and the duality exists. Let $\{a^n\}$ be a $\sigma(K_0^\beta, V)$ Cauchy sequence in K_0^β. Let A be the matrix whose n^{th} row is a^n.

Then $V \subset c_A$ so $V \subset W_A$. Let $a = \{\lim_A e^i\}$. If $v \in V$, then $P_n v \to v$ weakly in c_A, so $\lim_A P_n v \to \lim_A v$, i.e.

$$\sum_i a_i v_i = \lim_n \sum_i a_{ni} v_i.$$

Thus, $a^n \to a$ in $\sigma(K_0^\beta, V)$.

Conversely, suppose $V \subset c_A$. Let a^n be the n^{th} row of A. Then $\{a^n\}$ is $\sigma(K_0^\beta, V)$ Cauchy, so $a^n \to a$ in $\sigma(K_0^\beta, V)$. Let $f \in c_A'$. Now if $x \in K_0$ and $v \in V$, then $vx \in K_0$. Hence, $K_0 \subset c_{Av}$, so the matrix Av is semiconservative. Thus,

$$\sum_k \sum_i t_i a_{ik} v_k = \sum_i \sum_k t_i a_{ik} v_k.$$

The usual representation for c_A' yields

$$f(v) = \alpha \lim_A x + \sum_k t_k (Av)_k + \sum_i b_i v_i, \qquad v \in V$$

where α is a constant, $t \in l^1$ and $b \in c_A^\beta$. But then

$$f(v) = \alpha \sum_i a_i v_i + \sum_i v_i \Big(\sum_k t_k a_{ki}\Big) + \sum_i b_i v_i = \sum_i v_i f(e^i), \qquad v \in V$$

so $V \subset W_A$. ∎

LEMMA 8. *If K_0 is a semiconservative* FK *space and V is a sequence space with $K_0 \subset V \subset M(K_0)$, then $K_0 \subset M(V) \subset M(K_0^\beta)$.*

Proof. Let $x \in K_0$, $v \in V$. Then $v \in M(K_0)$, so $vx \in K_0 \subset V$. Therefore, $K_0 \subset M(V)$.

To obtain the second inclusion, observe that $M(V) \subset M(V^\beta)$. By Lemma 2, $V^\beta = K_0^\beta$, so $M(V) \subset M(K_0^\beta)$. ∎

THEOREM 4. *Let K_0 be a semiconservative* BK, AK *space such that φ is dense in K_0^β and $K_0 \subset M(K_0)$. Consider the following conditions on a sequence space V satisfying $K_0 \subset V \subset M(K_0)$.*

(i) *$M(V)$ has the gliding humps property;*

(ii) *For all conull spaces $E \supset K_0$, $W_E \cap M(V)$ is pseudoconull;*

(iii) *For all* FK *spaces $E \supset K_0$, $W_E \cap V \subset W_A$ whenever $W_E \cap V \subset c_A$;*

(iv) *For all* FK *spaces $E \supset K_0$, K_0^β is $\sigma(K_0^\beta, W_E \cap V)$ sequentially complete.*

Then (i) *implies* (ii), (ii) *implies* (iii), *and* (iii) *is equivalent to* (iv). *Also,* (ii) *is equivalent to* (iii) *if $V = M(V)$.*

Proof. Let $M(V)$ have the gliding humps property. Assume that E is conull, $K_0 \subset E$. According to Lemma 8, $K_0 \subset M(V) \subset M(K_0^\beta)$. By Theorem 2, $W_E \cap M(V)$ is pseudoconull, so (i) implies (ii).

Lemma 6 asserts that (ii) implies (iii) and Lemma 7 yields the equivalence of (iii) and (iv).

Finally, assume $V = M(V)$. If E is conull, $K_0 \subset E$, and $W_E \cap M(V) \subset c_A$, then $e \in W_A$ by (iii). Therefore, A is conull, so (iii) implies (ii). ∎

The relatively strong conditions imposed on the base space K_0 in Theorem 4 can be satisfied for $K_0 \neq c_0$. An elementary computation shows that

$$K_0 = \{x \in c_0 \colon \{x_{n+1} - x_n\} \in l^p\}, \qquad 1 < p < \infty$$

provides such an example. It should be noted however that if K_0 is semiconservative, $K_0 \subset M(K_0)$, and $\{e^n\}$ is an unconditional basis for K_0, then $K_0 = c_0$. Also note that if $bv \cap c_0 \subset K_0$ (for instance, K_0 semiconservative) and $K_0 \subset M(K_0)$, then K_0 is an AK space if and only if φ is dense in K_0.

It should be noted also that the techniques of [1] may be applied in Theorem 4 (iii) to replace the assumption that $W_E \cap V \subset c_A$ by $W_E \cap V \subset F$ where F is a separable FK space.

The following result is a semiconservative analog to Theorem 11 of [2]. The proof is omitted, being a more or less routine semiconservative extension of the proof of Bennett and Kalton.

THEOREM 5. *Let K_0 be a semiconservative* AK *space satisfying $K_0 \subset M(K_0)$, let V be a sequence space with $K_0 \subset V \subset M(K_0)$, and let A be a matrix with $K_0 \subset c_A$. Suppose that K_0^β is $\sigma(K_0^\beta, W_A \cap V)$ sequentially complete. If $c_A \cap V \subset c_B$, then there is a constant α such that*

$$\lim_B x - \sum_k x_k \lim_B e^k = \alpha \Big(\lim_A x - \sum_k x_k \lim_A e^k\Big)$$

for all $x \in c_A \cap V$.

THEOREM 6. *Let V be a sequence space satisfying $c_0 \subset V \subset m$, and let A be a matrix mapping c_0 into c_0. If $M(V)$ has the gliding humps property, then l^1 is $\sigma(l^1, V_A \cap m)$ sequentially complete.*

Proof. Assume that $V_A \cap m \subset c_B$. Now $c_0 \subset V_A \cap m$, so $c_0 \subset W_B$ since c_0 is an AK space. Let $u \in V_A \cap m$. As before, we may assume that $u_i \neq 0$ for all i. Then

$$\frac{1}{u} V_A \cap \frac{1}{u} m \subset \frac{1}{u} c_B,$$

so

$$V_{Au} \cap m \subset c_{Bu}.$$

According to Theorem 3, $V_{Au} \cap m$ is pseudoconull, so c_{Bu} is conull. Therefore, $u \in W_B$.

We have shown that $V_A \cap m \subset W_B$ whenever $V_A \cap m \subset c_B$. Since $c_0^\beta = l^1$, Lemma 7 implies that l^1 is $\sigma(l^1, V_A \cap m)$ sequentially complete. ∎

§ 4. The consistency theory for almost convergence. Let ac be the BK space of almost convergent sequences, and let ac_0 be the codimension one subspace of sequences which are almost convergent to zero. A nice discussion of almost convergence may be found in [2] along with proofs of the following known characterizations of the space ac_0.

LEMMA 9. (i) *ac_0 is the closure in m of bs;*

(ii) *$x \in ac_0$ if and only if*

$$\lim_p \frac{1}{p} \sum_{k=n}^{n+p-1} x_k = 0.$$

uniformly in n.

Lemma 9 (ii) is the original characterization of Lorentz in [4].

In this section we show that $M(ac_0)$ has the gliding humps property. This observation identifies the essential property of ac_0 on which the consistency and completeness theorems for almost convergence of G. Bennett and N. Kalton depend. See [2], Theorems 6 and 8.

If S is a set, let $|S|$ denote the cardinality of S. For positive integers n and p let J_{np} be the interval $[n, n+p-1]$ of positive integers.

LEMMA 10. *Every increasing sequence of intervals of positive integers has a subsequence $\{I_m\}$ such that*

$$\frac{1}{p} \sup_n |\{m\colon I_m \cap J_{np} \neq \varphi\}| \to 0$$

as $p \to \infty$.

Proof. Choose a subsequence $\{I_m\}$, $I_m = [i_m, j_m]$, of the given sequence of intervals such that

$$i_{m+1} - j_m \geqslant m$$

for all m.

For a given interval J_{np} suppose $I_M, I_{M+1}, \ldots, I_{M+k-1}$ are the members of $\{I_m\}$ which meet J_{np}. Then

$$p \geqslant \sum_{j=M+1}^{M+k-1} j \geqslant \tfrac{1}{2}(k-1)k.$$

Hence, $(k-1)^2 \leqslant (k-1)k \leqslant 2p$, so $k \leqslant 1+(2p)^{1/2}$. It follows that

$$\frac{1}{p} \sup_n |\{m\colon I_m \cap J_{np} \neq \varphi\}| \leqslant \frac{1}{p}\left(1+(2p)^{1/2}\right) \to 0$$

as $p \to \infty$. ∎

THEOREM 7. *$M(ac_0)$ has the gliding humps property.*

Proof. Let $\{p_n\}$ be an increasing sequence of positive integers. Using Lemma 10 let $\{I_n\}$ be a disjoint subsequence of $\{[p_n, p_{n+1}]\}$ satisfying $\frac{1}{p} \sup_n |\{m\colon I_m \cap J_{np} \neq \varphi\}| \to 0$ as $p \to \infty$. Let $\{u^n\}$ be a sequence in bv such that $\|u^n\|_{bv} \leqslant M$ for all n and $u_i^n = 0$ for all $i \notin I_n$. Let u be the pointwise sum $\sum_n u^n$.

Let $x \in bs$ be arbitrary. Now

$$\frac{1}{p}\Big|\sum_{i=n}^{n+p-1} u_i x_i\Big| \leqslant \frac{1}{p}\sum_{k=1}^{\infty}\Big|\sum_{i \in I_k \cap J_{np}} u_i^k x_i\Big| \leqslant \frac{1}{p}\sum_{I_k \cap J_{np} \neq \varphi} \|u^k\|_{bv}\|x\|_{bs}$$
$$\leqslant \frac{1}{p}|\{k\colon I_k \cap J_{np} \neq \varphi\}| M\|x\|_{bs}.$$

Thus, $\frac{1}{p}\sum_{i=n}^{n+p-1} u_i x_i \to 0$ uniformly in n as $p \to \infty$, so $ux \in ac_0$. We have shown that $u \in M(bs, ac_0)$.

Finally, let $x \in ac_0$ be arbitrary. By Lemma 9 (i), choose $x^n \in bs$ so that $x^n \to x$ in m. Then $ux^n \in ac_0$ and $ux^n \to ux$ in m. Therefore, $u \in M(ac_0)$. ∎

Taking $K_0 = c_0$ and $V = ac_0$ in Theorems 4 and 6, we obtain.

COROLLARY (Bennett–Kalton [2]). *If E is an FK space containing c_0 and A is a matrix mapping c_0 into c_0, then l^1 is sequentially complete under the topologies $\sigma(l^1, W_E \cap ac_0)$ and $\sigma(l^1, (ac_0)_A \cap m)$.*

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