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Weighted norm inequalities for generalized Hankel conjugate transformations

by

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Abstract. Weighted norm inequalities are considered for the operators H_λ, $\lambda > -1$, which for $\lambda > 0$ are the Hankel conjugate transformations introduced by Muckenhoupt and Stein. It is shown that for H_λ, $\lambda \neq -1/2$, to be of strong (equivalently, weak) type (p, p), $1 < p < \infty$, with respect to a given weight w, it is both necessary and sufficient that w belong to the class $A_{p,\lambda}$; that is, for $0 < a < b < \infty$,

$$\Big(\int_a^b t^p w(t)\,dt\Big)\Big(\int_a^b t^{2\lambda p'} w(t)^{-1/(p-1)}\,dt\Big)^{p-1} \leqslant C\,(b^{2(\lambda+1)} - a^{2(\lambda+1)})^p,$$

where $p' = p/(p-1)$ and C is a constant independent of a and b.

An interesting feature of the H_λ is that, unlike the situation for such operators as the Hilbert transformation, the necessary and sufficient condition on a weight for the weak type $(1, 1)$ inequality is not that obtained by taking the limit in the one for membership in $A_{p,\lambda}$.

1. Introduction. Our first concern is to characterize those nonnegative, Lebesgue-measurable functions w on $(0, \infty)$ for which the generalized Hankel conjugate transformation H_λ, $\lambda > -1$, is bounded from $L^p(w)$ into itself for a fixed p, $1 < p < \infty$; that is,

$$\int_0^\infty |(H_\lambda f)(y)|^p w(y)\,dy \leqslant C \int_0^\infty |f(z)|^p w(z)\,dz, \tag{1.1}$$

the constant C depending only on λ and p. Given a Lebesgue-measurable function f on $(0, \infty)$,

$$(H_\lambda f)(y) = \lim_{x \to 0+} \int_0^\infty Q_\lambda(x, y, z) f(z) z^{2\lambda}\,dz, \tag{1.2}$$

whenever there is a set E of Lebesgue measure zero so that the limit exists for all $y \notin E$; the kernel is defined in terms of the usual Bessel functions by

$$Q_\lambda(x, y, z) = -(yz)^{-\lambda+1/2} \int_0^\infty e^{-xt} J_{\lambda+1/2}(yt) J_{\lambda-1/2}(zt)\, t\,dt. \tag{1.3}$$

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The first result of this paper is

THEOREM 1. *Let w be a nonnegative Lebesgue-measurable function on $(0,\infty)$. Suppose $\lambda > -1$, $\lambda \neq -1/2$, and $1 < p < \infty$. The following statements are equivalent.*

(i) *w is in the class $A_{p,\lambda}$; that is, for all a, b, $0 < a < b < \infty$,*

$$(1.4)\qquad \Big(\int_a^b t^p w(t)\,dt\Big)\Big(\int_a^b t^{2\lambda p'} w(t)^{-1/(p-1)}dt\Big)^{p-1} \leqslant C(b^{2(\lambda+1)}-a^{2(\lambda+1)})^p,$$

where $p' = p/(p-1)$, C is a constant independent of a and b, and $0\cdot\infty$ is taken as 0.

(ii) $\int_0^\infty |(H_\lambda f)(y)|^p w(y)\,dy \leqslant C\int_0^\infty |f(z)|^p w(z)\,dz$, *the constant C depending only on λ and p.*

(iii) $\int_{E_t} w(y)\,dy \leqslant Ct^{-p}\int_0^\infty |f(z)|^p w(z)\,dz$, *where $t > 0$, $E_t = \{y: |(H_\lambda f)(y)| > t\}$, and the constant C depends only on λ and p.*

The case $\lambda = 0$ is that of the even Hilbert transformation; our $A_{p,\lambda}$ is then equivalent to the A_p condition of [1]. Again, for weights of the form $w(t) = t^{a-1}$, Theorem 1 above yields $-p < a < (2\lambda+1)p$, a result first proved for $\lambda > -1/2$ in [10] and later, in [4], extended to all $\lambda > -1$ by the use of other methods.

It is seen from Theorem 1 that, like other previously studied conjugate function operators, such as the Hilbert transformation, the conditions for the H_λ to be of weak and strong type are identical when $p > 1$. Unlike those operators, however, the condition for H_λ to be of weak type (1,1) is not the one obtained by letting $p\to 1$ in (1.4); thus $w(t) = t^{-2}$ does not satisfy the latter condition, though it is a weak type (1,1) weight for H_0. The correct result is given in

THEOREM 2. *Suppose $\lambda > -1$, $\lambda\neq -1/2$. Then w is a weak type (1,1) weight for H_λ if and only if w is in the class $A_{1,\lambda}$; that is, for some (equivalently, all) positive ε,*

$$(1.5)\qquad \Big(\int_a^b \Big(\frac{a}{t}+\frac{t}{b}\Big)^{\lambda+1+\varepsilon} t^{-\lambda} w(t)\,dt\Big)\Big(\operatorname*{ess\,sup}_{(a,b)}\frac{t^{\lambda-1}}{w(t)}\Big) \leqslant K_\varepsilon\frac{b^{2(\lambda+1)}-a^{2(\lambda+1)}}{(ab)^{\lambda+1}},$$

where K_ε is a positive constant independent of a, b, $0<a<b<\infty$.

It is a simple matter to show that, for fixed p in $[1,\infty)$, the class $A_{p,\lambda}$ increases with λ on $(-1,-1/2)\cup(-1/2,\infty)$. Again, by Hölder's inequality, $A_{p,\lambda}$ increases with p on $[1,\infty)$ for fixed $\lambda > -1$, $\lambda\neq -1/2$.

Basic to our analysis are estimates of $Q_\lambda(x,y,z)$ which are improvements of ones given in [9], p. 87 for $\lambda > 0$; their proofs are given in [4], Lemma 2.1 and [5], Theorem 2.1. They ensure that given $\lambda > -1$, $\lambda\neq -1/2$, there exist $k = k_\lambda > 1$ and $0 < K_1 = K_1(\lambda, k)$, $0 < K_2 = K_2(\lambda, k)$ such that $Q_\lambda(0,y,z)$ is of constant sign for z above and below $(y/k, ky)$ with

$$(1.6)\qquad K_1 y^{-2\lambda-1} \leqslant |Q_\lambda(0,y,z)|;\quad |Q_\lambda(x,y,z)| \leqslant K_2 y^{-2\lambda-1},\quad \text{if}\quad 0 < z < y/k$$

$$K_1 yz^{-2\lambda-2} \leqslant |Q_\lambda(0,y,z)|;\quad |Q_\lambda(x,y,z)| \leqslant K_2 yz^{-2\lambda-2},\quad \text{if}\quad z\geqslant ky$$

while

$$Q_\lambda(x,y,z) = C_\lambda(yz)^{-\lambda}\frac{y-z}{x^2+(y-z)^2} + O\Big((yz)^{-\lambda-1/2}\Big(1+\log^+\frac{yz}{(y-z)^2}\Big)\Big),\quad \text{if}\quad y/2k\leqslant z\leqslant 2ky.$$

Moreover, for any $k > 1$ there exists $0 < K_2 = K_2(\lambda, k)$ such that all the estimates in (1.6) hold, except for the lower bounds on $|Q_\lambda(0,y,z)|$. As pointed out in the remark following Lemma 2.1 of [4], sharper estimates are available for $Q_{-1/2}(x,y,z)$ than might be expected from (1.6). Our methods show that $H_{-1/2}$ behaves in the same way as $H_{1/2}$.

Section 2 shows that a weight's belonging to the class $A_{p,\lambda}$ is sufficient for the strong type inequality (ii) of Theorem 1 to hold; the necessity of membership in $A_{p,\lambda}$ given the weak type inequality (iii) is proved in Section 3. Theorem 2 is treated in Section 4. In the concluding section we briefly discuss the transformations C_λ, the analogue of the H_λ for ultraspherical series.

2. Sufficiency.

Fix an integer n and let $I_n = (2^n, 2^{n+1})$, $J_n = (2^{n-1}, 2^{n+2})$. For $y\in I_n$, express the integral defining H_λ in (1.2) as the sum of integrals over $(0, 2^{n-1})$, J_n, and $(2^{n+2},\infty)$. It is then seen, in view of the estimates (1.6), with $k = 2$, that w in $A_{p,\lambda}$ will imply (ii) of Theorem 1, if the same can be shown for the operators H_λ^j, $j = 1, 2, 3, 4$, where, for nonnegative $f\in L^p(w)$ and $f_n = f\chi_{J_n}$

$$(H_\lambda^1 f)(y) = y^{-2\lambda-1}\int_0^y f(z)z^{2\lambda}dz,$$

$$(H_\lambda^2 f)(y) = y\int_y^\infty f(z)\frac{dz}{z^2},$$

$$(H_\lambda^3 f)(y) = \sum_{n=-\infty}^{\infty}\chi_{I_n}(y)\int_0^\infty (yz)^{-1/2}\Big(1+\log^+\frac{yz}{(y-z)^2}\Big)f_n(z)\,dz,$$

$$(H_\lambda^4 f)(y) = \lim_{x\to 0+}\sum_{n=-\infty}^{\infty}\chi_{I_n}(y)y^{-\lambda}\int_0^\infty\frac{(y-z)z^\lambda f_n(z)}{x^2+(y-z)^2}\,dz.$$

We consider $H_\lambda^4 f$ first. Standard results concerning the Hilbert transformation will show that it exists and equals the principal value integral

$$y^{-\lambda}\int_0^\infty \frac{z^\lambda f_n(z)}{y-z}\,dz \tag{2.1}$$

for almost all $y \in I_n$, once it is verified that $\int_0^\infty f_n(z) z^\lambda dz$ is finite. But, by Hölder's inequality, this is dominated by a multiple of

$$\Big(\int_0^\infty f(z)^p w(z)\,dz\Big)^{1/p}\Big(\int_{I_n} z^{2\lambda p'} w(z)^{-1/(p-1)}\,dz\Big)^{1/p'}, \tag{2.2}$$

both factors of which are finite; the second since $w \in A_{p,\lambda}$ and $w \not\equiv 0$ implies $w > 0$ a.e., and hence the integrability of $z^p w(z)$ and $z^{2\lambda p'} w(z)^{-1/(p-1)}$ on every finite subinterval of $(0, \infty)$.

Now, an elementary estimate gives

$$\frac{(z/y)^\lambda}{y-z} = \frac{2(\lambda+1) z^{2\lambda} y}{y^{2(\lambda+1)} - z^{2(\lambda+1)}} + O\big((yz)^{-1/2}\big) \tag{2.3}$$

for $y/4 \leqslant z \leqslant 4y$. This means that the proof of the boundedness of H_λ^4 depends on showing that of H_λ^3 as well as

$$\int_{2^n}^{2^{n+1}} \Big|\int_0^\infty \frac{z^{2\lambda} y f_n(z)}{y^{2(\lambda+1)} - z^{2(\lambda+1)}}\,dz\Big|^p w(y)\,dy \leqslant C_{p,\lambda}\int_0^\infty f_n^p(z) w(z)\,dz \tag{2.4}$$

for $C_{p,\lambda}$ a positive constant independent of f and n. The changes of variable $z^{2(\lambda+1)} = z'$, $y^{2(\lambda+1)} = y'$ yield (2.4) equivalent to

$$\int_{2^{2(\lambda+1)n}}^{2^{2(\lambda+1)(n+1)}} |\tilde{g}_n(y)|^p W(y)\,dy \leqslant C_{p,\lambda}\int_0^\infty g_n(z)^p W(z)\,dz, \tag{2.5}$$

where, letting $\mu = 1/2(\lambda+1)$, $W(t) = t^{\mu(p-2\lambda-1)} w(t^\mu)$, $g_n(z) = z^{-\mu} f_n(z^\mu)$, and $\tilde{g}_n$ denotes the Hilbert transform of g_n. The same change of variable in (1.4) reveals that W, considered as an even function on $(-\infty, \infty)$, satisfies the A_p condition of R. Hunt, B. Muckenhoupt, and R. Wheeden [3], which means that (2.5) holds.

For H_λ^3, note first that $z \in J_n$, $y \in I_n$ implies $|y-z| \leqslant 3\cdot 2^n$, so that

$$(H_\lambda^3 f)(y) \leqslant (\varphi_{2^n} * f_n)(y), \tag{2.6}$$

where, in the notation of [11], § 2.2, p. 62,

$$\varphi(z) = \sqrt{2}\,(1+\log^+ 8z^{-2})\,\chi_{(0,3]}(|z|). \tag{2.7}$$

Hence,

$$(H_\lambda^3 f)(y) \leqslant C f_n^*(y) \qquad (y \in I_n), \tag{2.8}$$

f_n^* being the Hardy–Littlewood maximal function of f_n and $C = \int \varphi(z)\,dz$. Now, if $(a, b) \subset J_n$, (1.4) shows that w satisfies

$$\Big(\int_a^b w(t)\,dt\Big)\Big(\int_a^b w(t)^{-1/(p-1)}dt\Big)^{p-1} \leqslant K(b-a)^p, \tag{2.9}$$

with the constant K independent of n. A result of B. Muckenhoupt [7] then gives

$$\int_{I_n} |f_n^*(y)|^p w(y)\,dy \leqslant C_p \int_0^\infty |f_n(z)|^p w(z)\,dz, \tag{2.10}$$

and, hence, the result for H_λ^3.

Since $w \in A_{p,\lambda}$, we have

$$\Big(\int_r^s t^p w(t)\,dt\Big)\Big(\int_0^r t^{2\lambda p'} w(t)^{-1/(p-1)}dt\Big)^{p-1} \leqslant K s^{2(\lambda+1)p}, \tag{2.11}$$

for $s > r$. If this is multiplied by $s^{-2(\lambda+1)p-2}$ and the result integrated over (r, ∞), we obtain

$$\bigg(\int_r^\infty \Big(\frac{r}{t}\Big)\frac{w(t)}{t^{(2\lambda+1)p}}\,dt\bigg)\bigg(\int_0^r t^{2\lambda p'} w(t)^{-1/(p-1)}dt\bigg)^{p-1} \leqslant K, \tag{2.12}$$

by an application of Fubini's theorem. It is shown in [2], § 4, Lemma 2 that (2.12) is equivalent to

$$\sup_{r>0}\bigg(\int_r^\infty \frac{w(t)}{t^{(2\lambda+1)p}}\,dt\bigg)\bigg(\int_0^r t^{2\lambda p'} w(t)^{-1/(p-1)}dt\bigg)^{p-1} < \infty, \tag{2.13}$$

which is the required condition in order that H_λ^1 be defined and bounded; see [6].

It remains to consider H_λ^2. An argument similar to that which led to (2.12) shows $w \in A_{p,\lambda}$ implies

$$\Big(\int_0^r t^p w(t)\,dt\Big)^{1/(p-1)}\bigg(\int_r^\infty \Big(\frac{r}{t}\Big)\frac{w(t)^{-1/(p-1)}}{t^{2p'}}\,dt\bigg) \leqslant K. \tag{2.14}$$

By Theorems 2 and 3 of [2] this gives

$$\int_0^\infty \Big(\int_0^z h(y)\,dy\Big)^{p'} z^{-2p'} w(z)^{-1/(p-1)}\,dz \leqslant C_p \int_0^\infty h(y)^{p'} y^{-p'} w(y)^{-1/(p-1)}\,dy, \tag{2.15}$$

for $h \geqslant 0$. Thus, if $f \in L^p(w)$, $g \in L^{p'}(w)$ with $f, g \geqslant 0$, Fubini's theorem, followed by Hölder's inequality, yields

$$\int_0^\infty yg(y)w(y)\,dy \int_y^\infty f(z)\frac{dz}{z^2} = \int_0^\infty f(z)\frac{dz}{z^2}\int_0^z yg(y)w(y)\,dy \tag{2.16}$$
$$\leqslant \Big(\int_0^\infty f(z)^p w(z)\,dz\Big)^{1/p}\Big(\int_0^\infty \Big|\int_0^z yg(y)w(y)\,dy\Big|^{p'} z^{-2p'}w(z)^{-1/(p-1)}dz\Big)^{1/p'}$$
$$\leqslant C_p\Big(\int_0^\infty f(z)^p w(z)\,dz\Big)^{1/p}\Big(\int_0^\infty g(y)^{p'}w(y)\,dy\Big)^{1/p'}.$$

The converse of Hölder's inequality now gives the existence and boundedness of H_λ^2.

This completes the proof that (i) implies (ii).

3. Necessity. To prove (iii) implies (1.4), it suffices to fix on $R_\lambda > 1$ and to establish the result in case $I = (a, b)$ with

(i) $a = 0$, $b > 0$

or

(ii) $b \leqslant R_\lambda a$,

for the remaining case with $b > R_\lambda a$ reduces readily to case (i).

Now, there exist constants $r_\lambda > 1$ and $d_\lambda > 0$ so that

$$\operatorname{sgn}(y-z)Q_\lambda(0, y, z) \geqslant d_\lambda\frac{(yz)^{-\lambda}}{|y-z|}, \tag{3.1}$$

when $r_\lambda^{-1}y \leqslant z \leqslant r_\lambda y$. It will be seen that $R_\lambda = (r_\lambda+1)/2$ is what is needed in the proof given below.

Suppose, firstly, that $I = (0, b)$ and let $J = (k_\lambda b, \infty)$. For convenience, write $A = \int_I z^{2\lambda p'}w(z)^{-1/(p-1)}dz$, $B = \int_J z^{-2p'}w(z)^{-1/(p-1)}dz$. We first eliminate the pathological cases in which at least one of A, B fails to be a finite, positive number. If $A = 0$, (1.4) holds by the convention that $0\cdot\infty = 0$. If $A = \infty$, there exists a nonnegative $f \in L^p(w)$, supported on I, such that $\int_I z^{2\lambda}f(z)\,dz = \infty$. For this f, $|(H_\lambda f)(y)| = \infty$ when $y \in J$; as is seen from (1.6). Hence, the weak type inequality for H_λ shows

$$\int_J w(y)\,dy \leqslant C_{p,\lambda}t^{-p}\int_0^\infty f(z)^p w(z)\,dz \tag{3.2}$$

for all $t > 0$, which forces $w(y) = 0$ a.e. on J. But then, $\chi_J \in L^p(w)$ and, in the notation of (1.6),

$$|H_\lambda\chi_J(y)| \geqslant K_1(y/k_\lambda b) \tag{3.3}$$

for $y \in I$. Thus, for fixed $y \in I$

$$\int_y^b w(t)\,dt \leqslant C_{p,\lambda}(y/b)^{-p}\int_0^\infty \chi_J(z)w(z)\,dz = 0; \tag{3.4}$$

that is, $w = 0$ a.e. on I. In particular, $\int_I t^p w(t)\,dt = 0$, so that (1.4) holds by convention in this case also. Thus, we may assume $0 < A < \infty$ which means $f(z) = [z^{-2\lambda}w(z)]^{-1/(p-1)}\chi_I(z)$ belongs to $L^p(w)$. It follows from (1.6) that

$$|(H_\lambda f)(y)| \geqslant K_1 y^{-2\lambda-1}A, \qquad y \in J. \tag{3.5}$$

Therefore, (3.2) shows that for $y > k_\lambda b$

$$\int_{k_\lambda b}^y w(t)\,dt \leqslant C_{p,\lambda}\big(\min_{[k_\lambda b, y]} t^{-2\lambda-1}\big)^{-p}A^{1-p} < \infty, \tag{3.6}$$

and so $w < \infty$ a.e. on J, thereby forcing $B > 0$. If it be supposed that $B = \infty$, an argument similar to that which led to (3.4) shows $w = 0$ a.e. on I, contradicting $A < \infty$. Thus, $0 < B < \infty$, and so $g(z) = [z^2 w(z)]^{-1/(p-1)}\chi_J(z)$ belongs to $L^p(w)$.

Taking $0 < A$, $B < \infty$, we have from (1.6),

$$|(H_\lambda f)(y)| \geqslant K_1 y^{-2\lambda-1}A, \qquad y \in J \tag{3.7}$$

and

$$|(H_\lambda g)(y)| \geqslant K_1 yB, \qquad y \in I. \tag{3.8}$$

The weak type estimate for H_λ and (3.8) yield

$$\int_y^b w(t)\,dt \leqslant C_{p,\lambda}y^{-p}B^{1-p}, \qquad 0 < y < b. \tag{3.9}$$

If this is multiplied by $y^{p-1+\varepsilon}$, $\varepsilon > 0$, integrated over I, and Fubini's theorem applied on the left side, there results

$$\int_I t^{p+\varepsilon}w(t)\,dt \leqslant C_{p,\lambda,\varepsilon}b^\varepsilon B^{1-p}. \tag{3.10}$$

Similarly, (3.7) leads to

$$\int_J \frac{w(t)}{t^{(2\lambda+1)p+\varepsilon}}\,dt \leqslant C_{p,\lambda,\varepsilon}b^{-\varepsilon}A^{1-p}, \tag{3.11}$$

if $2\lambda+1 > 0$ and

$$\int_J w(t)\,dt \leqslant C_{p,\lambda}b^{(2\lambda+1)p}A^{1-p}, \tag{3.12}$$

if $2\lambda+1 < 0$. The proof of Lemma 2 of [2] may be adapted to show that (3.10) and (3.11) are equivalent, respectively, to

$$\int_I t^p w(t)\,dt \leqslant C_{p,\lambda}B^{1-p} \tag{3.13}$$

and

$$\int_J \frac{w(t)}{t^{(2\lambda+1)p}}\,dt \leqslant C_{p,\lambda}A^{1-p}. \tag{3.14}$$

Multiplying (3.13) and (3.14) and using Hölder's inequality on the integrals over J, yields

$$(3.15)\quad \Big(\int_0^b t^p w(t)\,dt\Big)\Big(\int_0^b t^{2\lambda p'} w(t)^{-1/(p-1)}\,dt\Big)^{p-1}$$
$$\leqslant C_{p,\lambda}\Big(\int_{k_\lambda b}^{\infty} t^{-2\lambda-3}\,dt\Big)^{-p} = C_{p,\lambda} b^{2(\lambda+1)p},$$

the required result for $2\lambda+1>0$. A similar argument involving (3.12) and (3.13) disposes of the case $2\lambda+1<0$.

Finally, consider $I=(a,b)$ with $b\leqslant R_\lambda a$; put $J=(b,2b-a)$. Let $A=\int_I w(z)^{-1/(p-1)}dz$, $B=\int_J w(z)^{-1/(p-1)}dz$. Arguments of the type used in case (i) above show it may be assumed that $0<A$, $B<\infty$, and so, in particular, that $f(z)=w(z)^{-1/(p-1)}\chi_I(z)$ and $g(z)=w(z)^{-1/(p-1)}\chi_J(z)$ belong to $L^p(w)$. The choice of $R_\lambda=(r_\lambda+1)/2$ ensures $r_\lambda^{-1}y\leqslant z\leqslant y$ when $z\in I$, $y\in J$ and $y\leqslant z\leqslant r_\lambda y$ when $z\in J$, $y\in I$. The estimate (3.1) and the weak type inequality for H_λ leads to

$$(3.16)\quad \int_J w(t)\,dt\leqslant C_{p,\lambda}(b-a)^p A^{1-p}$$

and

$$(3.17)\quad \int_I w(t)\,dt\leqslant C_{p,\lambda}(b-a)^p B^{1-p}.$$

Multiplying (3.16) and (3.17) and using Hölder's inequality on the integrals over J, we obtain

$$(3.18)\quad \Big(\int_a^b w(t)\,dt\Big)\Big(\int_a^b w(t)^{-1/(p-1)}dt\Big)^{p-1}\leqslant C_{p,\lambda}(b-a)^p.$$

Since $b\leqslant R_\lambda a$, (3.18) is readily seen to be equivalent to $A_{p,\lambda}$ on (a,b). This completes the proof.

4. The case $p=1$. We prove the sufficiency of (1.5) first. Observe that it implies the local A_1 condition; that is

$$(4.1)\quad \Big(\int_a^b w(t)\,dt\Big)\Big(\operatorname*{ess\,sup}_{(a,b)}\frac{1}{w(t)}\Big)\leqslant c(b-a),$$

whenever $0<a<b\leqslant ka$, k a fixed positive constant. Lemma 1 of [2] then yields H_λ^4 of weak type (1,1). For, letting $g(z)=z^\lambda f_n(z)$, we have $(H_\lambda^4 f)(y)=y^{-\lambda}\tilde g(y)$ when $y\in(2^n,2^{n+1})$, so that the integral of $w(y)$ over the set

$$(4.2)\quad \{y\in(2^n,2^{n+1}):\ |(H_\lambda^4)f(y)|>t\}$$

is bounded above by a constant multiple of $t^{-1}\int_0^\infty |f_n(z)|w(z)\,dz$; the result follows on summing over n. In view of (2.8), Theorem 1 of [7] shows H_λ^3 to be also of weak type (1,1) because of (4.1).

It remains to consider H_λ^2 and H_λ^1. As for H_λ^1, Theorems 1 and 2 of [2] give the desired result if (and only if)

$$(4.3)\quad \sup_{r>0}\Big(\int_r^\infty\Big(\frac{r}{t}\Big)^\delta\frac{w(t)}{t^{2\lambda+1}}\,dt\Big)\Big(\sup_{(0,r)}\frac{t^{2\lambda}}{w(t)}\Big)<\infty,$$

for, say, $\delta=|2\lambda+1|$. Now, (1.5), with $b\geqslant 2a$, leads to

$$(4.4)\quad 2^{(\lambda+1+\varepsilon)}\int_{b/2}^b t^{-\lambda}w(t)\,dt\leqslant K_\varepsilon(b/a)^{2(\lambda+1)}\operatorname*{ess\,inf}_{(a,2a)}\frac{w(t)}{t^{\lambda-1}}$$
$$\leqslant 2K_\varepsilon(b/a)^{2(\lambda+1)}\int_a^{2a} t^{-\lambda}w(t)\,dt;$$

that is,

$$(4.5)\quad 2b^{-1}\int_{b/2}^b t^{-2\lambda}w(t)\,dt\leqslant C_\varepsilon a^{-1}\int_a^{2a} t^{-2\lambda}w(t)\,dt,$$

whenever $b\geqslant 2a$. Thus, if $s\leqslant r$,

$$(4.6)\quad \int_r^\infty\Big(\frac{r}{t}\Big)^\delta\frac{w(t)}{t^{2\lambda+1}}\,dt=\sum_{k=0}^\infty\int_{2^k r}^{2^{k+1}r}\Big(\frac{r}{t}\Big)^\delta\frac{w(t)}{t^{2\lambda+1}}\,dt$$
$$\leqslant\sum_{k=0}^\infty 2^{-k\delta}\Big(2^{-k}r^{-1}\int_{2^k r}^{2^{k+1}r} t^{-2\lambda}w(t)\,dt\Big)$$
$$\leqslant\sum_{k=0}^\infty 2^{-k\delta}C_\varepsilon s^{-1}\int_s^{2s} t^{-2\lambda}w(t)\,dt$$
$$\leqslant C_{\varepsilon,\delta}\Big(s^{-1}\int_s^{2s} t^{-2\lambda}w(t)\,dt\Big).$$

From (4.1), the latter is bounded by a constant times $\operatorname*{ess\,inf}_{(s,2s)} t^{-2\lambda}w(t)$. Since $s\leqslant r$ is arbitrary,

$$(4.7)\quad \int_r^\infty\Big(\frac{r}{t}\Big)^\delta\frac{w(t)}{t^{2\lambda+1}}\,dt\leqslant C_{\varepsilon,\delta}\operatorname*{ess\,inf}_{(0,2r)} t^{-2\lambda}w(t),$$

which implies (4.3).

Finally, H_λ^2 will be of weak type (1, 1) if (and only if)

$$(4.8)\quad \sup_{r>0}\Big(\int_0^r\Big(\frac{t}{r}\Big)^\delta tw(t)\,dt\Big)\Big(\operatorname*{ess\,sup}_{(r,\infty)}\frac{1}{t^2w(t)}\Big)<\infty$$

for some $\delta>0$; see [2], Theorem 5. But, (4.8) may be obtained in a way similar to that which gave (4.3): firstly, (1.5) gives

$$a^2 \operatorname*{ess\,inf}_{(a,2a)} w(t) \leqslant C_\varepsilon b^2 \operatorname*{ess\,inf}_{(b,2b)} w(t), \tag{4.9}$$

whenever $b \geqslant a$; this leads to

$$\int_0^r \left(\frac{t}{r}\right)^\delta tw(t)\,dt \leqslant C_{\varepsilon,\delta} \operatorname*{ess\,inf}_{(r/2,\infty)} t^2 w(t), \tag{4.10}$$

just as (4.5) led to (4.7). This completes the proof of sufficiency.

To establish the necessity, we distinguish, as we did for $p > 1$, the cases $I = (a, b)$ with

(i) $a = 0$, $b > 0$

and

(ii) $b \leqslant R_\lambda a$.

In case (ii), (1.5) is equivalent to the local A_1 condition and the proof for $p > 1$ goes over with the usual changes. Consider, then, case (i). The proof for $p > 1$, suitably modified, leads to

$$\int_0^b t^{1+\varepsilon} w(t)\,dt \leqslant C_{p,\varepsilon} b^\varepsilon \operatorname*{ess\,inf}_{(k_\lambda b,\infty)} t^2 w(t) \tag{4.11}$$

for all $\varepsilon > 0$ and to

$$\int_{k_\lambda b}^\infty \frac{w(t)}{t^{2\lambda+1+\varepsilon}}\,dt \leqslant C_{p,\varepsilon} b^{-\varepsilon} \operatorname*{ess\,inf}_{(0,b)} t^{-2\lambda} w(t) \tag{4.12}$$

for arbitrary $\varepsilon > 0$ if $2\lambda+1 > 0$ and for $\varepsilon = -(2\lambda+1)$ otherwise. In fact, (4.12) always holds when $\varepsilon > 0$. For, if $2\lambda+1 < 0$ and $\varepsilon > 0$,

$$\begin{aligned}\int_{k_\lambda b}^\infty \frac{w(t)}{t^{2\lambda+1+\varepsilon}}\,dt &\leqslant C_{\lambda,\varepsilon} \sum_{k=0}^\infty [2^k k_\lambda b]^{-2\lambda-1-\varepsilon} \int_{2^k k_\lambda b}^{2^{k+1} k_\lambda b} w(t)\,dt \\ &\leqslant C_{\lambda,\varepsilon} k_\lambda^{-2\lambda-1-\varepsilon} b^{-\varepsilon} \sum_{k=1}^\infty 2^{-k\varepsilon} \operatorname*{ess\,inf}_{(0,2^k b)} t^{-2\lambda} w(t) \\ &\leqslant C_{\lambda,\delta} b^{-\delta} \Big(\sum_{k=0}^\infty 2^{-k\delta}\Big) \operatorname*{ess\,inf}_{(0,b)} t^{-2\lambda} w(t).\end{aligned} \tag{4.13}$$

Now, if (4.11) and (4.12) are multiplied together and Hölder's inequality applied twice in obvious ways, there results

$$\Big(\int_0^b t^{1+\varepsilon} w(t)\,dt\Big)\Big(\operatorname*{ess\,sup}_{(0,b)} \frac{1}{t^{-2\lambda} w(t)}\Big) \leqslant C_{p,\varepsilon,\lambda} b^{2(\lambda+1)+\varepsilon} \tag{4.14}$$

and

$$\Big(\int_b^\infty \frac{w(t)}{t^{2\lambda+1+\varepsilon}}\,dt\Big)\Big(\operatorname*{ess\,sup}_{(b,\infty)} \frac{1}{t^2 w(t)}\Big) \leqslant C_{p,\varepsilon,\lambda} b^{-[2(\lambda+1)+\varepsilon]} \tag{4.15}$$

for all $b, \varepsilon > 0$. Replacing b by a in (4.15), reducing the ranges of integration to (a, b) in both (4.14) and (4.15), and using elementary inequalities, results in

$$\Big(\int_a^b \left(\frac{t}{b}\right)^{\lambda+1+\varepsilon} t^{-\lambda} w(t)\,dt\Big)\Big(\operatorname*{ess\,sup}_{(a,b)} \frac{t^{\lambda-1}}{w(t)}\Big) \leqslant C_{p,\varepsilon,\lambda} (b/a)^{\lambda+1} \tag{4.16}$$

and

$$\Big(\int_a^b \left(\frac{a}{t}\right)^{\lambda+1+\varepsilon} t^{-\lambda} w(t)\,dt\Big)\Big(\operatorname*{ess\,sup}_{(a,b)} \frac{t^{\lambda-1}}{w(t)}\Big) \leqslant C_{p,\varepsilon,\lambda} (b/a)^{\lambda+1}. \tag{4.17}$$

Adding (4.16) and (4.17) completes the proof of necessity.

5. Untraspherical conjugate transformations. The generalized ultraspherical conjugate transformation C_λ, $\lambda > -1$, $\lambda \neq -1/2$, is defined, in the same manner as H_λ, by

$$\lim_{r\to 1-} \int_0^\pi Q_\lambda(r, \theta, \varphi) f(\varphi) \sin^{2\lambda}\varphi\,d\varphi, \quad 0 < \theta < \pi, \tag{5.1}$$

where f is Lebesgue-measurable on $(0, \pi)$. Here, the conjugate Poisson kernel has the form

$$Q_\lambda(r, \theta, \varphi) = \sum_{n=1}^\infty \frac{2\lambda}{n+2\lambda} r^n \gamma_n \sin\theta\, P_{n-1}^{\lambda+1}(\cos\theta) P_n^\lambda(\cos\varphi), \tag{5.2}$$

the ultraspherical polynomials of type P_n^λ being given by the generating relation

$$\sum_{n=0}^\infty s^n P_n^\lambda(t) = (1-2ts+s^2)^{-\lambda}; \tag{5.3}$$

the normalizing factors γ_n by $\dfrac{n!(n+\lambda)\Gamma(\lambda)\Gamma(2\lambda)}{\Gamma(n+2\lambda)\Gamma(1/2)\Gamma(\lambda+1/2)}$.

The methods of [8], whereby asymptotic expressions for the more general Jacobi polynomials are used to estimate transplantation kernels, may be applied to (5.2) to give a substitute for the estimates (1.6) for $Q_\lambda(r, \theta, \varphi)$ when $\lambda > -1/2$. This allows one to show that C_λ, $\lambda > -1/2$, satisfies the strong type inequality with respect to the weight w provided

$$\Big(\int_a^b \sin^p\theta\, w(\theta)\,d\theta\Big)\Big(\int_a^b \sin^{2\lambda p'}\theta\, w(\theta)^{-1/(p-1)}\,d\theta\Big)^{p-1} \leqslant C\Big[\sin^{2\lambda+1}\Big(\frac{b+a}{2}\Big)\sin\Big(\frac{b-a}{2}\Big)\Big]^p, \tag{5.4}$$

whenever $0 < a < b < \pi$.

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Almost sure summability of subsequences in Banach spaces

by

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Abstract. The property of almost everywhere summability, for subsequences of a sequence of vector-valued random variables, is considered. As a particular case of this, the convergence of sums $(N^{-1}\sum_{k=1}^{N}\varepsilon_k x_{n_k})$, for a bounded sequence (x_n) in a Banach space, and a sequence (ε_n) of signs, is examined; the results proved relate to the Banach–Saks and similar properties.

In this article, we will consider the notion of almost sure summability for subsequences of a sequence of Banach space valued random variables. Our first result extends the theorem of Erdös and Magidor [7] concerning the summability of subsequences in Banach spaces, and has relevance to several recent results in probability theory. We then employ similar techniques to prove results on the convergence of sums of the form $(N^{-1}\sum_{k=1}^{N}\pm x_k)$ in a Banach space: in particular we establish a conjecture of Beauzamy [4] relating to the alternating-signs Banach–Saks property.

Let (Ω, Σ, P) be a probability space; when X is a Banach space, $L^1(X)$ will denote the Banach space of all equivalence classes of Bochner integrable X-valued functions on Ω, as in the book of Diestel and Uhl [6], for example.

Let $(a_{ij})_{i,j=1}^{\infty}$ be an infinite matrix of scalars. The matrix (a_{ij}) is said to *define a regular method of summability* if, whenever (s_j) is a sequence in a Banach space with $s_j \to s$, then the sequence $(t_i) = \left(\sum_{j=1}^{\infty} a_{ij}s_j\right)$ also converges to s. This happens (cf. Hardy [11]) if and only if:

(I) There is a constant H such that $\sum_{j=1}^{\infty}|a_{ij}| \leqslant H$ for all i,

(II) $a_{ij} \to 0$ as $i \to \infty$ for each j, and

(III) $a_i = \sum_{j=1}^{\infty} a_{ij} \to 1$ as $i \to \infty$.