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Almost sure summability of subsequences in Banach spaces

by

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Abstract. The property of almost everywhere summability, for subsequences of a sequence of vector-valued random variables, is considered. As a particular case of this, the convergence of sums $(N^{-1}\sum_{k=1}^{N}\varepsilon_k x_{n_k})$, for a bounded sequence (x_n) in a Banach space, and a sequence (ε_n) of signs, is examined; the results proved relate to the Banach–Saks and similar properties.

In this article, we will consider the notion of almost sure summability for subsequences of a sequence of Banach space valued random variables. Our first result extends the theorem of Erdös and Magidor [7] concerning the summability of subsequences in Banach spaces, and has relevance to several recent results in probability theory. We then employ similar techniques to prove results on the convergence of sums of the form $(N^{-1}\sum_{k=1}^{N}\pm x_k)$ in a Banach space: in particular we establish a conjecture of Beauzamy [4] relating to the alternating-signs Banach–Saks property.

Let (Ω, Σ, P) be a probability space; when X is a Banach space, $L^1(X)$ will denote the Banach space of all equivalence classes of Bochner integrable X-valued functions on Ω, as in the book of Diestel and Uhl [6], for example.

Let $(a_{ij})_{i,j=1}^{\infty}$ be an infinite matrix of scalars. The matrix (a_{ij}) is said to *define a regular method of summability* if, whenever (s_j) is a sequence in a Banach space with $s_j \to s$, then the sequence $(t_i) = \left(\sum_{j=1}^{\infty} a_{ij}s_j\right)$ also converges to s. This happens (cf. Hardy [11]) if and only if:

(I) There is a constant H such that $\sum_{j=1}^{\infty}|a_{ij}| \leqslant H$ for all i,

(II) $a_{ij} \to 0$ as $i \to \infty$ for each j, and

(III) $a_i = \sum_{j=1}^{\infty} a_{ij} \to 1$ as $i \to \infty$.

Conditions (I) and (II) together are equivalent to the condition that if $s_j \to 0$, then $t_i = \sum a_{ij} s_j$ also converges to 0. We shall use the notation $(y_j) \subseteq (x_j)$ to indicate that (y_j) is a subsequence of (x_j).

Using the theory of Ramsey sets, Erdös and Magidor proved the following theorem. (See also Figiel and Sucheston [8].)

THEOREM A [7]. *Let (x_j) be a bounded sequence in a Banach space X, and (a_{ij}) a regular method of summability. Then there exists a subsequence (y_j) of (x_j) such that either*

(a) *every subsequence of (y_j) is summable with respect to (a_{ij}) and each to the same limit, or*

(b) *no subsequence of (y_j) is summable with respect to (a_{ij}).*

Our first result extends this by considering almost sure convergence in $L^1(X)$. When (a_{ij}) satisfies (I) and (f_j) is a bounded sequence in $L^1(X)$, let $g_i = \sum_{j=1}^{\infty} a_{ij} f_j$, the sum converging in the $L^1(X)$ norm. In many examples this is a finite sum for each i. We shall say that (f_j) is *almost surely summable* if the sequence (g_i) converges almost surely. As this mode of convergence cannot be specified by a metric, the arguments needed are slightly more complicated. We obtain Theorem A as an immediate corollary, restricting to constant functions. The theorem also has some bearing on results in probability theory to do with the convergence of subsequences of random variables, such as the results of Komlós [12], Aldous [1] and Garling [10].

THEOREM 1. *Let (f_j) be a norm-bounded sequence in $L^1(X)$, and (a_{ij}) a matrix satisfying conditions* (I) *and* (II) *above. Then there exists a subsequence (g_j) of (f_j) such that either*

(a) *every subsequence of (g_j) is almost surely summable with respect to (a_{ij}), and each to the same limit, or*

(b) *no subsequence of (g_j) is almost surely summable with respect to (a_{ij}).*

Proof. We may suppose that $\|f_j\| \leqslant 1$ for each j. As in [7], we will consider the set $[\boldsymbol{N}]$ of all infinite subsequences of the natural numbers $\boldsymbol{N}$, with the topology inherited from the product topology on $\{0,1\}^{\boldsymbol{N}}$ by identifying a subsequence $\boldsymbol{n} = (n_i)$ with the point $\boldsymbol{r}$ with $r(n_i) = 1$ for all i, $r(j) = 0$ otherwise. Given a sequence $\boldsymbol{q} = (q_i)$ in $[\boldsymbol{N}]$, let $[\boldsymbol{q}]$ denote the set of all infinite subsequences of $\boldsymbol{q}$.

A set $\boldsymbol{R} \subseteq [\boldsymbol{N}]$ is called a *Ramsey set* if there is a $\boldsymbol{q} \in [\boldsymbol{N}]$ such that $[\boldsymbol{q}] \subseteq \boldsymbol{R}$ or $[\boldsymbol{q}] \subseteq [\boldsymbol{N}] \setminus \boldsymbol{R}$.

The Galvin–Prikry partition theorem [9] states that all Borel sets in $[\boldsymbol{N}]$ are Ramsey sets. We shall show that

$$S = \{\boldsymbol{m} \in [\boldsymbol{N}]:\ (\sum_{j=1}^{\infty} a_{ij} f_{m_j})_{i=1}^{\infty} \text{ converges almost surely}\}$$

is a Borel set. We assert that $S = \bigcap_{R \in N} \bigcup_{N \in N} \bigcap_{Q \geqslant N} T_{QNR}$, where

$$T_{QNR} = \Big\{\boldsymbol{m} \in [\boldsymbol{N}]:\ P\Big(t:\ \sup_{Q \geqslant i,k \geqslant N} \Big\|\Big(\sum_{j=1}^{\infty} (a_{ij} - a_{kj}) f_{m_j}\Big)(t)\Big\| \geqslant 1/R\Big) < 1/R\Big\}.$$

For, letting $h_N(\boldsymbol{m})$ denote

$$\sup_{i,k \geqslant N} \Big\|\Big(\sum_{j=1}^{\infty} (a_{ij} - a_{kj}) f_{m_j}\Big)(t)\Big\|,$$

$$\begin{aligned} S &= \{m:\ h_N(\boldsymbol{m}) \to 0 \text{ almost surely}\} \\ &= \{m:\ h_N(\boldsymbol{m}) \to 0 \text{ in measure}\} \end{aligned}$$

because $h_N(\boldsymbol{m})$ is a decreasing sequence of functions.

Suppose now that $\boldsymbol{m} \in T_{QNR}$; we will show that there is a J such that, if $\boldsymbol{q} \in [\boldsymbol{N}]$ and $q_j = m_j$ for $j < J$, then $\boldsymbol{q} \in T_{QNR}$, and thus T_{QNR} is an open set.

There clearly exist $\delta_1, \delta_2 > 0$ such that

$$P\Big(t:\ \sup_{Q \geqslant i,k \geqslant N} \Big\|\Big(\sum_{j=1}^{\infty} (a_{ij} - a_{kj}) f_{m_j}\Big)(t)\Big\| > 1/R - \delta_1\Big) \leqslant 1/R - \delta_2.$$

Let $\alpha = \delta_1/4$, $M = \binom{Q-N+1}{2}$ and $\beta = \delta_2 \alpha / 8M$. There exists an integer J such that $\sum_{j=J}^{\infty} |a_{ij}| < \beta$ for all i with $N \leqslant i \leqslant Q$. So $\|\sum_{j=J}^{\infty} (a_{ij} - a_{kj}) f_{q_j}\|_{L^1(X)} < 2\beta$ for all $\boldsymbol{q} \in [\boldsymbol{N}]$ and $N \leqslant i,\ k \leqslant Q$. Thus

$$P\Big(t:\ \Big\|\Big(\sum_{j=J}^{\infty} (a_{ij} - a_{kj}) f_{q_j}\Big)(t)\Big\| > \alpha\Big) \leqslant 2\beta/\alpha = \delta_2/4M.$$

If $\boldsymbol{q} \in [\boldsymbol{N}]$ and $q_j = m_j$ for $j < J$, and $N \leqslant i,\ k \leqslant Q$,

$$P\Big(t:\ \Big\|\Big(\sum_{j=1}^{\infty} (a_{ij} - a_{kj})(f_{m_j} - f_{q_j})\Big)(t)\Big\| > 2\alpha\Big) \leqslant \delta_2/2M.$$

So

$$P\Big(t:\ \sup_{Q \geqslant i,k \geqslant N} \Big\|\Big(\sum_{j=1}^{\infty} (a_{ij} - a_{kj}) f_{m_j}\Big)(t)\Big\| - \\ - \sup_{Q \geqslant i,k \geqslant N} \Big\|\Big(\sum_{j=1}^{\infty} (a_{ij} - a_{kj}) f_{q_j}\Big)(t)\Big\| > 2\alpha\Big) \leqslant \delta_2/2.$$

So

$$P\Big(t:\ \sup_{Q \geqslant i,k \geqslant N} \Big\|\Big(\sum_{j=1}^{\infty} (a_{ij} - a_{kj}) f_{q_j}\Big)(t)\Big\| > 1/R - \delta_1 + \delta_1/2\Big) \leqslant 1/R - \delta_2 + \delta_2/2$$

and $q \in T_{QNR}$. Thus T_{QNR} is open, and S is Borel, hence Ramsey. If there is a q such that $[q] \subseteq N \setminus S$, then case (b) holds. To complete the proof, we need to show that if there is a q such that $[q] \subseteq S$, then there is an $r \in [q]$ such that the limit is the same for all subsequences of r, i.e. that case (a) holds. We remark that, if $[q] \subseteq S$, then every subsequence of $(h_j) = (f_{q_j})$ is summable with respect to the convergence in measure metric (d, say). Clearly we may assume that $L^1(X)$ is norm-separable and thus d-separable since the (h_j) are essentially separably-valued. We now adapt the corresponding argument of [7]: for $n = 1, 2, \ldots$ we may cover $L^1(X)$ by countably many open balls $(O_{n,j})$ of radius $1/n$, and find integers (k_n) and nested subsequences $(h_{n,j})$ such that every subsequence of $(h_{n,j})$ is d-summable to a limit in O_{n,k_n}. This is because the partition involved is again Borel. Clearly every subsequence of the diagonal subsequence $(h_{n,n})$ is summable almost surely to the same limit.

COROLLARY 1. *Let (f_j), (a_{ij}) be as above. Then there is a subsequence (g_j) of (f_j) such that either*

(a) *every subsequence of (g_j) is almost surely summable to zero with respect to (a_{ij}), or*

(b) *no subsequence of (g_j) is almost surely summable to zero with respect to (a_{ij}).*

Proof. Let

$$S' = \left\{m \in [N]: \left(\sum a_{ij} f_{m_j}\right) \to 0 \text{ almost surely}\right\} = \bigcap_{R \in N} \bigcup_{N \in N} \bigcap_{Q \geqslant N} T'_{QNR},$$

where

$$T'_{QNR} = \left\{m \in [N]: P\left(t: \sup_{Q \geqslant i \geqslant N} \left\|\left(\sum_{j=1}^{\infty} a_{ij} f_{m_j}\right)(t)\right\| \geqslant 1/R\right) < 1/R\right\}.$$

Similar arguments to the above show that T'_{QNR} is an open set and S' is a Ramsey set.

One particular case of this is the following, obtained by considering constant functions and a particular summability matrix satisfying conditions (I) and (II).

COROLLARY 2. *Let (x_j) be a bounded sequence in a Banach space X; then there is a subsequence (y_j) of (x_j) such that either*

(a) $\left\|N^{-1} \sum_{j=1}^{N} (-1)^j z_j\right\| \to 0$ *for all subsequence (z_j) of (y_j), or*

(b) $\left\|N^{-1} \sum_{j=1}^{N} (-1)^j z_j\right\| \to 0$ *for no subsequence (z_j) of (y_j).*

A Banach space X is said to have the *weak Banach–Saks property* (WBS) if for every weakly null sequence (x_j) in X there is a subsequence (y_j) such that $\left\|N^{-1} \sum_{j=1}^{N} y_j\right\| \to 0$. See [13], for example. If every bounded sequence (x_j) possesses a subsequence (y_j) for which $\left\|N^{-1} \sum_{j=1}^{N} (-1)^j y_j\right\| \to 0$, then X is said to have the *alternating-signs Banach–Saks property* (ABS). This property was extensively investigated by Beauzamy [3], using the "spreading-model" techniques of Brunel and Sucheston [5]. In particular Beauzamy proved that X has (ABS) if and only X has (WBS) and X does not contain an isomorphic copy of l_1; moreover, he obtained the following result.

THEOREM B [3]. *Let (x_j) be a bounded sequence in a Banach space X. Then there is a subsequence (y_j) of (x_j) such that either*

(a) $\left\|N^{-1} \sum_{j=1}^{N} (-1)^j y_j\right\| \to 0$, *or*

(b) *there is a $\delta > 0$ such that, for all subsequences (z_j) of (y_j) and for all choices of sign $(\varepsilon_j) = \pm 1$, we have*

$$\left\|N^{-1} \sum_{j=1}^{N} \varepsilon_j z_j\right\| \geqslant \delta.$$

In [4], Beauzamy improved this result to show that in case (a) one can have $N^{-1} \int_0^1 \left\|\sum_{j=1}^{N} r_j(t) z_j\right\| dt \to 0$ for all subsequences (z_j) of (y_j), where $(r_j(t))$ are the Rademacher functions on $[0, 1]$, and that this implies that

$$\left\|N^{-1} \sum_{j=1}^{N} r_j(t) z_j\right\| \to 0 \quad \text{almost surely for all } (z_j) \subseteq (y_j).$$

This is also expressed by saying that, for all $(z_j) \subseteq (y_j)$,

$$\left\|N^{-1} \sum_{j=1}^{N} \varepsilon_j z_j\right\| \to 0 \quad \text{for almost all choices of sign.}$$

Beauzamy then raised the question whether one could obtain that, for almost all choices of sign, $\left\|N^{-1} \sum_{j=}^{N} \varepsilon_j z_j\right\| \to 0$ for each of the uncountably many subsequences of (y_j).

In the remainder of this article we shall answer this question in the affirmative: indeed, we shall provide a simple characterization of the set of sequences of signs (ε_j) for which the convergence of $\left(N^{-1} \sum_{j=1}^{N} \varepsilon_j z_j\right)$ holds.

We will find it convenient to identify a sequence (ε_m) taking values ± 1 with the corresponding sequence of plus and minus signs. Given such a sequence, let P_m and N_m denote respectively the number of plus and minus signs occurring amongst the first m signs. Thus $P_m+N_m=m$. The following combinatorial result will be used to extract strictly alternating subsequences from a sequence $(\varepsilon_n x_n)$.

THEOREM 2. *Let (ε_m) be a sequence of signs such that $P_m/m\to 1/2$ as $m\to\infty$. Then given any positive integer k, it is possible to partition $\{\varepsilon_1,\ldots,\varepsilon_m\}$ into disjoint strictly alternating subsequences of length at least k, together with a residual subsequence of s_m terms, where $s_m/m\to 0$ as $m\to\infty$.*

Proof. It is sufficient to show that, given $\gamma>0$, there is an integer N such that if $n\geqslant N$, we can partition $\{\varepsilon_1,\ldots,\varepsilon_{kn}\}$ into disjoint strictly alternating subsequences of length k, together with at most γkn residual terms.

Let N be chosen so that $|p_m/m-1/2|<\gamma/6k^2$ for $m\geqslant N$, and $N\geqslant 6/\gamma$. Let $n\geqslant N$ and consider the partition of $\{\varepsilon_1,\ldots,\varepsilon_{kn}\}$ into k blocks of n consecutive terms.

Now for $0\leqslant r\leqslant k-1$, $P_{rn}<rn(1/2+\gamma/6k^2)$ and $P_{(r+1)n}>(r+1)n(1/2-\gamma/6k^2)$; thus the $(r+1)^{\text{st}}$ block contains at least $n/2-(2r+1)\gamma n/6k^2$ plus signs. The same is true for minus signs. Hence we may select $[n/2-\gamma n/3k]$ plus signs and the same number of minus signs from each block, leaving at most $k(2(\gamma n/3k+1))$ residual terms. The selected terms can be arranged into $2[n/2-\gamma n/3k]$ disjoint strictly alternating subsequences of length k, for each subsequence choosing one term from each block in such a way that the signs alternate.

Since the residual terms number at most $\frac{2}{3}n\gamma+2k\leqslant\gamma kn(\frac{2}{3}+\frac{1}{3})=\gamma kn$, the result follows.

We will now require a simple technical lemma, related to a remark of Szlenk [14], which will enable us to obtain results giving uniformity of convergence among all subsequences of a given sequence.

LEMMA. *Suppose that (x_n) is a bounded sequence in a Banach space such that $\|N^{-1}\sum_{n=1}^{N}(-1)^n y_n\|\to 0$ for all subsequences (y_n) of (X_n). Then*

$$\sup_{(y_n)\subseteq(x_n)}\Big\|N^{-1}\sum_{n=1}^{N}(-1)^n y_n\Big\|\to 0\quad\textit{as}\quad N\to\infty.$$

Proof. We may assume that $\|x_n\|\leqslant 1$ for all n. If the assertion fails, there exist $\delta>0$, a sequence $N_i\to\infty$ and subsequences $(x_{i,n})$ of (x_n), $i=1,2,\ldots$, such that, for each i,

$$\Big\|\sum_{n=1}^{N_i}(-1)^n x_{i,n}\Big\|\geqslant N_i\delta.$$

Given any $k_1\leqslant\ldots\leqslant k_M$, it is possible to select integers i and j with $j>k_M$ and $N_i\delta/2\geqslant M(1+\delta/2)+j(1-\delta/2)$. Then

$$\Big\|\sum_{m=1}^{M}(-1)^m x_{k_m}\pm\sum_{n=j+1}^{N_i}(-1)^n x_{i,n}\Big\|\geqslant N_i\delta-j-M\geqslant(N_i-j+M)\delta/2.$$

Defining $k_{M+1},\ldots,k_{N_i-j+M}$ so that

$$\{x_{i,j+1},\ldots,x_{i,N_i}\}=\{x_{k_{M+1}},\ldots,x_{k_J}\},\qquad J=N_i+M-j,$$

we have

$$\Big\|\sum_{m=1}^{J}(-1)^m x_{k_m}\Big\|\geqslant J\delta/2.$$

Hence it is possible to construct a subsequence $(y_n)=(x_{k_n})$ with $\limsup_{N\to\infty}\|N^{-1}\sum_{n=1}^{N}(-1)^n y_n\|\geqslant\delta/2$, a contradiction. This completes the proof.

We are now ready to prove our main result in this direction, relating the convergence of $N^{-1}\sum_{n=1}^{N}(-1)^n y_n$ and $N^{-1}\sum_{n=1}^{N}\varepsilon_n y_n$.

THEOREM 3. (i) *Let (ε_m) be a sequence of plus and minus signs such that $P_m/m\to 1/2$ as $m\to\infty$. If (x_n) is any bounded sequence in a Banach space such that $\|N^{-1}\sum_{n=1}^{N}(-1)^n y_n\|\to 0$ for all $(y_n)\subseteq(x_n)$, then*

$$\sup_{(y_n)\subseteq(x_n)}\Big\|N^{-1}\sum_{n=1}^{N}\varepsilon_n y_n\Big\|\to 0\quad\textit{as}\quad N\to\infty.$$

(ii) *There is a bounded sequence (x_n) in a Banach space X, for which $\|N^{-1}\sum_{n=1}^{N}(-1)^n y_n\|\to 0$ for all $(y_n)\subseteq(x_n)$, but $N^{-1}\sum_{n=1}^{N}\varepsilon_n y_n$ fails to converge for all $(y_n)\subseteq(x_n)$ and all sequences (ε_n) for which $P_n/n\nrightarrow 1/2$.*

Proof. (i): By the lemma, given $\gamma>0$, there is an integer k such that

$$\sup_{(y_n)\subseteq(x_n)}\Big\|N^{-1}\sum_{n=1}^{N}(-1)^n y_n\Big\|<\gamma/2\quad\text{ if }\quad N\geqslant k.$$

Using Theorem 2, we can partition $\{\varepsilon_1,\ldots,\varepsilon_m\}$ into disjoint alternating subsequences of length at least k, together with a residual subsequence of length s_m, where $s_m/m\to 0$ as $m\to\infty$. Hence, for any $(y_n)\subseteq(x_n)$, $\|\sum_{n=1}^{m}\varepsilon_n y_n\|\leqslant\gamma m/2+s_m$ and so

$$\sup_{(y_n)\subseteq(x_n)}\Big\|m^{-1}\sum_{n=1}^{m}\varepsilon_n y_n\Big\|\leqslant\gamma/2+s_m/m\leqslant\gamma$$

for sufficiently large m. This completes the proof.

(ii): Let $X = c_0$, (e_n) the standard basis in c_0; let $x_n = e_1 + \ldots + e_n$. Then (x_n) is a bounded sequence and, for all $(y_n) \subseteq (x_n)$, $a_1, \ldots, a_N$ scalars, $\|\sum a_n y_n\| = \|\sum a_n x_n\|$, since it is easily seen that the linear map taking $(y_{i+1} - y_i)$ $(i = 1, \ldots, N)$ to $(x_{i+1} - x_i)$ and y_1 to x_1 is an isometry. Thus $\|N^{-1} \sum_{n=1}^{N} (-1)^n y_n\| \to 0$ for all $(y_n) \subseteq (x_n)$.

Suppose we are given a sequence (ε_n) for which $N^{-1} \sum_{n=1}^{N} \varepsilon_n x_n$ is a Cauchy sequence. Then given $\gamma > 0$ there is an integer M for which

$$\left\| N^{-1} \sum_{n=1}^{N} \varepsilon_n x_n - \frac{1}{M} \sum_{n=1}^{M} \varepsilon_n x_n \right\| < \gamma \quad \text{for all } N > M.$$

Looking at the $(M+1)^{\text{st}}$ coordinate, we see that $|N^{-1} \sum_{n=M+1}^{N} \varepsilon_n| < \gamma$ for all $N > M$, and hence $|N^{-1} \sum_{n=1}^{N} \varepsilon_n| < \gamma$ for all sufficiently large N. Hence $|N^{-1}(\varepsilon_1 + \ldots + \varepsilon_N)| \to 0$ if $(N^{-1} \sum_{n=1}^{N} \varepsilon_n y_n)$ is Cauchy for any $(y_n) \subseteq (x_n)$, and so $P_N/N \to 1/2$ as $N \to \infty$, as required.

COROLLARY 3. *Let (x_n) be a bounded sequence in a Banach space such that $\|N^{-1} \sum_{n=1}^{N} (-1)^n y_n\| \to 0$ for all $(y_n) \subseteq (x_n)$. Then* $\sup_{(y_n) \subseteq (x_n)} \|N^{-1} \sum_{n=1}^{N} r_n(t) y_n\| \to 0$ *almost surely and in L^p $(1 \leqslant p < \infty)$.*

Proof. For almost all t, the sequence $(\varepsilon_n) = (r_n(t))$ satisfies $P_m/m \to 1/2$ as $m \to \infty$. Now use Theorem 3 (i) and the dominated convergence theorem.

COROLLARY 4. *Let X be a space with the* (ABS) *property, and (x_n) a bounded sequence in X. Then there is a subsequence (y_n) of (x_n) for which* $\sup_{(z_n) \subseteq (y_n)} \|N^{-1} \sum_{n=1}^{N} r_n(t) z_n\| \to 0$ *almost surely and in L^p $(1 \leqslant p < \infty)$.*

Proof. Use Corollary 3 and Corollary 2.

Clearly it follows also from Corollaries 2 and 3 that in case (a) of Theorem B one can have $\sup_{(z_n) \subseteq (y_n)} \|N^{-1} \sum_{n=1}^{N} r_n(t) z_n\| \to 0$ almost surely and in L^p $(1 \leqslant p < \infty)$.

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