

Travaux cités

- [1] T. Leżański, *Sur les équations du type: $\mathcal{P}(w, h) = 0$ (I)* Studia Math. 59 (1976), 155–175.
- [2] F. Riesz et B. Sz. Nagy *Leçons d'analyse fonctionnelle*, Budapest 1952.
- [3] [M. M. Smirnov] М. М. Смирнов, *Вырождающиеся эллиптические и гиперболические уравнения*, Москва 1966.
- [4] [M. I. Višik] М. И. Вишик, *Красное задание для эллиптических уравнений вырождающихся на границе области*, Mat. Sb. 35/77.3 (1954), 513–568.

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A generalization of the Shimogaki theorem

by

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Abstract. There are given a necessary condition and a sufficient condition for a rearrangement invariant space on $(0, l)$, $0 < l < \infty$ to have an interpolation property for the class of Lipschitz operators which are of weak type (X, X) and type (L^∞, L^∞) where X is any fixed rearrangement invariant space on $(0, l)$ from the class \mathcal{L} .

§ 1. Introduction. An operator T that maps a quasi-Banach space X to a quasi-Banach space Y is called a *Lipschitz operator* if $T0 = 0$ and if $\|Tf - Tg\|_Y \leq K\|f - g\|_X$, $f, g \in X$ for some $K > 0$. The smallest K in this inequality is called the *bound* of T . By $\text{Lip}(X, Y; K)$ we denote the class of all Lipschitz operators T from X to Y with the bound not exceeding K . If $X = Y$, we write briefly $\text{Lip}(X; K)$.

Now, let X be a rearrangement invariant space, $\Lambda(X)$ and $M^*(X)$ rearrangement invariant Lorentz spaces (see § 2), and let $\omega\text{-Lip}(X; K_X) = \text{Lip}(\Lambda(X), M^*(X); K_X)$ be a class of Lipschitz operators of weak type (X, X) with bound not exceeding K_X . We say that a Banach space Y such that $Y \subset \Lambda(X) + L^\infty$ has the *interpolation property for the class* $\omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$ if, for each $T \in \omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$, T can be considered as a Lipschitz operator from Y into itself.

W. Orlicz [13] has proved that any Orlicz space L^ϕ on $(0, l)$ has the interpolation property for the class $\text{Lip}(L^1; K_1) \cap \text{Lip}(L^\infty; K_\infty)$ in the case when $l < \infty$.

G. G. Lorentz and T. Shimogaki [8] have generalized this theorem to the case of any rearrangement invariant space on $(0, l)$. They have also obtained this theorem in the case when $l = \infty$, under an additional assumption of continuity of the norm. In [10] an analogous theorem was obtained for the class $\text{Lip}(l^1; K_1) \cap \text{Lip}(e_0; K_0)$ and for rearrangement invariant sequence spaces. Next, a sufficient condition was given in [11] for a Banach function space Y on $(0, l)$ to have the interpolation property for the class $\text{Lip}(L^p; K_p) \cap \text{Lip}(L^\infty; K_\infty)$ and similarly for a Banach sequence space and for the class $\text{Lip}(l^p; K_p) \cap \text{Lip}(e_0; K_0)$. All theorems mentioned above deal with Lipschitz operators of strong type. Interpolation of Lipschitz operators of weak type has been considered by T. Shimogaki [15] who has given the following

THEOREM A (Shimogaki). *A rearrangement invariant space Y on $(0, t)$ has the interpolation property for the class $\omega\text{-Lip}(L^1; K_1) \cap \text{Lip}(L^\infty; K_\infty)$ if and only if $\alpha(Y) < 1$, where $\alpha(Y)$ denotes the upper Boyd of the space Y . If $l = \infty$, one additionally assumes in the sufficient condition that the norm on Y is continuous.*

In this paper, in Section 4, Theorem A is generalized by giving a necessary condition and a sufficient condition for a rearrangement invariant space Y to have the interpolation property for the class $\omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$, where X is any fixed rearrangement invariant space from the class \mathcal{L} (Def. 2.20). In the case when X is an Orlicz space L^ϕ , the symbol $L^\phi \in \mathcal{L}$ means that ϕ satisfies the condition Δ_2 (see [6], p. 23). The necessary condition and the sufficient condition of this theorem are expressed as in the Shimogaki theorem, but by means of indices of Boyd and Zippin type (Th. 4.6).

In Section 3 of this paper there is described the X -Hardy property (Def. 3.1) which is a fundamental one for interpolation. There is also given a generalization of Hardy's inequality. Namely, if $\alpha(Y) < p_0(X)$ (where $p_0(X)$ denotes the lower Zippin index of X), then Y has X -Hardy property and

$$(1.1) \quad \left\| \varphi_X(t)^{-1} \int_0^t f(s) d\varphi_X(s) \right\|_Y \leq \int_0^1 h(s, Y) dM(s, X) \|f\|_X.$$

These considerations have been inspired by the paper by P. L. Butzer and F. Fehér [5] concerning, among other things, $L^{1/\theta}$ -Hardy property ($0 < \theta \leq 1$) and Hardy's inequality; in this paper the following theorem is proved:

THEOREM B (Boyd, see also Butzer-Fehér). *A rearrangement invariant space Y has the $L^{1/\theta}$ -Hardy property if and only if $\alpha(Y) < \theta$. Moreover,*

$$(1.2) \quad \left\| t^{-\theta} \int_0^t s^{\theta-1} f(s) ds \right\|_Y \leq \int_0^1 s^{\theta-1} h(s, Y) ds \|f\|_Y.$$

Using Boyd's [2] considerations it is also proved that if Y has the X -Hardy property, then $\alpha(Y) < p_\infty(X)$. It can be seen that in case where $\varphi_X(t) = t^\theta$ then from the above facts we obtain Theorem B and inequality (1.2) as the particular case of (1.1).

In Section 5, a majorant for the class $\mathcal{C} = \omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$ is found in the case where $\varphi_X(t) = t^\theta$, $0 < \theta \leq 1$. The general case remains open.

§ 2. Rearrangement invariant spaces. Let \mathcal{M} be the space of real-valued Lebesgue measurable functions on the interval $I = (0, t)$, $0 < t$

$\leq \infty$. A Banach subspace X of \mathcal{M} satisfying (m denoting the Lebesgue measure)

$$(2.1) \quad |g| \leq |f| \text{ a.e., and } f \in X \text{ implies } g \in X \text{ and } \|g\|_X \leq \|f\|_X;$$

$$(2.2) \quad 0 \leq f_n \uparrow, \sup_{n \in \mathbb{N}} \|f_n\|_X \leq M,$$

$$\text{implies } f = \sup_{n \in \mathbb{N}} f_n \in X \text{ and } \|f\|_X = \sup_{n \in \mathbb{N}} \|f_n\|_X;$$

$$(2.3) \quad mA < \infty \text{ implies } \int_A f(s) ds \leq C_A \|f\|_X,$$

with $C_A > 0$ independent of $f \in X$;

$$(2.4) \quad mA < \infty \text{ implies } \|\chi_A\|_X < \infty,$$

where χ_A is the characteristic function of the set A

is usually referred to as a *Banach function space* (see [7], [9]). A Banach function space X is called a *rearrangement invariant* (r.i.) space or *symmetric* in the terminology of Semenov if $d_f(y) = m\{s \in I: |f(s)| > y\} < \infty$ for each $f \in X$, $y > 0$, and if

$$(2.5) \quad \text{for any } g \in \mathcal{M} \text{ equimeasurable to } f \in X, \text{ i.e., } d_g = d_f, \text{ one has } g \in X \text{ and } \|g\|_X = \|f\|_X.$$

The nonincreasing rearrangement f^* of a function $f \in X$ is an almost everywhere unique positive nonincreasing function on I which is equimeasurable to $|f|$. So, for an r.i. space we have

$$\|f\|_X = \|f^*\|_X.$$

Examples of r.i. spaces include the Lebesgue L^p -spaces, the Orlicz L^ϕ -spaces and the Lorentz spaces A , M and $L^{p,q}$. Also if X , Y are r.i. spaces, so is $X \cap Y$, $X + Y$, and \bar{A}^X satisfy (2.1) and (2.5), where A is symmetric linear subset of X (is symmetric if $f \in A$ and $g^*(t) \leq f^*(t)$ imply $g \in A$).

We shall write $\langle f, g \rangle$ for the inner product $\int f(t)g(t)dt$.

The associated space X' of an r.i. space X is defined by

$$X' = \{g \in \mathcal{M}: \|g\|_{X'} = \sup_{\|f\|_X \leq 1} \langle f, g \rangle < \infty\}.$$

It follows that X' is also an r.i. space under the norm $\|\cdot\|_{X'}$. A result due to Lorentz and Luxemburg [9] is that X is isometrically isomorphic to X'' , and, in particular, the norm has the representation

$$(2.6) \quad \|f\|_X = \sup \{\langle f^*, g^* \rangle: \|g\|_{X'} \leq 1\}$$

from which the important Hölder inequality

$$(2.7) \quad |\langle f, g \rangle| \leq \langle |f|, |g| \rangle \leq \langle f^*, g^* \rangle \leq \|f\|_X \|g\|_{X'}$$

follows.

The smallest and the largest of the r.i. spaces are $L^1 \cap L^\infty$ and $L^1 + L^\infty$, respectively, in the sense that the continuous embeddings

$$(2.8) \quad L^1 \cap L^\infty \subset X \subset L^1 + L^\infty$$

hold for any r.i. space X .

The fundamental function φ_X of an r.i. space X is given by

$$(2.9) \quad \varphi_X(t) = \|\chi_{(0,t)}\|_X, \quad t \in I.$$

The fundamental function is a positive nondecreasing function on I which is absolutely continuous on $[s, t]$ for each $s > 0$ and satisfies

$$(2.10) \quad \varphi_X(t)\varphi_{X'}(t) = t \quad \text{for all } t \in I,$$

$$(2.11) \quad \frac{d\varphi_X(t)}{dt} \leq \frac{\varphi_X(t)}{t} \quad \text{a.e. on } I.$$

A straightforward computation shows that (i) for the Orlicz space with Orlicz norm L^Φ one has $\varphi_{L^\Phi}(t) = t\mathcal{P}^{-1}(1/t)$, where \mathcal{P} is complementary to the Orlicz function Φ , (ii) for the Orlicz space with Luxemburg norm L^{Φ} one has $\varphi_{L^{\Phi}}(t) = 1/\Phi^{-1}(1/t)$, (iii) φ_{L^Φ} is concave function, (iv) $\varphi_{L^{\Phi}}$ does not have to be concave function; it is sufficient to take $\Phi(u) = u^2$ for $0 < u \leq 1$ and u^3 for $u > 1$.

In [16] and [18] it was shown that X can be equivalently renormed so that the resulting fundamental function is concave. We assume throughout this paper that X is renormed in this manner, i.e., φ_X is concave.

The rearrangement invariant Lorentz spaces associated with r.i. space X , defined by

$$\Lambda(X) = \left\{ f \in \mathcal{M} : \|f\|_{\Lambda(X)} = \int_0^t f^*(s) d\varphi_X(s) < \infty \right\},$$

and

$$M(X) = \left\{ f \in \mathcal{M} : \|f\|_{M(X)} = \sup_{t \in I} \left(\frac{\varphi_X(t)}{t} \int_0^t f^*(s) ds \right) < \infty \right\},$$

are extremal in the sense that

$$(2.12) \quad \Lambda(X) \subset X \subset M(X)$$

with normal continuous embeddings. We consider also a rearrangement invariant quasi-Banach function space (Marcinkiewicz space)

$$M^*(X) = \left\{ f \in \mathcal{M} : \|f\|_{M^*(X)} = \sup_{t \in I} (\varphi_X(t) f^*(t)) < \infty \right\}.$$

For example, if $X = L^{p,q}$, then $\varphi_X(t) = t^{1/p}$, so that $\Lambda(X) = L^{p,1}$ and $M^*(X) = L^{p,\infty}$, the “weak” L^p -spaces.

The fundamental Zippin indices of an r.i. space X (see [18]) are given by

$$p_0(X) = p_0(X, I) = \sup_{0 < s < 1} \theta(s) = \lim_{s \rightarrow 0^+} \theta(s),$$

$$p_\infty(X) = p_\infty(X, I) = \inf_{s > 1} \theta(s) = \lim_{s \rightarrow \infty} \theta(s),$$

where $\theta(s) = \log M(s, X)/\log s$, and for $s > 0$,

$$M(s, X) = \sup_{t \in I, t \leq s} (\varphi_X(ts)/\varphi_X(t)).$$

If $m(s; X) = \inf_{t \in I, t \leq s} (\varphi_X(ts)/\varphi_X(t))$, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(2.13) \quad s^{p_0(X)} \leq M(s, X) \leq s^{p_0(X)-\varepsilon}, \quad 0 < s < \delta,$$

$$(2.14) \quad s^{p_\infty(X)+\varepsilon} \leq m(s, X) \leq s^{p_\infty(X)}, \quad 0 < s < \delta.$$

If \tilde{X} is the space X with an equivalent norm then $p_0(\tilde{X}) = p_0(X)$ and $p_\infty(\tilde{X}) = p_\infty(X)$.

The upper and lower Boyd indices of an r.i. space X (see [3]) are given by

$$\alpha(X) = \alpha(X, I) = \inf_{0 < s < 1} \theta_1(s) = \lim_{s \rightarrow 0^+} \theta_1(s),$$

$$\beta(X) = \beta(X, I) = \sup_{s > 1} \theta_1(s) = \lim_{s \rightarrow \infty} \theta_1(s),$$

where $\theta_1(s) = -\log h(s, X)/\log s$ and $h(s, X)$ is the operator norm on X of the dilation operator

$$(2.15) \quad (E_s f)(t) = \begin{cases} f(ts) & \text{if } ts \in I, \\ 0 & \text{elsewhere.} \end{cases}$$

From the definition of α we have that for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(2.16) \quad s^{-\alpha(X)} \leq h(s, X) \leq s^{-\alpha(X)-\varepsilon}, \quad 0 < s < \delta.$$

These indices are known to satisfy:

$$(2.17) \quad 0 \leq \beta(X) \leq p_0(X) \leq p_\infty(X) \leq \alpha(X) \leq 1,$$

$$(2.18) \quad p_0(X') = 1 - p_\infty(X), \quad p_\infty(X') = 1 - p_0(X),$$

$$(2.19) \quad \beta(X') = 1 - \alpha(X), \quad \alpha(X') = 1 - \beta(X).$$

For familiar r.i. spaces such as the Lebesgue spaces L^p , $1 \leq p \leq \infty$, the Orlicz space L^Φ , the Lorentz spaces $\Lambda(\psi)$ and $M(\psi)$, both $p_0(\cdot) = \beta(\cdot)$ and $p_\infty(\cdot) = \alpha(\cdot)$ hold. T. Shimogaki [16] has given an example of r.i. spaces X_a such that $p_0(X_a) = p_\infty(X_a) = a$ and $\beta(X_a) = 0$ or $\alpha(X_a) = 1$ for every $0 < a < 1$.

The following class of spaces will be useful in this paper ([14]):

(2.20) $X \in \mathcal{L} = \mathcal{L}(I)$, if for some $0 < b < 1$ there is a pair of positive constants C and δ such that for $u, v \in I$

$$\varphi_X(v)/\varphi_X(u) \leq C(v/u)^b \quad \text{if} \quad v/u \leq \delta.$$

Let us notice that $X \in \mathcal{L}$ if and only if $p_0(X) > 0$; in the case of an Orlicz space, the condition $L^\Phi \in \mathcal{L}(I)$ is equivalent to the condition: Φ satisfies $\Delta_2(I)$ condition.

We shall use the following simple inequality repeatedly.

LEMMA (2.1). If $f_1(s), f_2(s)$ are nondecreasing absolutely continuous functions such that $f_1(0) = f_2(0) = 0$ and if $f_1(s) \leq f_2(s)$ for all $s \in I$, then for any nonnegative, nonincreasing function g

$$(2.21) \quad \int_I g(s) df_1(s) \leq \int_I g(s) df_2(s).$$

§ 3. X -Hardy property. Whereas in [3] and [5] the $L^{1/\theta}$ -Hardy property ($0 < \theta \leq 1$) is considered, we now want to replace $L^{1/\theta}$ by any r.i. space X . The aim of this section is therefore to introduce the " X -Hardy property" and give necessary and sufficient conditions for this property to hold. For this purpose, let X, Y denote any r.i. spaces on I with $X \in \mathcal{L}$, and $[Y]$ the space of bounded linear operators from Y into itself.

We define an operator P_X associated with an r.i. space X for $f \in \mathcal{M}$ by

$$(3.1) \quad (P_X f)(t) := \varphi_X(t)^{-1} \int_0^t f(s) d\varphi_X(s), \quad t \in I.$$

whenever the integral exists a.e.

DEFINITION (3.1). We shall say that Y has the X -Hardy property (we write $Y \in \text{HP}_X$), if the operator $P_X \in [Y]$.

If $P_X \in [Y]$ and $|\lambda| < r(P_X)^{-1}$, where $r(P_X)$ denotes the spectral radius of the operator P_X , and I is the identity operator, then $(I - \lambda P_X)^{-1}$ exists and $(I - \lambda P_X)^{-1} \in [Y]$, and

$$(3.2) \quad (I - \lambda P_X)^{-1} = \sum_{n=0}^{\infty} \lambda^n P_X^n,$$

where the series converges in operator norm, and P_X^n denotes the n th iterate of P_X . Since

$$(3.3) \quad (P_X^n f)(t) = \varphi_X(t)^{-1} \int_0^t \frac{1}{(n-1)!} \left(\ln \frac{\varphi_X(t)}{\varphi_X(ts)} \right)^{n-1} f(st) d\varphi_X(ts),$$

one has

$$(3.4) \quad (P_{X(\lambda)} f)(t) := P_X(I - \lambda P_X)^{-1} f(t)$$

$$= \varphi_X(t)^{-1} \int_0^1 f(st) \left(\frac{\varphi_X(t)}{\varphi_X(ts)} \right)^\lambda d\varphi_X(ts).$$

LEMMA (3.2). If $P_{X(\lambda)} \in [Y]$, and $\lambda > 0$, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $u \in (0, \delta)$

$$(3.5) \quad h(u, Y) \int_0^u M(s, X)^{-\lambda} ds^{p_\infty(X)+\varepsilon} \leq \|P_{X(\lambda)}\|_{[Y]}.$$

Proof (see [1]). Since $P_{X(\lambda)} \in [Y]$ and $s^{p_\infty(X)+\varepsilon} \leq \varphi_X(ts)/\varphi_X(t)$ for $0 < s < \delta$, we have for every nonnegative and nonincreasing f, g

$$\begin{aligned} \langle E_u f, g \rangle &\int_0^u M(s, X)^{-\lambda} ds^{p_\infty(X)+\varepsilon} \leq \int_0^u M(s, X)^{-\lambda} \langle E_s f, g \rangle ds^{p_\infty(X)+\varepsilon} \\ &\leq \int_0^u (\varphi_X(ts)/\varphi_X(t))^{-\lambda} \langle E_s f, g \rangle ds^{p_\infty(X)+\varepsilon} \\ &= \left\langle \int_0^u (\varphi_X(\cdot s)/\varphi_X(\cdot))^{-\lambda} f(\cdot s) \chi_I(\cdot s) ds^{p_\infty(X)+\varepsilon}, g \right\rangle \\ &\leq \left\langle \int_0^u (\varphi_X(\cdot s)/\varphi_X(\cdot))^{-\lambda} f(\cdot s) \chi_I(\cdot s) d(\varphi_X(\cdot s)/\varphi_X(\cdot)), g \right\rangle \\ &\leq \langle P_{X(\lambda)} f, g \rangle \leq \|P_{X(\lambda)}\|_{[Y]} \|f\|_{[Y]} \|g\|_{[Y]}. \end{aligned}$$

Taking the supremum over all $g \in Y'$, $\|g\|_{Y'} \leq 1$, and then over all $f \in Y$, $\|f\|_Y \leq 1$, we obtain (3.5).

LEMMA (3.3). Let $p_0(X) > 0$. The following assertions are equivalent:

$$(3.6) \quad \alpha(Y) < p_0(X);$$

$$(3.7) \quad \text{There exist positive numbers } K \text{ and } \gamma \text{ } (0 \leq \gamma < 1) \text{ such that } h(s, Y) \leq K s^{1-p_0(X)-\gamma}, \quad 0 < s \leq 1;$$

$$(3.8) \quad s^{p_0(X)} h(s, Y) \rightarrow 0 \text{ as } s \rightarrow 0^+;$$

$$(3.9) \quad A_{p_0(X)} := \int_0^1 s^{p_0(X)-1} h(s, Y) ds < \infty;$$

$$(3.10) \quad C_0(X, Y) := \int_0^1 h(s, Y) dM(s, X) < \infty.$$

Proof. For (3.6) \Leftrightarrow (3.7) \Leftrightarrow (3.8) \Leftrightarrow (3.9) see [1] and [5].

(3.6) \Rightarrow (3.10): Let us take $\varepsilon > 0$ such that $a(Y) + 2\varepsilon < p_0(X)$, then by Lemma (2.1)

$$\begin{aligned} C_0(X, Y) &\leq \int_0^\delta s^{-a(Y)-\varepsilon} dM(s, X) + \int_\delta^1 (1/s) dM(s, X) \\ &\leq \int_0^\delta s^{-a(Y)-\varepsilon} ds^{p_0(X)-\varepsilon} + \text{const} < \infty. \end{aligned}$$

(3.10) \Rightarrow (3.6): Evidently since by Lemma (2.1)

$$0 < \int_0^\delta s^{-a(Y)} ds^{p_0(X)} \leq \int_0^\delta s^{-a(Y)} dM(s, X) \leq C_0(X, Y).$$

THEOREM (3.4). (a) If $C_0(X, Y) < \infty$, then $P_X \in [Y]$ and $\|P_X\|_{[Y]} \leq C_0(X, Y)$.

(b) If $P_X \in [Y]$, then $A_{p_\infty(X)} := \int_0^1 s^{p_\infty(X)-1} h(s, Y) ds < \infty$.

Proof. (a): Using the Fubini theorem, Lemma (2.1) and the Hölder inequality, we have

$$\begin{aligned} \langle (P_X f)^*, g^* \rangle &\leq \langle P_X f^*, g^* \rangle = \int_0^1 \left(\varphi_X(t)^{-1} \int_0^1 f^*(ts) d\varphi_X(ts) \right) g^*(t) dt \\ &\leq \int_0^1 \left(\int_0^1 E_s f^*(t) g^*(t) dt \right) dM(s, X) \\ &\leq \int_0^1 \|E_s f^*\|_{Y'} \|g^*\|_{Y'} dM(s, X) \leq \int_0^1 h(s, Y) dM(s, X) \|f^*\|_{Y'} \|g^*\|_{Y'}. \end{aligned}$$

Taking the supremum over all $g \in Y'$, $\|g\|_{Y'} \leq 1$, we obtain by (2.6) the generalized inequality of the Hardy type:

$$(3.11) \quad \left\| \varphi_X(t)^{-1} \int_0^t f(s) d\varphi_X(s) \right\|_{Y'} \leq \int_0^1 h(s, Y) dM(s, X) \|f\|_{Y'}.$$

(b): Let $0 < \lambda < 1$ and $\lambda < r(P_X)^{-1}$. Then $P_{X(\lambda)} \in [Y]$ and, by Lemma (3.2),

$$h(u, Y) \int_0^u M(s, X)^{-\lambda} ds^{p_\infty(X)+\varepsilon} \leq \|P_{X(\lambda)}\|_{[Y]},$$

i.e.

$$h(u, Y) \leq \text{const } u^{-(p_\infty(X)+\varepsilon-\lambda p_0(X)+\varepsilon\lambda)} \quad \text{for } u \in (0, \delta).$$

Taking $\varepsilon < \lambda p_0(X)/(1+\lambda)$, we obtain

$$\begin{aligned} A_{p_\infty(X)} &\leq \text{const} \int_0^\delta s^{p_\infty(X)-1} s^{-(p_\infty(X)-\lambda p_0(X)+\varepsilon+\varepsilon\lambda)} ds + \\ &\quad + \text{const} \int_\delta^1 s^{p_\infty(X)-2} ds < \infty. \end{aligned}$$

According to Lemma (3.3), Theorem (3.4), and fact that $A_{p_\infty(X)} < \infty$ implies $a(Y) < p_\infty(X)$, we obtain

COROLLARY (3.5). If $a(Y) < p_0(X)$, then $Y \in \text{HP}_X$; if $Y \in \text{HP}_X$, then $a(Y) < p_\infty(X)$.

Hence the following follows directly:

COROLLARY (3.6). Let $p_0(X) = p_\infty(X) > 0$. $Y \in \text{HP}_X$ if and only if $a(Y) < p_0(X)$.

In particular, if $\varphi_X(t) = t^\theta$, $0 < \theta \leq 1$, then $Y \in \text{HP}_X$ if and only if $a(Y) < \theta$ (Theorem B).

EXAMPLE (3.7). We consider Orlicz space $L \log^+ L$ on $(0, 1)$ generated by the Orlicz function $\Phi(u) = u \log^+ u$ with Orlicz norm. Then

$$\varphi_{L \log^+ L}(t) = t(1 - \log t), \quad p_0(L \log^+ L) = 1,$$

$$(P_{L \log^+ L} f)(t) = [t(1 - \log t)]^{-1} \int_0^t f(s) \log 1/s ds,$$

and

$$C_0(L \log^+ L, Y) = \int_0^1 h(s, Y) \log 1/s ds.$$

By Corollary (3.6), $Y \in \text{HP}_{L \log^+ L}$ if and only if $a(Y) < 1$.

CONJECTURE (3.8). There exist r.i. spaces X_i, Y_i such that $0 < p_0(X_i) \leq a(Y_i) < p_\infty(X_i)$ ($i = 1, 2$) and $Y_1 \notin \text{HP}_{X_1}$, and $Y_2 \in \text{HP}_{X_2}$.

§ 4. Interpolation of Lipschitz operators of weak type (X, X) and type (L^∞, L^∞) . Let (X_1, X_2) be a quasi-Banach couple, i.e., there is a Hausdorff topological vector space \mathfrak{X} and continuous embeddings $X_1 \subset \mathfrak{X}, X_2 \subset \mathfrak{X}$. The K -functional of Peetre is defined for $0 < t < \infty$ on $X_1 + X_2$ (the hull of X_1 and X_2 in \mathfrak{X}) by (cf. [4])

$$(4.1) \quad K(t, f) = K(t, f; X_1, X_2) = \inf \{ \|f_1\|_{X_1} + t \|f_2\|_{X_2}; f_1 \in X_1, f_2 \in X_2, f = f_1 + f_2 \}.$$

We assume throughout this section as well as the rest of the paper that $X \in \mathcal{L}$ has a strictly increasing concave fundamental function φ_X ; this is no restriction if $l = \infty$.

Since $X \in \mathcal{L}$, we have $\lim_{t \rightarrow 0^+} \varphi_X(t) = 0$, and therefore

$$(4.2) \quad K(t, f; \Lambda(X), L^\infty) = \int_0^{\min(\varphi_X^{-1}(t), l)} f^*(s) d\varphi_X(s) \quad (\text{see [14]}).$$

For $\alpha > 0$, $f^{(\alpha)}$ will denote the α -truncation of f , that is,

$$(4.3) \quad f^{(\alpha)}(x) = \min(|f(x)|, \alpha) \operatorname{sgn} f(x).$$

LEMMA (4.1). We have

$$\frac{1}{2} \sup_{s \in S_t} \varphi_X(s) f^*(s) \leq K(t, f; M^*(X), L^\infty) \leq 2 \sup_{s \in S_t} \varphi_X(s) f^*(s),$$

where

$$S_t = \{s \in I: s \leq \min(\varphi_X^{-1}(t), l)\}.$$

Proof. If $l < \infty$, then $\|f\|_{M^*(X)} \leq \varphi_X(l) \|f\|_{L^\infty}$ for $f \in L^\infty$, so $\frac{1}{2} \|f\|_{M^*(X)} \leq K(t, f) \leq \|f\|_{M^*(X)}$ for $t \geq \varphi_X(l)$. Let $\varphi_X^{-1}(t) < l$ and $\alpha = f^*(\varphi_X^{-1}(t))$, and let $f_1 = f - f^{(\alpha)}$, $f_2 = f^{(\alpha)}$. Since $(f - f^{(\alpha)})^* = f^* - (f^*)^{(\alpha)}$, we have

$$\begin{aligned} K(t, f) &\leq \|f_1\|_{M^*(X)} + t \|f_2\|_{L^\infty} \\ &\leq \sup_{s \in I} \varphi_X(s) [f^*(s) - (f^*)^{(\alpha)}(s)] + t f^*(\varphi_X^{-1}(t)) \\ &= \sup_{0 < s \leq \varphi_X^{-1}(t)} \varphi_X(s) [f^*(s) - f^*(\varphi_X^{-1}(t))] + t f^*(\varphi_X^{-1}(t)) \\ &\leq 2 \sup_{0 < s \leq \varphi_X^{-1}(t)} \varphi_X(s) f^*(s). \end{aligned}$$

On the other hand, let $\varepsilon \in (0, 1)$ be an arbitrary fixed number, and let $f = f_1 + f_2$ be an arbitrary decomposition of $f \in M^*(X) + L^\infty$. Then

$$\begin{aligned} \|f_1\|_{M^*(X)} + t \|f_2\|_{L^\infty} &\geq \sup_{0 < s \leq (1-\varepsilon)\varphi_X^{-1}(t)} \varphi_X(s) f_1^*(s) + t \sup_{0 < s \leq \varepsilon\varphi_X^{-1}(t)} f_2^*(s) \\ &= \sup_{0 < v \leq \varphi_X^{-1}(t)} \varphi_X((1-\varepsilon)v) f_1^*((1-\varepsilon)v) + \sup_{0 < v \leq \varphi_X^{-1}(t)} \varphi_X(v) f_2^*(\varepsilon v) \\ &\geq (1-\varepsilon) \sup_{0 < v \leq \varphi_X^{-1}(t)} \varphi_X(v) f_1^*((1-\varepsilon)v) \\ &\quad + \sup_{0 < v \leq \varphi_X^{-1}(t)} \varphi_X(v) f_2^*(\varepsilon v) \geq (1-\varepsilon) \sup_{0 < v \leq \varphi_X^{-1}(t)} \varphi_X(v) f^*(v). \end{aligned}$$

Since the decomposition and $\varepsilon > 0$ are arbitrary, we obtain the left-hand inequality.

LEMMA (4.2). If $p_\infty(Y) < p_0(X)$, then Y is continuously embedded in $\Lambda(X) + L^\infty$.

Proof. Since

$$f^*(s) \leq s^{-1} \int_0^s f^*(t) dt \leq \varphi_X(s)^{-1} \|f\|_{M(X)} \quad \text{for all } s \in I,$$

we have

$$\|f\|_{\Lambda(X) + L^\infty} = K(1, f; \Lambda(X), L^\infty) = \int_0^{\min(\varphi_X^{-1}(1), l)} f^*(s) d\varphi_X(s)$$

$$\leq \int_0^{\min(\varphi_X^{-1}(1), l)} \varphi_X(s)^{-1} d\varphi_X(s) \|f\|_{M(X)}.$$

By relations (2.11), (2.13), and (2.14) we obtain

$$\begin{aligned} \int_0^{\min(\varphi_X^{-1}(1), l)} \varphi_X(s)^{-1} d\varphi_X(s) &\leq \int_0^{\min(\varphi_X^{-1}(1), l)} \varphi_X(s)^{-1} \varphi_X(s) ds/s \\ &\leq \operatorname{const} \left(\int_0^{\delta} s^{p_0(X)-\varepsilon-(p_\infty(X)+\varepsilon)} ds/s + \int_0^{\min(\varphi_X^{-1}(1), l)} \varphi_X(s)^{-1} \varphi_X(s) ds/s \right) < \infty. \end{aligned}$$

By (2.12), we have $Y \subset \Lambda(X) + L^\infty$.

We denote $K_0(t, f; M^*(X), L^\infty) = \sup_{s \in S_t} \varphi_X(s) f^*(s)$.

A rearrangement invariant space Y , $Y \subset \Lambda(X) + L^\infty$, is called K_0 -monotone with regard to couples $(\Lambda(X), L^\infty)$, $(M^*(X), L^\infty)$, if $f \in Y$, $g \in M^*(X) + L^\infty$, $K_0(t, g; M^*(X), L^\infty) \leq K(t, f; \Lambda(X), L^\infty)$ for $t > 0$, imply $g \in Y$ and $\|g\|_Y \leq C \|f\|_Y$ with some $C = C(Y) > 0$.

LEMMA (4.3). Y is K_0 -monotone with regard to couples $(\Lambda(X), L^\infty)$, $(M^*(X), L^\infty)$ if and only if $Y \in \text{HP}_X$.

Proof. The necessity follows from the fact that

$$\begin{aligned} K_0(t, P_X f; M^*(X), L^\infty) &\leq \sup_{s \in S_t} \varphi_X(s) (P_X f)^*(s) \\ &= \int_0^{\min(\varphi_X^{-1}(t), l)} f^*(s) d\varphi_X(s) = K(t, f; \Lambda(X), L^\infty). \end{aligned}$$

Concerning the sufficiency, for $t > 0$ such that $\varphi_X^{-1}(t) < l$ we have

$$\begin{aligned} t g^*(\varphi_X^{-1}(t)) &\leq \sup_{0 < s \leq \varphi_X^{-1}(t)} \varphi_X(s) g^*(s) \\ &= K_0(t, g; M^*(X), L^\infty) \leq K(t, f; \Lambda(X), L^\infty). \end{aligned}$$

Hence

$$g^*(u) \leq \varphi_X(u)^{-1} \int_0^u f^*(s) d\varphi_X(s) = (P_X f^*)(u).$$

LEMMA (4.4). If $T \in \omega\text{-Lip}(X; 1) \cap \text{Lip}(L^\infty; 1)$, then $K_0(t, Tf; M^*(X), L^\infty) \leq K(t, f; A(X), L^\infty)$ for $t > 0$ and $f \in A(X)$.

Proof. For each $t > 0$ we put $\alpha = f^*(\min(\varphi_X^{-1}(t), l))$. We have

$$\begin{aligned} K_0(t, Tf; M^*(X), L^\infty) &\leq \sup_{s \in S_t} \varphi_X(s) \{ (Tf)^*(s) - [(Tf)^*]^{(\alpha)}(s) \} + \sup_{s \in S_t} \varphi_X(s) [(Tf)^*]^{(\alpha)}(s) \\ &\leq \|Tf - (Tf)^{(\alpha)}\|_{M^*(X)} + t'\alpha, \quad \text{where } t' = \varphi_X(\min(\varphi_X^{-1}(t), l)). \end{aligned}$$

Since $T \in \text{Lip}(L^\infty; 1)$, we get $\|T(f^{(\alpha)})\|_{L^\infty} \leq \|f^{(\alpha)}\|_{L^\infty} \leq \alpha$, which implies

$$(4.4) \quad |Tf - (Tf)^{(\alpha)}| \leq |Tf - T(f^{(\alpha)})|.$$

Therefore we obtain

$$\begin{aligned} K_0(t, Tf; M^*(X), L^\infty) &\leq \|Tf - T(f^{(\alpha)})\|_{M^*(X)} + t'\alpha \\ &\leq \|f - f^{(\alpha)}\|_{A(X)} + t'\alpha = \int_0^{\min(\varphi_X^{-1}(t), l)} (f^*(s) - \alpha) d\varphi_X(s) + t'\alpha \\ &= \int_0^{\min(\varphi_X^{-1}(t), l)} f^*(s) d\varphi_X(s) = K(t, f; A(X), L^\infty). \end{aligned}$$

This lemma can be proved also as in [15] by means of the fact that

$$(4.5) \quad \|f\|_{M^*(X)} = \sup_{y>0} y \varphi_X^{-1}(d_f(y)).$$

In [17], p. 215, one can find the proof of (4.5) in the case $X = L^p$; the proof for any $X \in \mathcal{L}$ is similar.

THEOREM (4.5). Let $l < \infty$. Y has the interpolation property for the class $\omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$ if and only if $Y \in \text{HP}_X$.

Proof. Sufficiency. Suppose that $Y \in \text{HP}_X$ and let $T \in \omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$. We fix $h \in A(X) \cap L^\infty$, and define the operator S by

$$(4.6) \quad Sf = \frac{T(f+h) - Th}{\max(K_X, K_\infty)}, \quad f \in A(X) \cup L^\infty.$$

Then $S \in \omega\text{-Lip}(X; 1) \cap \text{Lip}(L^\infty; 1)$. Hence, by Lemma (4.4), $K_0(t, Sf; M^*(X), L^\infty) \leq K(t, f; A(X), L^\infty)$, and by Lemma (4.3), $\|Sf\|_Y \leq \|P_X\|_{[Y]} \|f\|_Y$. This means that

$$\|Tf - Th\|_Y = \max(K_X, K_\infty) \|S(f-h)\|_Y \leq \max(K_X, K_\infty) \|P_X\|_{[Y]} \|f-h\|_Y$$

for $f \in A(X) \cup L^\infty$, $h \in A(X) \cap L^\infty$. For arbitrary $f, g \in Y$ we consider the truncations $f^{(n)}, g^{(n)}$. Then $T(f^{(n)})$ and $T(g^{(n)})$ converge in the $M^*(X)$ -quasi-norm to Tf and Tg , respectively, since $T \in \omega\text{-Lip}(X; K_X)$ and $Y \subset A(X)$. Therefore, for a properly chosen sequence (n_i) , $T(f^{(n_i)})$ and

$T(g^{(n_i)})$, converge almost everywhere to Tf and Tg . Since $f^{(n)} \in L^\infty$ and $|f^{(n)} - g^{(n)}| \leq |f - g|$, we obtain

$$\|T(f^{(n_i)}) - T(g^{(n_i)})\|_Y \leq \max(K_X, K_\infty) \|P_X\|_{[Y]} \|f - g\|_Y.$$

Hence, by virtue of property (2.2),

$$\|Tf - Tg\|_Y \leq \max(K_X, K_\infty) \|P_X\|_{[Y]} \|f - g\|_Y, \quad f, g \in Y.$$

Necessity. This follows from the fact that

$$(4.7) \quad P_X \in \omega\text{-Lip}(X; 1) \cap \text{Lip}(L^\infty; 1).$$

Remark. Since (4.7) is also true for $l = \infty$, the necessary condition of the Theorem 4.5 includes the case $l = \infty$.

Theorem (4.5) and Corollary (3.5) imply

THEOREM (4.6). (a) Let $l < \infty$. If $\alpha(Y) < p_0(X)$, then Y has the interpolation property for the class $\omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$ and

$$(4.8) \quad \|T\|_{[Y]} \leq \max(K_X, K_\infty) \|P_X\|_{[Y]}.$$

(b) If Y has the interpolation property for the class $\omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$, then $\alpha(Y) < p_\infty(X)$.

COROLLARY (4.7). Let $l < \infty$ and let $p_\infty(X) = p_0(X) > 0$. Y has the interpolation property for the class $\omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$ if and only if $\alpha(Y) < p_0(X)$.

If the norm of Y is continuous, that is $0 \leq f_n \downarrow 0$ implies $\|f_n\| \downarrow 0$, we can extend Theorems (4.5) and (4.6) to spaces on $(0, \infty)$. In fact, we have

THEOREM (4.8). Let $l = \infty$ and let the norm on Y be continuous. Then the statements of sufficiency of Theorems (4.5) and (4.6) remain valid if T is replaced by the unique extension of T onto Y .

The proof is derived from the facts that T is also a Lipschitz operator from $Z = Y \cap A(X) \cap L^\infty$ (with the norm of Y) into Y , and Z is dense in Y .

In particular, in case $X = L^1$ by Corollary (4.7) and Theorem (4.8) we have Theorem A.

§ 5. Majorant for interpolation theorem. By Theorem (4.6) we see that each operator T of the class

$$O = \omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$$

is a Lipschitz operator on Y with the bound not exceeding $\max(K_X, K_\infty) \|P_X\|_{[Y]}$ if $\alpha(Y) < p_0(X)$. In [15] it is shown that if $T \in \omega\text{-Lip}(L^1; K_1) \cap \text{Lip}(L^\infty; K_\infty)$ and $\alpha(Y) < 1$, then $\|T\|_{\text{Lip}[Y]} \leq K_\infty h(\alpha, Y) \|P_{L^1}\|_{[Y]}$, where $\alpha = K_\infty K_1^{-1}$. For the class $\omega\text{-Lip}(X; K_X) \cap \text{Lip}(L^\infty; K_\infty)$ we have

THEOREM (5.1). (a) If $\alpha(Y) < p_0(X)$, then

$$(5.1) \quad \|T\|_{\text{Lip}[Y]} \leq K_\infty h(a, Y) \|P_X\|_{[Y]}, \quad \text{where } a = n(K_\infty K_X^{-1}, X)$$

and

$$n(s, X) = \inf_{t \in I_\varphi, t \leq I_\varphi} \frac{\varphi_X^{-1}(ts)}{\varphi_X^{-1}(t)}, \quad I_\varphi = (0, \varphi(l)).$$

Proof. Since $\varphi_X(n(s, X)u) \leq s\varphi_X(u)$, for any $T \in C$ the operator $T' = K_\infty^{-1}E_{a-1}T$ belongs to $\omega\text{-Lip}(X; 1) \cap \text{Lip}(L^\infty; 1)$ if $0 < a < 1$, and the operator $T'' = K_\infty^{-1}TE_{a-1}$ belongs to $\omega\text{-Lip}(X; 1) \cap \text{Lip}(L^\infty; 1)$ if $a \geq 1$. Now, 5.1 is proved in the same manner as in [8].

For example, if $\varphi_X(t) = t^\theta$, $0 < \theta \leq 1$ and $\alpha(Y) < \theta$, then

$$(5.2) \quad \|T\|_{\text{Lip}[Y]} \leq K_\infty h(a, Y) \|P_X\|_{[Y]}, \quad \text{where } a = (K_\infty \cdot K_X^{-1})^{1/\theta}$$

and

$$(5.3) \quad \sup \{\|T\|_{\text{Lip}[Y]} : T \in C\} = K_\infty h(a, Y) \|P_X\|_{[Y]}.$$

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Added in proof. After this paper was submitted for publication, the author answered Conjecture (3.8) in the affirmative, see *Indices and interpolation*, preprint.

References

- [1] D. W. Boyd, *The Hilbert transform on rearrangement-invariant spaces*, Canad. J. Math. 19 (1967), 599–616.
- [2] — *The spectral radius of averaging operators*, Pacific J. Math. 24 (1968), 19–28.
- [3] — *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. 21 (1969), 1245–1254.
- [4] P. L. Butzer and H. Berens, *Semi-groups of operators and approximation*, Grundlehren Math. Wiss. 145, Berlin 1967.
- [5] P. L. Butzer and F. Fehér, *Generalized Hardy and Hardy–Littlewood inequalities in rearrangement invariant spaces*, Comment. Math. Prace Mat. Tomus Specialis in Honorem Ladislai Orlicz I (1978), 41–64.
- [6] M. A. Krasnosel'skii and Ya. B. Rutickii, *Convex functions and Orlicz spaces*, Noordhoff, Groningen 1961.
- [7] G. G. Lorentz, *Bernstein polynomials*, University of Toronto Press, Toronto 1953.
- [8] G. G. Lorentz and T. Shimogaki, *Majorants for interpolation theorems*, Publ. Ramanujan Inst. 1 (1969), 115–122.
- [9] W. A. J. Luxemburg, *Banach function spaces*, Thesis, Van Goroum, Assen 1955.

- [10] L. Maligranda, *Interpolation of Lipschitz operators for the pairs of spaces (L^p, L^∞) and (l^p, c_0)* , Comment. Math. Prace Mat. 21 (1979), 327–339.
- [11] — *Interpolation of Lipschitz operators for the pairs of spaces (L^p, L^∞) and (l^p, c_0) , $0 < p < \infty$* , Funct. Approximatio 9 (1980), 107–115.
- [12] W. Matuszewska and W. Orlicz, *On certain properties of φ -functions*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8 (1960), 439–443.
- [13] W. Orlicz, *On a class of operations over the space of integrable functions*, Studia Math. 14 (1954), 302–309.
- [14] R. C. Sharpley, *Spaces $A_\alpha(X)$ and interpolation*, J. Functional Analysis 11 (1972), 479–513.
- [15] T. Shimogaki, *An interpolation theorem on Banach function spaces*, Studia Math. 31 (1968), 233–240.
- [16] — *Indices of function spaces*, preprint.
- [17] E. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces* (Russian), Mir, Moscow 1974.
- [18] M. Zippin, *Interpolation of operators of weak type between rearrangement invariant function spaces*, J. Functional Analysis 7 (1971), 267–284.

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