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Nuclear Fréchet spaces without the bounded approximation property

by

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Abstract. We construct such spaces, thereby solving a problem of Grothendieck. The main point is to construct a nuclear Fréchet space which admits a continuous norm but is not countably normed in the sense of Gelfand and Šilov. In addition, we show that a nuclear Fréchet space with a basis and a continuous norm can have a quotient space which fails the bounded approximation property.

Since the appearance of Grothendieck's memoir [9], many authors have studied variations of approximation properties on linear topological spaces. The two most important classes of spaces which have been considered are Banach spaces and nuclear Fréchet spaces. For Banach spaces the definitions and basic properties of the most interesting variants are found in [10] (see [17] for an additional variation). An analogous discussion for nuclear Fréchet spaces can be found in [5], Ch. VI.

The most important result in the Banach space case is the counterexample of Enflo [7] who constructed a Banach space which fails the approximation property. Since this is the weakest property that has been considered, one thus has a counterexample for all of the approximation properties.

For nuclear Fréchet spaces the situation is different. It is easy to see [20], p. 110 that every nuclear Fréchet space has the approximation property. At the other extreme, the existence of a Schauder basis is the strongest approximation property which has been studied and here there is, again, a counterexample due to Mitiagin and Zobin [15].

The approximation property is not very interesting for nuclear Fréchet spaces, and in this case, Grothendieck emphasizes a stronger property—the *bounded approximation property* (BAP). A nuclear Fréchet space E is said to have this property provided there is a sequence of finite rank operators on E which converge pointwise to the identity. The main purpose of this paper is to construct a nuclear Fréchet space which does not have BAP.

It turns out that a property closely connected to BAP in nuclear Fréchet spaces is that of countably normed space (CN). This was introduced by Gelfand and Šilov [8]. Roughly speaking (see below for complete definitions), this means that the space is an intersection of Banach spaces and its topology is determined by the norms of these spaces. Most spaces which occur naturally are of this type. For example, the space $\mathscr{E}[-1,1]$ of real valued functions, all of whose derivatives exist and are continuous on $[-1,1]$, is (along with its topology) the interesection of the spaces $\mathscr{E}^p[-1,1]$ $(p = 0, 1, \ldots)$ of real valued functions, all of whose derivatives up to order p exist and are continuous on $[-1,1]$.

It is easy to see that a countably normed space is a Fréchet space, but it is on the question of the converse that our entire construction will be based.

Inasmuch as it turns out that an approximation property is usually not satisfied by all spaces in a particular class, the question of permanence becomes important. That is, if a space has a certain approximation property, what can be said of its subspaces and its quotient spaces? For all of the properties except BAP, it is known in the case of nuclear Fréchet spaces that the results are always negative in a very strong sense. The known facts are summarized in [1]. For BAP, we show in this paper that there exist nuclear Fréchet spaces with bases, indeed power series spaces, which have quotient spaces that do not have BAP. But the only thing we know about subspaces is that the approach in this paper cannot be helpful.

One interesting aspect of our result for quotient spaces is that the proof makes use of the theorem of Vogt and Wagner [21] characterizing quotient spaces of the nuclear Fréchet space (s).

An interesting question about which not much is known is whether all of the nuclear Fréchet spaces which occur naturally as function spaces have BAP. For example, $C^\infty(\Omega)$ or $\mathscr{H}(D)$ where Ω is a C^∞-manifold and D an analytic manifold.

The proof of the main theorem given here is not the shortest possible. In fact, we want to make our construction using the most general possible parameters. The reason for this is to make it easier to use our construction in other contexts—for example, quotients of given spaces. A streamlined version of the proof appears in [5].

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Definitions and terminology. We denote by $\boldsymbol{N}$ the set of positive integers and by δ_{ij} the Kronecker delta. If an index is not explicitly restricted, it is assumed to run through $\boldsymbol{N}$.

We denote by l_2 the separable Hilbert space of square summable sequences and by $\|\cdot\|$ the norm and by $(\cdot,\cdot)$ the inner product on l_2. If $A\colon l_2 \to l_2$ is a linear operator, we say that A is *Hilbert–Schmidt* if for some (equivalently for all) orthonormal basis (e_n) we have $\sum_{n,k} |(Ae_n, e_k)|^2 < \infty$.

A Fréchet space is a complete metrizable locally convex space. Thus, its topology is defined by an increasing sequence of seminorms $(\|\cdot\|_k)_k$ in the sense that a sequence (x_n) converges to 0 iff $\lim_n \|x_n\|_k = 0$ for each $k \in \boldsymbol{N}$. We call $(\|\cdot\|_k)$ a *fundamental sequence of seminorms* for the space. If a Fréchet space E has a fundamental sequence of norms (that is, each $\|\cdot\|_k$ is a norm), then we say that E *admits a continuous norm*. The reason for this terminology is that, as is easy to see, this is equivalent to the existence of a continuous norm on E.

If E is a Fréchet space, $(\|\cdot\|_k)$ is a fundamental sequence of norms, and $k < j$, then since $\|x\|_k \leqslant \|x\|_j$ $(x \in E)$, the identity map $(E, \|\cdot\|_j) \to (E, \|\cdot\|_k)$ of normed spaces is continuous so it has a unique extension to the completion, $(E, \|\cdot\|_j)^\Delta \to (E, \|\cdot\|_k)^\Delta$. We call these extensions the *canonical maps* corresponding to $(\|\cdot\|_k)$.

We say that a Fréchet space E with continuous norm is *nuclear* if it has a fundamental sequence of norms $(\|\cdot\|_k)$ such that each $(E, \|\cdot\|_k)^\Delta$ is isomorphic to l_2 and each canonical map $(E, \|\cdot\|_{k+1})^\Delta \to (E, \|\cdot\|_k)^\Delta$ is Hilbert–Schmidt $(k \in \boldsymbol{N})$.

In this paper, the most important example of a nuclear Fréchet space is the space (s) given by

$$(s) = \Big\{\xi = (\xi_n)\colon \sum_n |\xi_n| n^k < \infty, \quad k \in \boldsymbol{N}\Big\}$$

with fundamental sequence of norms given by $(\|\cdot\|_k)$ where

$$\|\xi\|_k = \sum_n |\xi_n| n^k.$$

This space is isomorphic to the space $\mathscr{E}[-1,1]$ mentioned above.

A *basis* in a Fréchet space E is a sequence (x_n) in E such that for each $x \in E$ there is a unique sequence of scalars (t_n) such that $x = \sum_n t_n x_n$.

For the elementary facts about bases and nuclear Fréchet spaces, we refer to [13] and [19]. In particular, we will mention power series spaces briefly.

A *finite dimensional decomposition* in a Fréchet space E is a sequence $B_n\colon E \to E$ $(n \in \boldsymbol{N})$ of continuous linear operators such that each $B_n(E)$ is finite dimensional, $B_n B_k = \delta_{nk}$ $(n, k \in \boldsymbol{N})$ and $x = \sum_n B_n x$ $(x \in E)$.

By subspace we shall mean closed, infinite dimensional subspace. In particular, a quotient space will always be Hausdorff.

Finally, we define the property that, next to BAP, is central to this paper. Let E be a Fréchet space which admits a continuous norm. We say that E is *countably normed* if it has a fundamental system of norms $(\|\cdot\|_k)$ such that if (x_n) is a sequence in E which is Cauchy with respect to both $\|\cdot\|_k$ and $\|\cdot\|_j$, then $\lim_n \|x_n\|_k = 0$ iff $\lim_n \|x_n\|_j = 0$. It is obvious that this requirement is equivalent to the condition that each canonical map $(E, \|\cdot\|_j)^{\Lambda} \to (E, \|\cdot\|_k)^{\Lambda}$ $(k < j)$ is injective.

Connections between BAP **and** CN. In this section we show that, in the presence of a continuous norm, CN is a necessary condition for BAP. Thus, our problem is changed to that of constructing a nuclear Fréchet space with a continuous norm that is not CN, and this is precisely what will be done.

Our first result is due to Pełczyński and Wojtaszczyk [18]. We include the proof here for completeness and also to point out that the space they construct does indeed admit a continuous norm.

PROPOSITION 1. *If E is a Fréchet space which admits a continuous norm and has* BAP, *then there exists a Fréchet space F which admits a continuous norm, has a finite dimensional decomposition and contains a subspace isomorphic to E.*

Proof. Let (A_n) be a sequence of finite rank operators on E which converge pointwise to the identity and let $(\|\cdot\|_k)$ be a sequence of norms which defines the topology of E. Write $B_1 = A_1$, $B_{n+1} = A_{n+1} - A_n$ $(n \in \boldsymbol{N})$ so that each $B_n(E)$ is finite dimensional and $x = \sum_n B_n x$ $(x \in E)$.

We define a new sequence of norms $(|\cdot|_k)$ on E by

$$|x|_k = \sup_n \Big\| \sum_{i=1}^{n} B_i x \Big\|_k \quad (x \in E,\ k \in \boldsymbol{N}).$$

Clearly, $\|x\|_k \leqslant |x|_k$ and the unit ball of $|\cdot|_k$ is a barrel in E so it is a neighborhood of 0. Hence, $(|\cdot|_k)$ also defines the topology of E.

Now we define

$$F = \Big\{ (y_n) \colon\ y_n \in B_n(E),\ n \in \boldsymbol{N} \text{ and } \sum_n y_n \text{ converges in } E \Big\}$$

with topology given by the sequence of norms $(|\cdot|_k)$ where

$$|(y_n)|_k = \sup_n \Big\| \sum_{i=1}^{n} y_i \Big\|_k .$$

It is easy to check that F is a Fréchet space, each $|\cdot|_k$ is a continuous norm and the coordinate projections onto the subspaces $B_n(E)$ form a finite dimensional decomposition. Finally, the mapping which sends x to $(B_n x)_n$ is clearly an isomorphism of E into F. ∎

Actually, more is true. Pełczyński and Wojtaszczyk show that in fact E is complemented in F which we do not need. Also, Pełczyński has shown ([16] and also [12]) that such an F can be chosen to have a basis. Using this result would make the proof of our next proposition a little simpler but then the proof of Proposition 1 would be considerably more complicated.

We come now to the main result in this section. It was originally suggested by A. Pełczyński.

PROPOSITION 2. *If a Fréchet space E admits a continuous norm and has* BAP, *then it is* CN.

Proof. Let F be the space of Proposition 1. It is immediate from the first definition of CN that this property is preserved by subspaces. Thus it suffices to show that F is CN.

Let (A_n) be a finite dimensional decomposition for F and $(\|\cdot\|_k)$ a sequence of norms which defines the topology of F. As in Proposition 1, the topology of F is also defined by the sequence of norms $(|\cdot|_k)$ where

$$|y|_k = \sup_n \Big\| \sum_{i=1}^{n} A_i y \Big\|_k \quad (y \in F,\ k \in \boldsymbol{N}).$$

It then follows that (A_n) gives a finite dimensional decomposition for each of the Banach spaces $F_k = (F, |\cdot|_k)^{\Lambda}$, $(k \in \boldsymbol{N})$. Indeed, using standard ideas from the theory of Banach spaces, it suffices to show that $(A_n)_n$ is a uniformly bounded sequence of operators on the normed space $(F, |\cdot|_k)$. In fact, since $A_i A_j = \delta_{ij} A_i$ we have,

$$|A_n y|_k = \sup_m \Big\| \sum_{i=1}^{m} A_i A_n y \Big\|_k = \|A_n y\|_k \leqslant |y|_k .$$

Now, since $A_n(F)$ is finite dimensional, $A_n(F_k) = A_n(F) \subset F$. Thus if $J_k \colon F_{k+1} \to F_k$ is the canonical map and $x = \sum_n A_n x$ in F_{k+1}, then $J_k x = \sum_n A_n x$ in F_k because J_k is the identity on F. It then follows from the uniqueness of the expansion that if $J_k x = 0$, then $A_n x = 0$ for all $n \in \boldsymbol{N}$ so $x = 0$. Thus, J_k is 1-1 and F is CN. ∎

This result will be the basis for our construction. We remark in passing that our method cannot help to find a subspace E of a space F such that F has a basis and a continuous norm and E does not have BAP. The reason is that our method would be to show that E is not CN. However, as above, F is CN and so its subspace E is also. It remains an open question whether any nuclear Fréchet space with a basis and a continuous norm has a subspace which does not have BAP.

Weakly injective sequences of functions. As we saw in the last section, our problem has been reduced to finding a nuclear Fréchet space with continuous norm that is not CN. Because of the form of the definition of CN, it would be very difficult to do this directly. That is, the definition of CN only requires that it be possible to select one fundamental sequence of norms for which the canonical maps are injective. It is easy to see, however, that the canonical maps may be injective for one choice and fail to be injective for another choice of the norms. Thus, to show directly that a space is *not* CN, it would be necessary to prove that the canonical maps fail to be injective for every choice of norms. We know of no way to do this directly.

An alternative is to find some property which is independent of the choice of the norms and which is a consequence of CN. Then we can try to find a space which fails this property. We will do exactly this with the notion of weakly injective.

Let $A_k\colon S_{k+1}\to S_k\,(k\in \boldsymbol{N})$ be a sequence of functions. We say that $(A_k)_{k\in\boldsymbol{N}}$ is *weakly injective* if the following condition holds:

$$\forall x\in \bigcap_k A_1\ldots A_k(S_k)\exists (x_k)_{k\in\boldsymbol{N}}\ni x_1 = x \text{ and } x_k = A_k x_{k+1}\ (k\in\boldsymbol{N}).$$

The following diagram may motivate this definition:

$$\begin{tikzcd}
\cdots \arrow[r] & S_{k+1} \arrow[r, "A_k"] & S_k \arrow[r] & \cdots \arrow[r] & S_2 \arrow[r, "A_1"] & S_1 \\
& y_{k+1} \arrow[rrrr, bend right] & y_k \arrow[rrr, bend right] & & y_2 \arrow[r] & y_1 = x
\end{tikzcd}$$

The hypothesis in the definition is that for each k we can begin in S_1 with $y_1 = x$ and "go back" to S_{k+1} and find $y_{k+1}\in S_{k+1}$ such that $y_1 = A_1\ldots A_k y_{k+1}$. The conclusion is that we can actually do this step-by-step, that is, $y_k = A_k y_{k+1}$. Of course, this is immediate if each A_k is injective, which explains the name we have chosen.

Here is another way to think about weak injectivity. The projective limit of the sequence of maps (A_k) is the set $S = \{(x_k)_{k\in\boldsymbol{N}}\colon x_k = A_k x_{k+1};\ k\in\boldsymbol{N}\}$. On the other hand, it is often useful to be able to represent the projective limit more concretely as the set $T = \bigcap_k A_1\ldots A_k(S_k)$. For example, the space $\mathscr{E}[-1,1]$ mentioned in the Introduction is the intersection of the spaces $\mathscr{E}^p[-1,1]$. Here, the maps A_k are the inclusions. In general, it is obvious that $S\subset T$ and weak injectivity says exactly that $T\subset S$.

The main result in this section is that, essentially, in a CN space, the canonical maps are weakly injective.

PROPOSITION 3. *Let E be CN and $(\|\cdot\|_k)$ any fundamental sequence of norms with canonical maps $A_k\colon (E,\|\cdot\|_{k+1})^\Lambda\to (E,\|\cdot\|_k)^\Lambda$ $(k\in\boldsymbol{N})$. Then there exists k_0 such that $(A_k)_{k\geqslant k_0}$ is weakly injective.*

Proof. Let $(|\cdot|_k)$ be a fundamental sequence of norms for E such that each canonical map $B_k\colon (E,|\cdot|_{k+1})^\Lambda\to (E,|\cdot|_k)^\Lambda$ is injective. For convenience, we will write $E_k = (E,\|\cdot\|_k)^\Lambda$ and $F_k = (E,|\cdot|_k)^\Lambda$.

Since $(\|\cdot\|_k)$ and $(|\cdot|_k)$ both define the topology of E, we have increasing sequences of indices (j_k), (l_k) and positive constants (α_k), (β_k) such that

$$|x|_{l_k}\leqslant \alpha_k\|x\|_{j_k}\leqslant \beta_k|x|_{l_{k+1}}\qquad (x\in E,\ k\in\boldsymbol{N}).$$

We will show that $(A_k)_{k\geqslant j_1}$ is weakly injective.

Write $C_k = A_{j_k}\ldots A_{j_{k+1}-1}$, $D_k = B_{l_k}\ldots B_{l_{k+1}-1}$ $(k\in\boldsymbol{N})$ so that each D_k is 1-1. Now, the above inequalities relate norms on E so the corresponding-identity maps have continuous extensions. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
& {} \arrow[r] & F_{l_{k+2}} \arrow[rr, "D_{k+1}"] \arrow[dr, "Q_{k+1}"] & & F_{l_{k+1}} \arrow[rr, "D_k"] \arrow[dr, "Q_k"] & & F_{l_k} \arrow[r] & \cdots \arrow[rr] & & F_{l_2} \arrow[rr, "D_1"] \arrow[dr, "Q_1"] & & F_{l_1} \\
{} \arrow[r] & E_{j_{k+2}} \arrow[ur, "P_{k+2}"] \arrow[rr, "C_{k+1}"] & & E_{j_{k+1}} \arrow[ur, "P_{k+1}"] \arrow[rr, "C_k"] & & E_{j_k} \arrow[ur, "P_k"] \arrow[rr] & & \cdots \arrow[r] & E_{j_2} \arrow[ur, "P_2"] \arrow[rr, "C_1"] & & E_{j_1} \arrow[ur, "P_1"] & \\
& y_{k+2} \arrow[rrrrrrrrr, bend right] & & y_{k+1} \arrow[rrrrrrr, bend right] & & y_k \arrow[rrrrr, bend right] & & & y_2 \arrow[rr] & & y_1 = x &
\end{tikzcd}$$

First, we show that (C_k) is weakly injective. Thus, we assume that $x = C_1\ldots C_k y_{k+1}$ $(k\in\boldsymbol{N})$. We define $x_k = C_k y_{k+1}$ so $x_1 = C_1 y_2 = x$. Moreover,

$$\begin{aligned} D_1\ldots D_k P_{k+1}C_{k+1}y_{k+2} &= P_1 C_1\ldots C_{k+1}y_{k+2} = P_1 x \\ &= P_1 C_1\ldots C_k y_{k+1} = D_1\ldots D_k P_{k+1}y_{k+1},\end{aligned}$$

and, since each D_k is 1-1, $P_{k+1}C_{k+1}y_{k+2} = P_{k+1}y_{k+1}$, so we have

$$C_k x_{k+1} = C_k C_{k+1}y_{k+2} = Q_k P_{k+1}C_{k+1}y_{k+2} = Q_k P_{k+1}y_{k+1} = C_k y_{k+1} = x_k.$$

Hence, (C_k) is weakly injective.

Finally, if $x\in\bigcap_{k\geqslant j_1} A_{j_1}\ldots A_k(E_{k+1})$, then $x\in\bigcap_k C_1\ldots C_k(E_{j_{k+1}})$, so we have (y_k) with $y_1 = x$ and $y_k = C_k y_{k+1}$. Define (x_k) by setting $x_{j_k} = y_k$ $(k\in\boldsymbol{N})$ and $x_l = A_l\ldots A_{j_{k+1}-1}y_{k+1}$ for $j_k < l < j_{k+1}$. This shows that $(A_k)_{k\geqslant j_1}$ is weakly injective. ■

We do not know if the converse of Proposition 3 is true. That is, if E is a Fréchet space with a fundamental sequence of norms for which the canonical maps form a weakly injective sequence, does it follow that E is CN? [Added in proof: The author was recently shown that this converse does not hold.]

The construction. Our space E will be a projective limit of a sequence of operators A_k on l_2. First, we develop conditions on A_k which guarantee that E has all the desired properties, then we construct the infinite matrices which determine the A_k and finally we show that these A_k satisfy the conditions.

Given a sequence $A_k\colon l_2 \to l_2$ $(k \in \boldsymbol{N})$ of continuous linear operators, the projective limit E with a fundamental sequence of semi-norms $(\|\cdot\|_k)$ is given by

$$E = \{(x_k)\colon x_k = A_k x_{k+1} \ (k \in \boldsymbol{N})\}, \qquad \|(x_j)\|_k = \|x_k\|.$$

It is elementary to check that E is a Fréchet space.

PROPOSITION 4. *In the above context, assume that*

(i) $(\ker A_k) \cap \bigcap_{j>k} A_{k+1} \ldots A_j(l_2) = \{0\}$ $(k \in \boldsymbol{N})$;

(ii) *Each A_k has dense range;*

(iii) *Each A_k is Hilbert–Schmidt;*

(iv) *For each $k \in \boldsymbol{N}$ there exists $x^{k+1} \in l_2$ such that*

$$(x^{k+1} + \ker A_k) \cap A_{k+1} \ldots A_j(l_2) \neq \emptyset \qquad (j > k)$$

but

$$(x^{k+1} + \ker A_k) \cap \bigcap_{j>k} A_{k+1} \ldots A_j(l_2) = \emptyset.$$

Then E is a nuclear Fréchet space with a continuous norm that is not CN *so E does not have* BAP.

Proof. Before giving the details, it may be useful to explain the separate roles which each of these conditions play. From (i) it follows immediately that E admits a continuous norm. We obtain from (ii) that the canonical maps for E are, essentially, the A_k $(k \in \boldsymbol{N})$ and thus it follows from (iii) that E is nuclear. Finally, the first part of (iv) says that, for each k one can begin with $A_k x^{k+1}$ and "go back" any finite number of steps, while the second part says that one cannot go "all of the way back". This will imply that for any k, $(A_j)_{j\geqslant k}$ is not weakly injective.

So we begin by showing that each $\|\cdot\|_k$ is a norm. Let $x = (x_k) \in E$ and suppose $\|x\|_k = 0$. Then $\|x_k\| = 0$, so $x_k = 0$. But $x_{k+1} = A_{k+1} \ldots A_j x_{j+1}$ $(j > k)$ and $A_k x_{k+1} = x_k = 0$, so by (i), $x_{k+1} = 0$. Similarly, $x_j = 0$ for all $j > k$ and clearly $x_j = 0$ for all $j < k$. Hence, $x = 0$.

Now we define $\pi_k\colon (E, \|\cdot\|_k) \to l_2$ by $\pi_k(x) \doteq x_k$. It is clear that π_k is 1-1 and norm preserving. We want to show that (ii) implies that π_k

has dense range. This argument was originally due to W. Wojtyński (unpublished). Let $\varepsilon > 0$ and $y \in l_2$. We define the quantities in the following layout:

$$\begin{array}{ccccccccc}
\ldots \to & l_2 & \xrightarrow{A_{k+2}} & l_2 & \xrightarrow{A_{k+1}} & l_2 & \xrightarrow{A_k} & l_2 \\
 & & & & & & & y \\
 & & & & & x_1^2 & \to & x_1^1 \\
 & & & x_2^3 & \longrightarrow & x_2^2 & \to & x_2^1 \\
\ldots \to & x_3^4 & \longrightarrow & x_3^3 & \longrightarrow & x_3^2 & \to & x_3^1 \\
 & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

That is, we define $x_n^j \in l_2$ $(j = 1, \ldots, n+1$ and $n \in \boldsymbol{N})$ in such a way that $x_n^j = A_{k+j-1} x_n^{j+1}$ and, because of (ii), $\|x_n^j - x_{n+1}^j\| \leqslant 2^{-n}\varepsilon$, and $\|x_1^1 - y\| \leqslant \varepsilon$. It follows that $u^{k+j-1} = \lim_n x_n^j$ exists and $\|u^k - y\| \leqslant 2\varepsilon$.

Hence, if we define $u^l = A_l u^{l+1}$ $(l < k)$, it follows that $u = (u^l) \in E$ and $\|\pi_k u - y\| \leqslant 2\varepsilon$.

Thus, π_k has dense range so it has a unique extension to an isometry $\pi_k\colon (E, \|\cdot\|_k)^\Lambda \to l_2$. It then follows that the canonical map $(E, \|\cdot\|_{k+1})^\Lambda \to (E, \|\cdot\|_k)^\Lambda$ is equal to $\pi_k^{-1} A_k \pi_{k+1}$. In particular, it then follows from (iii) that E is nuclear.

Moreover, the weak injectivity of $(\pi_j^{-1} A_j \pi_{j+1})_{j\geqslant k}$ would imply the weak injectivity of $(A_j)_{j\geqslant k}$ so it suffices to show that for each k, $(A_j)_{j\geqslant k}$ is not weakly injective.

Given k we have x^{k+1} from (iv) so for $j > k$ we have y_j and z_{j+1} such that

$$A_k x^{k+1} = A_k y_j \quad \text{and} \quad y_j = A_{k+1} \ldots A_j z_{j+1}.$$

Hence, $A_k x^{k+1} \in \bigcap_{j\geqslant k} A_k \ldots A_j(l_2)$ so if $(A_j)_{j\geqslant k}$ were weakly injective we would have $(v_j)_{j\geqslant k}$ with $v_k = A_k x^{k+1}$ and $v_j = A_j v_{j+1}$ $(j \geqslant k)$. But then

$$A_k v_{k+1} = v_k = A_k x^{k+1}$$

so $v_{k+1} \in x^{k+1} + \ker A_k$ and, for $j > k$,

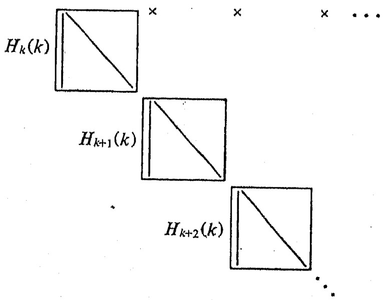
$$v_{k+1} = A_{k+1} \ldots A_j v_{j+1}$$

so $v_{k+1} \in \bigcap_{j>k} A_{k+1} \ldots A_j(l_2)$ which contradicts the second statement in (iv). Our result then follows from Propositions 2, 3. ∎

Now, we construct the infinite matrices which determine the operators $A_k\colon l_2 \to l_2$ $(k \in \boldsymbol{N})$. We begin by decomposing l_2 into infinitely many pairwise orthogonal infinite dimensional subspaces $H(\nu)$ $(\nu \in \boldsymbol{N})$. We call $H(\nu)$ the ν^{th} *block*. Each $H(\nu)$ is similarly decomposed into *sub-blocks* $H_m(\nu)$ $(m \in \boldsymbol{N})$. For each m, ν we fix an orthonormal basis $(e_{mn}(\nu))_{n\in\boldsymbol{N}}$ for $H_m(\nu)$. Thus, $\{e_{mn\nu}\colon m, n, \nu \in \boldsymbol{N}\}$ is an orthonormal basis for l_2.

First, we give a "geometrical" description of the matrix of A_k with respect to the basis. Each block $H(\nu)$ is invariant under A_k. For each

$\nu \neq k$, A_k is a diagonal operator on $H(\nu)$. Now we consider the action of A_k on $H(k)$. Each sub-block $H_m(k)$ $(m = 1, \ldots, k-1)$ is invariant under A_k and A_k is a diagonal operator on it. The subspace generated by the sub-blocks $H_m(k)$ $(m \geqslant k)$ is invariant and on it the matrix of A_k has the following form:



(All positions where nothing is indicated have 0). All of the sub-blocks $H_m(k)$ $(m > k)$ have the same form. Of course, it must be remembered that this picture is misleading in that each sub-block is, in fact, infinite.

With this pattern in mind, we now give the precise value of $A_k e_{mn}(\nu)$. Let $\{\lambda_{mn}^k(\nu)\colon m, n, k, \nu \in \boldsymbol{N}\}$ be a set of positive real numbers such that

$$\sum_{m,n,\nu} \big(\lambda_{mn}^k(\nu)\big)^2 < \infty \qquad (k \in \boldsymbol{N}).$$

We define part of each A_k $(k \in \boldsymbol{N})$ as follows:

If

$$\left.\begin{array}{l} \nu \neq k \\ \text{or } \nu = k > m \\ \text{or } \nu = k = m \text{ and } n > 1 \end{array}\right\}, \quad \text{set} \quad A_k e_{mn}(\nu) = \lambda_{mn}^k(\nu) e_{mn}(\nu)$$

and if

$$\nu = k < m \text{ and } n > 1, \quad \text{set} \quad A_k e_{mn}(\nu) = \lambda_{mn}^k(\nu) e_{m,n-1}(\nu).$$

Extending A_k by linearity and continuity, we have each A_k defined on the subspace generated by the blocks $H(\nu)$ $(\nu \neq k)$, the sub-blocks $H_m(k)$ $(m < k)$, and a subspace of codimension 1 of each sub-block $H_m(k)$ $(m \geqslant k)$. Moreover, on each sub-block $H(\nu)$ $(\nu \neq k)$ the matrix of A_k is diagonal with a non-vanishing diagonal sequence $(\lambda_{mn}^k(\nu))_\nu$ and $\lim\limits_n \lambda_{mn}^k(\nu) = 0$.

For convenience of notation, we will write $A_{k+1} \ldots A_m$ when $m \geqslant k$ and consider this to be the identity when $m = k$.

Fix $m \geqslant k$. If we consider the operators $A_{k+1} \ldots A_m$ and $A_{k+1} \ldots A_{m+1}$ restricted to the subspace H of l_2 generated by $e_{mn}(k)$ $(n > 1)$, then these two operators (as far as they have been defined) have the property that H and its orthogonal complement are invariant under both of them, and on H they are diagonal. Moreover, if $(\alpha_n), (\beta_n)$ are the respective diagonal sequences, then $\alpha_n > 0, \beta_n > 0$ $(n > 1)$ and $\lim\limits_n (\beta_n/\alpha_n) = 0$. Hence, there exists $w_m^k \in H$ such that w_m^k is in the range of $A_{k+1} \ldots A_m$ but not in the range of $A_{k+1} \ldots A_{m+1}$. We write

$$w_m^k = \sum_{n=2}^{\infty} s_{mn}^k e_{mn}(k).$$

It follows that $A_k w_m^k$ has been defined so we may write,

$$s_{m1}^k = \frac{1 + \|A_k w_m^k\|}{\lambda_{m1}^k(k)} \quad \text{and} \quad u_m^{k+1} = s_{m1}^k e_{m1}(k) + w_m^k,$$

and set

$$A_k\big(e_{m1}(k)\big) = \frac{1}{s_{m1}^k}\big(e_{k1}(k) - A_k w_m^k\big).$$

Notice that if this is done for all $k \in \boldsymbol{N}$, none of the properties of $A_{k+1} \ldots A_m$, $A_{k+1} \ldots A_{m+1}$ used above are affected.

This completes the definition of the operator A_k $(k \in \boldsymbol{N})$, and we summarize by writing, for any family of scalars $\{v_{mn}(\nu)\colon m, n, \nu \in \boldsymbol{N}\}$ such that $\sum\limits_{m,n,\nu} \big(v_{mn}(\nu)\big)^2 < \infty$, the formula

$$\text{(1)} \quad A_k\Big(\sum_{m,n,\nu} v_{mn}(\nu) e_{mn}(\nu)\Big) = \sum_{\substack{\nu \neq k \\ \text{or } \nu = k > m}} \lambda_{mn}^k(\nu) v_{mn}(\nu) e_{mn}(\nu) + $$
$$+ \Big(\sum_{m \geqslant k} \frac{1}{s_{m1}^k} v_{m1}(k)\Big) e_{k1}(k) + \sum_{n \geqslant 2} \lambda_{kn}^k(k) \Big(v_{kn}(k) - \frac{s_{kn}^k}{s_{k1}^k} v_1(k)\Big) e_{kn}(k) +$$
$$+ \sum_{m > k} \sum_{n \geqslant 1} \lambda_{m,n+1}^k(k) \Big(v_{m,n+1}(k) - \frac{s_{m,n+1}^k}{s_{m1}^k} v_{m1}(k)\Big) e_{mn}(k).$$

PROPOSITION 5. *In the above context, we have*

(i) $\|A_k e_{mn}(\nu)\| \leqslant \lambda_{mn}^k(\nu)$ *for all* $m, n, \nu \in \boldsymbol{N}$;

(ii) $A_k u_m^{k+1} = e_{k1}(k)$ *for all* $m \geqslant k$;

(iii) *For each* $m \geqslant k$, $u_m^{k+1} \in A_{k+1} \ldots A_m(l_2)$ *but* $u_m^{k+1} \notin A_{k+1} \ldots A_{m+1}(l_2)$.

Proof. From the construction we trivially have (i) for all cases except $\nu = k \leqslant m$ and $n = 1$. For this case we have

$$\|A_k e_{m1}(k)\| \leqslant \frac{1 + \|A_k w_m^k\|}{s_{m1}^k} = \lambda_{m1}^k(k).$$

To verify (ii), we compute,

$$A_k u_m^{k+1} = s_{m1}^k A_k e_{m1}(k) + A_k w_m^k = e_{k1}(k).$$

Finally, we observe that from the construction it follows that $e_{m1}(k)$ and w_m^k are both in $A_{k+1}\dots A_m(l_2)$ and so u_m^{k+1} is also. On the other hand, $e_{m1}(k)$ is in $A_{k+1}\dots A_{m+1}(l_2)$ but w_m^k is not, so u_m^{k+1} is not. Thus we have (iii). ■

PROPOSITION 6. *In the above context, we have*

$$\ker A_k = \Big\{v = \sum_{m=k}^{\infty} \xi_m u_m^{k+1}\colon \sum_{m=k}^{\infty} (|\xi_m|\,\|u_m^{k+1}\|)^2 < \infty \text{ and } \sum_{m=k}^{\infty} \xi_m = 0\Big\}.$$

Proof. Suppose v is in the given set. Now the sequence $(u_m^{k+1})_{m\geqslant k}$ is orthogonal and A_k is continuous by Proposition 5 (i) and the fact that $\sum_{m,n,\nu}(\lambda_{mn}^k(\nu))^2 < \infty$. Hence, by Proposition 5 (ii) we have

$$A_k v = \sum_{m=k}^{\infty} \xi_m A_k(u_m^{k+1}) = \Big(\sum_{m=k}^{\infty} \xi_m\Big) e_{k1}(k) = 0.$$

On the other hand, if $v \in \ker A_k$, we first expand v as follows:

$$v = \sum_{\nu\neq k}\sum_{m,n} \eta_{mn}(\nu)\, e_{mn}(\nu) + \sum_{m=1}^{k-1}\sum_{n} \eta_{mn} e_{mn}(k) + \sum_{m\geqslant k}\Big(\xi_m u_m^{k+1} + \sum_{n\geqslant 2} \zeta_{mn} e_{mn}(k)\Big).$$

That is, on the blocks $H(\nu)$ $(\nu\neq k)$ and the sub-blocks $H_m(k)$ $(m<k)$ we have used the orthonormal basis, $e_{mn}(\nu)$ while on each sub-block $H_m(k)$ $(m\geqslant k)$, since the component of u_m^{k+1} corresponding to $e_{m1}(k)$ is $s_{m1}^k \neq 0$, we can replace $e_{m1}(k)$ in the orthonormal basis $\big(e_{mn}(k)\big)_n$ by u_m^{k+1} and we still have a basis.

Thus if we apply A_k to v, we obtain from Proposition 5 (ii)

$$0 = A_k v = \sum_{\nu\neq k}\sum_{m,n} \eta_{mn}(\nu)\,\lambda_{mn}^k(\nu)\, e_{mn}(\nu) + \sum_{m=1}^{k-1}\sum_{n} \eta_{mn}\lambda_{mn}^k(k)\, e_{mn}(k) +$$

$$+ \xi_k e_{k1}(k) + \sum_{n\geqslant 2} \zeta_{kn}\lambda_{kn}^k(k)\, e_{kn}(k) + \sum_{m>k} \xi_m e_{k1}(k) + \sum_{n\geqslant 2} \zeta_{mn}\lambda_{mn}^k(k)\, e_{m,n-1}(k).$$

But this is now an expansion in the orthonormal basis $\big(e_{mn}(\nu)\big)_{m,n,\nu}$ so the coefficient of each $e_{mn}(\nu)$ is 0. Hence, all η, ζ terms vanish and $\sum_{m\geqslant k}\xi_m = 0$. It therefore follows that $v = \sum_{m\geqslant k}\xi_m u_m^{k+1}$ and by the orthogonality,

$$\sum_{m\geqslant k}\|\xi_m u_m^{k+1}\|^2 < \infty. \ ■$$

All of the ingredients are now present and we are ready to prove the main result of this paper.

THEOREM. *The space E constructed above is a nuclear Fréchet space with a continuous norm but E is not CN. Hence, E does not have* BAP.

Proof. We obtain our result by using Propositions 5, 6 to show that the operators constructed satisfy the four conditions of Proposition 4.

Condition (iii) of Proposition 4 follows from Proposition 5 (i) and the fact that $\sum_{m,n,\nu}(\lambda_{mn}^k(\nu))^2 < \infty$.

To show that A_k has dense range it suffices to show that every $e_{mn}(\nu)$ $(m,n,\nu\in N)$ is in the range of A_k. For $\nu\neq k$ or $\nu = k > m$, this is immediate from (1). Now in (1) set $v_{mn}(\nu) = 0$ if $\nu\neq k$ or $\nu = k > m$ or both $\nu = k\leqslant m$ and $n = 1$. Inspection of the resulting relation shows immediately that $e_{mn}(k)$ is in the range of A_k if $k < m$ or both $k = m$ and $n\geqslant 2$. Finally, $e_{k1}(k) = A_k u_k^{k+1}$ by Proposition 5 (ii).

We turn now to condition (i) of Proposition 4. Let $v\in\ker A_k$ so by Proposition 6, $v = \sum_{m=k}^{\infty}\xi_m u_m^{k+1}$. If $\xi_m = 0$ for all $m > k$, then since $\sum_{m=k}^{\infty}\xi_m = 0$, it would follow that $v = 0$. Let $m > k$ with $\xi_m\neq 0$. Then if $v\in A_{k+1}\dots A_{m+1}(l_2)$, it would follow from the diagonality of this operator on each $H_m(k)$ and the fact that the orthogonal complement of $H(k)$ is invariant under this operator that u_m^{k+1} is in the range of $A_{k+1}\dots A_{m+1}(l_2)$. By Proposition 5 (iii) this is not the case.

Condition (iv) of Proposition 4 is obtained similarly. We set $x^{k+1} = u_k^{k+1}$ $(k\in N)$. Let $v\in\ker A_k$ so again $v = \sum_{m=k}^{\infty}\xi_m u_m^{k+1}$ and if $\xi_m = 0$ for all $m > k$, it would follow that $v = 0$ and by Proposition 5 (iii), x^{k+1} is not in the range of A_{k+1}. Hence, if $x^{k+1} + v\in\bigcap_{j>k} A_{k+1}\dots A_j(l_2)$, we have $m > k$ with $\xi_m\neq 0$. Exactly as in the previous paragraph it follows that if $x^{k+1}+v$ were in the range of $A_{k+1}\dots A_{m+1}$, then so would be u_m^{k+1} which it is not. Thus, the second part of Proposition 4 (iv) holds. To verify the first part we use Proposition 5 (iii) to conclude that $x^{k+1} + u_m^{k+1} - x^{k+1}$ is in $A_{k+1}\dots A_m(l_2)$ $(m > k)$ and by Proposition 5 (ii), $u_m^{k+1} - x^{k+1}\in\ker A_k$. ■

It should be noted that this construction corrects an error in [3], Theorem 1, where it is asserted, essentially, that such a space cannot exist. The error in the proof occurs on page 151, line 10 from the bottom where it is erroneously claimed (and subsequently used) that each f_k is an isomorphism.

A further observation is that the nuclearity does not play a primary role in the theorem. It was easy to construct E to be nuclear and this, perhaps, is the most important case. But it was unnecessary and our result is really about Fréchet spaces.

Quotient spaces without BAP. After the first construction of a nuclear Fréchet space without basis by Mitiagin and Zobin [15] and especially with the simplified version of Djakov and Mitiagin [2] it was not difficult to show that this pathology is ubiquitous — indeed, every nuclear Fréchet (except ω) has both a subspace and a quotient space which has no basis [4], [6].

Analogous results for BAP seem to be more difficult. Of course, by the embedding theorem of Kōmura and Kōmura [11], the space $(s)^N$ has a subspace without BAP. But we do not know a single example of a nuclear Fréchet space with a basis and a continuous norm that has a subspace without BAP.

For quotients we know a few such spaces which have a quotient without BAP but we do not have a general result. In this section we show that (s) has a quotient without BAP and then we indicate how this could be done for a few other infinite type power series spaces.

Our problem is to show that one of the spaces we constructed is isomorphic to a quotient of (s) and for this we use a result of Vogt and Wagner [21] who showed that a Fréchet space E is isomorphic to a quotient of (s) iff E satisfied the following condition, called (Ω):

There is a fundamental sequence of norms $(\|\cdot\|_p)_{p\in N} \ni \forall p \exists q \geqslant p \ni \forall r \geqslant q \exists C, D > 0 \ni$

$$(\|u\|_q^*)^{D+1} \leqslant C\|u\|_r^*(\|u\|_p^*)^D \qquad (u \in E').$$

Here $\|\cdot\|^*$ refers to the dual of $\|\cdot\|$ given by $\|u\|^* = \sup\{|u(x)|: \|x\| \leqslant 1\}$.

LEMMA 1. *If we set*

$$\lambda_{mn}^k(\nu) = \lambda_{mn}(\nu) = \frac{1}{2(mn\nu)^2} \qquad (m, n, \nu \in N),$$

$$s_{mn}^k = (\lambda_{mn}(k))^{m-k+1} \qquad (n \geqslant 2, m \geqslant k),$$

then all of the requirements of the above construction hold and, moreover, we have

(i) $\lambda_{mn}(\nu) \leqslant \lambda_{m1}(\nu)$ $(m, n, \nu \in N)$;

(ii) $\lambda_{mn}(\nu) \leqslant 4\lambda_{m,n+1}(\nu)$ $(m, n, \nu \in N)$;

(iii) $\sum_{n=2}^{\infty} s_{mn}^k \lambda_{mn}(k) \leqslant 1/2$ $(k \leqslant m)$.

Proof. It is obvious that $\sum_{m,n,\nu} (\lambda_{mn}^k(\nu))^2 < \infty$ and it is only necessary to show that w_m^k is in $A_{k+1} \ldots A_m(l_2)$ but not in $A_{k+1} \ldots A_{m+1}(l_2)$ $(m \geqslant k)$ where $w_m^k = \sum_{n\geqslant 2} s_{mn}^k e_{mn}(k)$. Now $\sum_{n\geqslant 2} \lambda_{mn}(k) e_{mn}(k) \in l_2$ and

$$A_{k+1} \ldots A_m\Big(\sum_{n\geqslant 2} \lambda_{mn}(k) e_{mn}(k)\Big) = \sum_{n\geqslant 2} (\lambda_{mn}(k))^{m-k+1} e_{mn}(k) = w_m^k.$$

On the other hand, it suffices to show that if $\sum_{n\geqslant 2} t_n e_{mn}(k) \in l_2$, then applying $A_{k+1} \ldots A_{m+1}$ to this vector cannot yield w_m^k. Indeed, this application yields $\sum_{n\geqslant 2} t_n (\lambda_{mn}(k))^{m-k+1} e_{mn}(k)$ which gives w_m^k iff $t_n = 1$ $(n \geqslant 2)$ and this is impossible since $\|e_{mn}(k)\| = 1$.

Now we turn to the three conditions. Conditions (i), (ii) are immediate from our definition and for condition (iii) we have, for $k \leqslant m$ since $\lambda_{mn}(k) \leqslant 1$,

$$\sum_{n=2}^{\infty} s_{mn}^k \lambda_{mn}(k) = \sum_{n=2}^{\infty} (\lambda_{mn} k))^{m-k+2} \leqslant \sum_{n=2}^{\infty} (\lambda_{mn}(k))^2 \leqslant \frac{1}{4} \sum_{n=2}^{\infty} \frac{1}{n^2} \leqslant \frac{1}{2}. \blacksquare$$

For the remainder of the paper we adopt the values assigned in the above Lemma. In particular, we write $\lambda_{mn}(\nu)$ instead of $\lambda_{mn}^k(\nu)$. Of course, this does not mean that A_k is independent of k.

Our next step is to write the formula for A_k^*, the adjoint of A_k. If $\{w_{mn}(\nu): m, n, \nu \in N\}$ is a square summable sequence, then we have

$$(2)\qquad A_k^*\Big(\sum_{m,n,\nu} w_{mn}(\nu) e_{mn}(\nu)\Big) = \sum_{\substack{\nu \neq k \\ \text{or } \nu = k > m \\ \text{or } \nu = k = m, n > 1}} \lambda_{mn}(\nu) w_{mn}(\nu) e_{mn}(\nu) + $$
$$+ \Big(\frac{1}{s_{k1}^k} w_{k1}(k) - \sum_{n\geqslant 2} \frac{s_{kn}^k}{s_{k1}^k} \lambda_{kn}(k) w_{kn}(k)\Big) e_{k1}(k) +$$
$$+ \sum_{m>k} \Big(\frac{1}{s_{m1}^k} w_{k1}(k) - \sum_{n} \frac{s_{m,n+1}^k}{s_{m1}^k} \lambda_{m,n+1}(k) w_{mn}(k)\Big) e_{m1}(k) +$$
$$+ \sum_{m>k} \sum_{n\geqslant 2} \lambda_{mn}(k) w_{m,n-1}(k) e_{mn}(k).$$

The best way to verify this formula is to use it along with formula (1) for A_k to check that $(A_k v, w) = (v, A_k^* w)$ for each $v, w \in l_2$. The details are lengthy but completely straightforward so we omit them.

LEMMA 2. *In the above context we have*

$$\sum_{m,n,\nu} \|A_k^* e_{mn}(\nu)\| < \infty.$$

Proof. For simplicity of computation we omit the term $\|A_k^* e_{k1}(k)\|$. For the others, we have immediately from (2) that

if $\nu \neq k$ or $\nu = k > m$, then

$$A_k^* e_{mn}(\nu) = \lambda_{mn}(\nu) e_{mn}(\nu),$$

if $\nu = k = m$ and $n \geqslant 2$, then

$$A_k^* e_{mn}(\nu) = -\frac{s_{kn}^k}{s_{k1}^k} \lambda_{kn}(k) e_{k1}(k) + \lambda_{kn}(k) e_{kn}(k),$$

if $\nu = k < m$ and $n \geqslant 1$, then

$$A_k^* e_{mn}(\nu) = -\frac{s^k_{m,n+1}}{s^k_{m1}} \lambda_{m,n+1}(k)\, e_{m1}(k) + \lambda_{m,n+1}(k)\, e_{m,n+1}(k).$$

Hence,

$$\sum_{m,n,\nu} \|A_k^* e_{mn}(\nu)\| \leqslant \|A_k^* e_{k1}(k)\| + \sum_{m,n,\nu} \lambda_{mn}(\nu) + \sum_{\substack{m\geqslant k\\ n\geqslant 1}} \frac{s^k_{mn}}{s^k_{m1}} \lambda_{mn}(k)$$

and since $1/s^k_{m1} \leqslant \lambda^k_{m1}(k) < 1$ and $s_{mn} \leqslant \lambda_{mn}(k)$, this quantity is finite. ■

In our context and with the values assigned it is now possible to relax condition (Ω) a little.

LEMMA 3. *If $v \in l_2$, let $|v| = \sup_{m,n,\nu} |(v, e_{mn}(\nu))|$. Assume that the operators (A_k) satisfy the condition $(\tilde{\Omega})$:*

$$\forall p < r \exists C, D > 0 \ni |A_p^* v|^{D+1} \leqslant C|A_r^* \dots A_p^* v| \qquad (|v| \leqslant 1).$$

Then the space in our construction satisfies condition (Ω).

Proof. Our first step is to interpret the inequality in (Ω) in terms of the operators A_k^*. Returning to the proof of Proposition 4 we have isometries π_k: $(E, \|\cdot\|_k)^\Delta \to l_2$ and, writing $E_k = (E, \|\cdot\|_k)^\Delta$ we have, for $k < j$ the commuting diagram

$$\begin{tikzcd} l_2 \arrow[r, "A_k \dots A_{j-1}"] & l_2 \\ E_j \arrow[u, "\pi_j"] \arrow[r, "I_{kj}"'] & E_k \arrow[u, "\pi_k"'] \end{tikzcd}$$

where I_{kj} is the canonical map. Dualizing the diagram we obtain,

$$\begin{tikzcd} l_2 \arrow[r, "A^*_{j-1} \dots A^*_k"] \arrow[d, "\pi^*_k"'] & l_2 \arrow[d, "\pi^*_j"] \\ E'_k \arrow[r, "I^*_{kj}"'] & E'_j \end{tikzcd}$$

where π_k^*, π_j^* are still isometries and I^*_{kj} is the injection which is usually interpreted by identification to be the inclusion. Without this interpretation, the inequality in (Ω) really means:

$$(\|I^*_{pq}u\|^*_q)^{D+1} \leqslant C\|I^*_{pr}u\|^*_r (\|u\|^*_p)^D \qquad (u \in E'_p).$$

Now if $k < j$, we have for $v \in l_2$,

$$\|I^*_{kj}\pi^*_k v\|^*_j = \|\pi^*_j A^*_{j-1} \dots A^*_k v\|^*_j = \|A^*_{j-1} \dots A^*_k v\|$$

and since each π_k^* is an isometry onto, the inequality in (Ω) is equivalent to:

$$(\|A^*_{q-1} \dots A^*_p v\|)^{D+1} \leqslant C\|A^*_{r-1} \dots A^*_p v\| (\|v\|)^D \qquad (v \in l_2).$$

Now suppose that $(\tilde{\Omega})$ holds. Given p take $q = p+2$ so if $r \geqslant q$, then $r-1 > p$ and we have C, D such if $v \in l_2$, we may compute, using Lemma 2,

$$\begin{aligned}(\|A^*_{q-1} \dots A^*_p v\|)^{D+1} &= (\|A^*_{p+1} A^*_p v\|)^{D+1} \\ &\leqslant \Big(\sum_{m,n,\nu} \big|(A^*_p v, e_{mn}(\nu))\big|\, \big\|A^*_{p+1}(e_{mn}(\nu))\big\|\Big)^{D+1} \\ &\leqslant C_1 |A^*_p v|^{D+1} = C_1 \left|A^*_p \frac{v}{|v|}\right|^{D+1} |v|^{D+1} \\ &\leqslant C_2 \left|A^*_{r-1} \dots A^*_p \frac{v}{|v|}\right| |v|^{D+1} \\ &= C_2 |A^*_{r-1} \dots A^*_p v|\, |v|^D \\ &\leqslant C_2 \|A^*_{r-1} \dots A^*_p v\|\, \|v\|^D. \ ■\end{aligned}$$

We are ready now for the final computations. They are not completely straightforward and so we give them in detail.

PROPOSITION 7. *The space (s) has a quotient space which does not have* BAP.

Proof. In view of Lemma 3 and the result of Vogt and Wagner, it suffices to show that the operators A_k^* (constructed with the values given in Lemma 1) satisfy $(\tilde{\Omega})$. Given $p < r$, we take $C = 4^{r-p}$, $D = r-p$. Now condition $(\tilde{\Omega})$ has the form,

$$\sup_{m,n,\nu} |(A^*_p w, e_{mn}(\nu))|^{r-p+1} \leqslant 4^{r-p} \sup_{m,n,\nu} \big|(A^*_r \dots A^*_p w, e_{mn}(\nu))\big|. \tag{3}$$

We will establish this by showing that for each m, n, ν the quantity $|(A^*_p w, e_{mn}(\nu))|^{r-p+1}$ is dominated by the sup on the right-hand side. All of this need be done only for $|w_{mn}(\nu)| \leqslant 1$.

We will consider three cases corresponding to the value of ν.

Case I. $\nu < p$ or $\nu > r$. In this case all operators $A^*_p, \dots, A^*_r$ are diagonal and we have

$$|\lambda_{mn}(\nu) w_{mn}(\nu)|^{r-p+1} \leqslant C\big|(\lambda_{mn}(\nu))^{r-p+1} w_{mn}(\nu)\big|,$$

because $|w_{mn}(\nu)| \leqslant 1$.

Case II. $\nu = p$. The formula for A^*_p is obtained from (2) by replacing k, ν with p. The operator $A^*_r \dots A^*_p$ is A^*_p followed by a diagonal operator. Thus, its formula is obtained from (2) by replacing k, ν with p and multiplying each $e_{mn}(p)$ by $(\lambda_{mn}(p))^{r-p}$. We consider separately four cases corresponding to where the max occurs in computing $|A^*_p w| = \sup_{m,n,\nu} |(A^*_p w, e_{mn\nu})|$.

Max *occurs at* $(m < p)$ *or* $(m = p$ *and* $n > 1)$. This leads to exactly the situation on Case I.

Max *occurs at* $m = p$ *and* $n = 1$. We will show

$$\left|\frac{1}{s_{p1}^{p}} w_{p1}(p) - \sum_{n\geqslant 2} \frac{s_{pn}^{p}}{s_{p1}^{p}} \lambda_{pn}(p) w_{pn}\right|^{r-p+1} \leqslant C\big(\lambda_{p1}(p)\big)^{r-p}\left|\frac{1}{s_{p1}^{p}} w_{p1}(p) - \sum_{n\geqslant 2}\frac{s_{pn}^{p}}{s_{p1}^{p}}\lambda_{pn}(p) w_{pn}(p)\right|.$$

After cancelling this reduces to showing

$$\frac{1}{s_{p1}^{p}}\Big(1+\sum_{n\geqslant 2} s_{pn}^{p}\lambda_{pn}(p)\Big) \leqslant 4\lambda_{p1}(p).$$

By definition of s_{p1}^{p} and Lemma 1 (iii) the left-hand side is dominated by $\frac{3}{2}\lambda_{p1}(p)$ so the inequality holds.

Max *occurs at* $m > p$ *and* $n = 1$. We will show

$$\left|\frac{1}{s_{m1}^{p}} w_{p1}(p) - \sum_{n} \frac{s_{m,n+1}^{p}}{s_{m1}^{p}} \lambda_{m,n+1}(p) w_{mn}(p)\right|^{r-p+1} \leqslant C(\lambda_{m1})^{r-p}\left|\frac{1}{s_{m1}^{p}} w_{m1}(p) - \sum_{n}\frac{s_{m,n+1}^{p}}{s_{m1}^{p}}\lambda_{m,n+1}(p) w_{pn}(p)\right|.$$

The argument is identical to the previous case.

Max *occurs at* $m > p$ *and* $n \geqslant 2$. Again, we have the "diagonal" situation and the argument is identical to Case I.

Case III. $p < \nu \leqslant r$. The operator A_p^* is diagonal. The operator $A_r^* \dots A_p^*$ is the operator A_ν^* preceded by the diagonal operator $A_{\nu-1}^* \dots$ $\dots A_p^*$ and followed by the diagonal operator $A_r^* \dots A_{\nu+1}^*$. Thus, the formula for $A_r^* \dots A_p^*$ is obtained from (2) by first replacing k by ν, second multiplying each $w_{mn}(\nu)$ by $\big(\lambda_{mn}(\nu)\big)^{\nu-p}$ and finally multiplying each $e_{mn}(\nu)$ by $\big(\lambda_{mn}(\nu)\big)^{r-\nu}$. We obtain the formula for the action of $A_r^* \dots A_p^*$ on the block $H(\nu)$ as follows:

$$\begin{aligned} A_r^* \dots A_p^*(w) = &\sum_{\substack{\nu>m \\ \text{or } \nu=m,n>1}} \big(\lambda_{mn}(\nu)\big)^{r-p+1} w_{mn}(\nu) e_{mn}(\nu) + \\ &+ \big(\lambda_{\nu1}(\nu)\big)^{r-\nu}\left(\frac{1}{s_{\nu1}^{\nu}}\big(\lambda_{\nu1}(\nu)\big)^{\nu-p} w_{\nu1}(\nu) - \sum_{n\geqslant 2}\frac{s_{\nu m}^{\nu}}{s_{\nu1}^{\nu}}\big(\lambda_{\nu n}(\nu)\big)^{\nu-p+1} w_{\nu n}(\nu)\right) e_{\nu1}(\nu) + \\ &+ \sum_{m>\nu}\Bigg(\big(\lambda_{m1}(\nu)\big)^{r-\nu}\Bigg(\frac{1}{s_{m1}^{\nu}}\big(\lambda_{\nu1}(\nu)\big)^{\nu-p} w_{\nu1}(\nu) - \\ &- \sum_{n}\frac{s_{m,n+1}^{\nu}}{s_{m1}^{\nu}}\big(\lambda_{mn}(\nu)\big)^{\nu-p}\lambda_{m,n+1}(\nu) w_{mn}(\nu)\Bigg) e_{m1}(\nu) + \\ &+ \sum_{m>\nu}\sum_{n\geqslant 2}\big(\lambda_{m,n-1}(\nu)\big)^{\nu-p}\big(\lambda_{mn}(\nu)\big)^{r-\nu+1} w_{m,n-1}(\nu) e_{mn}(\nu). \end{aligned}$$

Again, we consider separately four cases corresponding to the index at which the max occurs in computing $|A_p^* w|$. But this time we cannot always compare with the corresponding term in $|A_r^* \dots A_p^* w|$. Sometimes a different index will have to be chosen. This is okay because the right-hand side of (3) is a sup. We will indicate in each case the choice of index for $|A_r^* \dots A_p^* w|$.

Max *occurs at* $(m < \nu)$ *or* $(m = \nu$ *and* $n > 1)$. We choose the same index for $|A_r^* \dots A_p^* w|$. This is the diagonal case and the argument is identical to Case I.

Max *occurs at* $m = \nu$ *and* $n = 1$. We choose the same index for $|A_r^* \dots$ $\dots A_p^* w|$ and we will show

$$\big(\lambda_{\nu1}(\nu)|w_{\nu1}(\nu)|\big)^{r-p+1} \leqslant C\,\frac{|\lambda_{\nu1}(\nu)|^{r-p}}{s_{\nu1}^{\nu}}\left|w_{\nu1}(\nu) - \sum_{n\geqslant 2}\left(\frac{\lambda_{\nu n}(\nu)}{\lambda_{\nu1}(\nu)}\right)^{\nu-p} s_{\nu n}^{\nu}\lambda_{\nu n}(\nu) w_{\nu n}(\nu)\right|.$$

Because the max occurs at $m = \nu$, $n = 1$, we have $\lambda_{\nu n}(\nu)|w_{\nu n}(\nu)|$ $\leqslant \lambda_{\nu1}(\nu)|w_{\nu1}(\nu)|$ $(n \geqslant 1)$ so by Lemma 1 (i), (iii) and the fact that $\nu - p \geqslant 1$,

$$\begin{aligned} \left|\sum_{n\geqslant 2}\left(\frac{\lambda_{\nu n}(\nu)}{\lambda_{\nu1}(\nu)}\right)^{\nu-p} s_{\nu n}^{\nu}\lambda_n(\nu) w_{\nu n}(\nu)\right| &\leqslant \sum_{n\geqslant 2}\left(\frac{\lambda_{\nu n}(\nu)}{\lambda_{\nu1}(\nu)}\right)^{\nu-p} s_{\nu n}^{\nu}\lambda_{\nu1}(\nu)|w_{\nu1}(\nu)| \\ &\leqslant \sum_{n\geqslant 2}\lambda_{\nu n}(\nu) s_{\nu n}^{\nu}|w_{\nu1}(\nu)| \leqslant \tfrac{1}{2}|w_{\nu1}(\nu)|. \end{aligned}$$

Hence, it suffices to show that

$$\big(\lambda_{\nu1}(\nu)|w_{\nu1}(\nu)|\big)^{r-p+1} \leqslant \frac{C}{2}\,\frac{\big(\lambda_{\nu1}(\nu)\big)^{r-p}}{s_{\nu1}^{\nu}},$$

or, that

$$\lambda_{\nu1}(\nu) s_{\nu1}^{\nu} \leqslant C/2.$$

Now by our construction and Lemma 1, we have

$$\begin{aligned} \lambda_{\nu1}(\nu) s_{\nu1}^{\nu} &= 1 + \|A_\nu w_\nu^{\nu}\| = 1 + \Big\|\sum_{n\geqslant 2} s_{\nu n}^{\nu} A_\nu e_{\nu n}(\nu)\Big\| \\ &= 1 + \Big\|\sum_{n\geqslant 2} s_{\nu n}^{\nu}\lambda_{\nu n}(\nu) e_{\nu n}(\nu)\Big\| = 1 + \Big(\sum_{n\geqslant 2}\big(s_{\nu n}^{\nu}\lambda_{\nu n}(\nu)\big)^2\Big)^{1/2} \\ &= 1 + \Big(\sum_{n\geqslant 2}\big(\lambda_{\nu n}(\nu)\big)^4\Big)^{1/2} \leqslant 1 + \sqrt{2}/2 \leqslant C/2. \end{aligned}$$

Max *occurs at* $m > \nu$ *and* $n = 1$. This time we choose for $|A_r^* \dots A_p^* w|$ the index corresponding to $e_{m2}(\nu)$ and we will show

$$\big(\lambda_{m1}(\nu)|w_{m1}(\nu)|\big)^{r-p+1} \leqslant C\big(\lambda_{m1}(\nu)\big)^{\nu-p}\big(\lambda_{m2}(\nu)\big)^{r-\nu+1}|w_{m1}(\nu)|.$$

We have, using the fact that $|w_{m1}(\nu)| \leqslant 1$ and Lemma 1 (ii)

$$\begin{aligned} \big(\lambda_{m1}(\nu)|w_{m1}(\nu)|\big)^{r-p+1} &\leqslant \big(\lambda_{m1}(\nu)\big)^{\nu-p}\big(\lambda_{m1}(\nu)\big)^{r-\nu+1}|w_{m1}(\nu)| \\ &\leqslant 4^{r-\nu+1}\big(\lambda_{m1}(\nu)\big)^{\nu-p}\big(\lambda_{m2}(\nu)\big)^{r-\nu+1}|w_{m1}(\nu)| \\ &\leqslant C\big(\lambda_{m1}(\nu)\big)^{\nu-p}\big(\lambda_{m2}(\nu)\big)^{r-\nu+1}|w_{m1}(\nu)|. \end{aligned}$$

Max *occurs at* $m > \nu$ *and* $n \geqslant 2$. This time we choose for $|A_r^* \ldots$ $\ldots A_p^* w|$ the index corresponding to $e_{m,n+1}(\nu)$ and we will show

$$\left(\lambda_{mn}(\nu)|w_{mn}(\nu)|\right)^{r-p+1} \leqslant C\left(\lambda_{mn}(\nu)\right)^{\nu-p}\left(\lambda_{m,n+1}(\nu)\right)^{r-\nu+1}|w_{mn}(\nu)|.$$

The argument is identical to the previous case with 1 replaced by n. ∎

It seems likely that the above result can be proved for spaces other than (s). For example, in [22], Vogt and Wagner extend their characterizations to infinite type power series spaces which are stable—that is, isomorphic to their Cartesian square. Using this result, one could try to repeat the above proof with (s) replaced by a power series space. Of course, more would have to be shown. Specifically, it would be necessary to further restrict $\lambda_{mn}(\nu)$ so that the space E is sufficiently strongly nuclear (so that it can be a quotient of a power series space). This would lead to an inequality that would have to be balanced against condition (ii) of Lemma 1. Probably it would work for some cases and not for others, but the computations required seem formidable. In any case, it is unlikely that this approach can lead to a determination of whether every nuclear Fréchet space (other than ω) has a quotient space without BAP and so this remains an open question.

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