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A generalization of the Yosida–Kakutani ergodic theorem

by

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Abstract. We prove a mean ergodic theorem for the sequence $\frac{1}{n}\sum_{k=0}^{n-1} T_k \dots T_1$ where $\{T_k\}$ is a suitable convergent sequence of operators of a Banach space X into itself.

The purpose of this note is to prove an ergodic theorem of the Yosida–Kakutani type where the ergodic sum involving a single operator T is replaced by $\frac{1}{n}\sum_{k=0}^{n-1} T_k \dots T_1$ where $\{T_i\}$ is a convergent sequence of operators. The ergodic theorem remains true under a suitable convergence of the sequence $\{T_i\}$.

We will prove our result in a Banach space set-up. We denote by $\| \ \|$ either the norm in the space or the operator norm.

We now extend the mean ergodic theorem:

THEOREM. *Let X be a Banach space and $\{T_i\}_{i=1}^{\infty}$ be a sequence of bounded linear operators of X into itself, satisfying:*

(i) *There exists a constant $c > 0$ such that for each n and $i_1 < i_2 < \dots$ $\dots < i_n$,*

$$\|T_{i_n} \dots T_{i_2} T_{i_1}\| \leqslant c.$$

(ii) *$T_i \to T$ in the operator norm and $T_i T = T T_i$ for each i.*

(iii) *For each $x \in X$, the sequences*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T_k \dots T_1 x \quad \text{and} \quad x_n = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

contain subsequences weakly convergent to elements of X, $\overline{T}x$ and $P_T x$, respectively.

(iv) *For each $x \in P_T(X)$,*

$$Q(x) = \sum_{k=1}^{\infty} \sum_{i_1=k}^{\infty} \sum_{i_2=k-1}^{i_1-1} \dots \sum_{i_{k-1}=2}^{i_{k-2}-1} \sum_{i_k=1}^{i_{k-1}-1} \|\varepsilon_{i_1} \dots \varepsilon_{i_k} x\| < \infty$$

where $\varepsilon_i = T_i - T$.

Then

$$\|S_n x - \overline{T}x\| \to 0$$

where $n \to \infty$ *and where the expression of* $\overline{T}x$ *is given by*

$$\overline{T}x = P_T x + \sum_{k=1}^{\infty} \sum_{i_1=k}^{\infty} \sum_{i_2=k-1}^{i_1-1} \cdots \sum_{i_{k-1}=2}^{i_{k-2}-1} \sum_{i_k=1}^{i_{k-1}-1} \varepsilon_{i_1} \ldots \varepsilon_{i_k} P_T x.$$

Moreover,

$$T\overline{T} = \overline{T}T = \overline{T}, \quad P_T\overline{T} = \overline{T}P_T = \overline{T}$$

and $\|\overline{T}\| \leqslant c$.

Proof. We note that T indeed fulfills the requirements of Yosida–Kakutani Theorem since by (i)

$$\|T_n \ldots T_{n-k}\| \leqslant c,$$

but on the other hand $T_n \to T$ in the operator norm, which implies that for any k, $\|T^k\| \leqslant c$. Therefore the projector P_T is well defined as a strong limit of x_n.

From the following inequality

$$\|T(S_n x) - S_n x\| \leqslant \frac{2c}{n}\|x\| + c\sum_{i=1}^{n-1} \frac{\|T - T_i\|\,\|x\|}{n}$$

and recalling that a subsequence of $S_n(x)$ converges weakly to $\overline{T}x$, we have that $T\overline{T} = \overline{T}$. On the other hand, from the fact that $T_i T = TT_i$, we can easily derive $\overline{T}T = \overline{T}$. It is also seen from here that

$$P_T\overline{T} = \overline{T}P_T = \overline{T}.$$

Now we will prove that $S_n(x)$ converges strongly. For this, we consider the decomposition

$$x = P_T x + (x - P_T x).$$

We remind that

$$R(I - P_T) \subset \overline{R(I-T)}$$

where R stands for the range of the operator. Indeed, it is easy to verify that

$$R(I - T^k) \subset R(I-T)$$

for any $k \geqslant 1$, and from here,

$$R(I - P_T) = R\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}(I - T^k)\right) \subset \overline{R(I-T)}.$$

In order to prove the theorem, it remains to show that for any $y \in \overline{R(I-T)}$, $\|S_n y\| \to 0$ and

$$S_n P_T x \to \overline{T}x$$

for any $x \in X$.

At first we show that $\|S_n y\| \to 0$. For a given $\varepsilon > 0$, and $y \in \overline{R(I-T)}$, we write $y = z - Tz + z_\varepsilon$ where $\|z_\varepsilon\| < \varepsilon$. We have

$$\begin{aligned}
\|S_n Y\| &= \|S_n(z - Tz + z_\varepsilon)\| \\
&= \|S_n(z - Tz) + S_n z_\varepsilon\| \\
&\leqslant \frac{\|z - TT_{n-1}\ldots T_1 z\|}{n} + C\sum_{i=1}^{n-1}\frac{\|T_i - T\|\,\|z\|}{n} + \|S_n z_\varepsilon\|.
\end{aligned}$$

This implies our claim. For the remaining convergence, we use the following computations. For $T_i = T + \varepsilon_i$, we have $\varepsilon_i T = T\varepsilon_i$ for each i. Besides we also have $\varepsilon_i \to 0$. With this nomenclature, we get

$$\begin{aligned}
T_{n-1} \ldots T_1 &= (T + \varepsilon_{n-1}) \ldots (T + \varepsilon_1) \\
&= T^{n-1} + \delta_1^{n-1}T^{n-2} + \delta_2^{n-1}T^{n-3} + \ldots + \delta_{n-2}^{n-1}T + \delta_{n-1}^{n-1}
\end{aligned}$$

where

$$\delta_k^{n-1} = \sum_{i_1=k}^{n-1} \sum_{i_2=k-1}^{i_1-1} \cdots \sum_{i_{k-1}=2}^{i_{k-2}-1} \sum_{i_k=1}^{i_{k-1}-1} \varepsilon_{i_1}\varepsilon_{i_2} \ldots \varepsilon_{i_k}$$

for $k \leqslant n-1$. Therefore, since $TP_T = P_T$, we have

$$\begin{aligned}
S_n P_T x &= \frac{1}{n}\Big[\sum_{k=0}^{n-1} T^k + \sum_{k=1}^{n-1}\delta_1^k T^{k-1} + \sum_{k=2}^{n-1}\delta_2^k T^{k-2} + \ldots \\
&\qquad \ldots + \sum_{k=n-2}^{n-1}\delta_{n-2}^k T^{k-(n-2)} + \sum_{k=n-1}^{n-1}\delta_{n-1}^k T^{k-(n-1)}\Big] P_T x \\
&= \frac{1}{n}\Big[nP_T x + \sum_{k=1}^{n-1}\delta_1^k P_T x + \sum_{k=2}^{n-1}\delta_2^k P_T x + \ldots \\
&\qquad \ldots + \sum_{k=n-2}^{n-1}\delta_{n-2}^k P_T x + \sum_{k=n-1}^{n-1}\delta_{n-1}^k P_T x\Big].
\end{aligned}$$

On the other hand, one can easily compute that

$$\frac{\sum_{k=l}^{n-1}\delta_l^k}{n} = \delta_l^{n-1} - \frac{1}{n}\Big[\sum_{i_1=l}^{n-1}\sum_{i_2=l-1}^{i_1-1} \cdots \sum_{i_{l-1}=2}^{i_{l-2}-1}\sum_{i_l=1}^{i_{l-1}-1} i_1\varepsilon_{i_1}\ldots\varepsilon_{i_l}\Big]$$

for $1 \leqslant l \leqslant n-1$. Replacing these expressions in the previous one, it turns out

$$S_n P_T x = P_T x + \sum_{k=1}^{n-1} \delta_k^{n-1} P_T x -$$

$$-\frac{1}{n}\sum_{k=1}^{n-1}\Big[\sum_{i_1=k}^{n-1}\sum_{i_2=k-1}^{i_1-1}\cdots\sum_{i_{k-1}=2}^{i_{k-2}-1}\sum_{i_k=1}^{i_{k-1}-1} i_1\varepsilon_{i_1}\ldots\varepsilon_{i_k}\Big]P_T x$$

$$= P_T x + \Big[\sum_{k=1}^{n-1}\sum_{i_1=k}^{n-1}\sum_{i_2=k-1}^{i_1-1}\cdots\sum_{i_{k-1}=2}^{i_{k-2}-1}\sum_{i_k=1}^{i_{k-1}-1}\varepsilon_{i_1}\ldots\varepsilon_{i_k}\Big]P_T x -$$

$$-\frac{1}{n}\sum_{k=1}^{n-1}\Big[\sum_{i_1=k}^{n-1}\sum_{i_2=k-1}^{i_1-1}\cdots\sum_{i_{k-1}=2}^{i_{k-2}-1}\sum_{i_k=1}^{i_{k-1}-1} i_1\varepsilon_{i_1}\ldots\varepsilon_{i_k}\Big]P_T x.$$

We now will show that the last term on the right hand tends to zero for $n\to\infty$. In the last term of the previous equality a change of indices gives

$$R_n(P_T x) = \frac{1}{n}\Big[\sum_{k=1}^{n-1}\sum_{i_1=k}^{n-1}\sum_{i_2=k-1}^{i_1-1}\cdots\sum_{i_{k-1}=2}^{i_{k-2}-1}\sum_{i_k=1}^{i_{k-1}-1} i_1\varepsilon_{i_1}\ldots\varepsilon_{i_k}\Big]P_T x$$

$$= \frac{1}{n}\Big[\sum_{i=1}^{n-1} i\,\varepsilon_i\sum_{k=1}^{i}\sum_{i_2=k-1}^{i-1}\cdots\sum_{i_{k-1}=2}^{i_{k-2}-1}\sum_{i_k=1}^{i_{k-1}-1}\varepsilon_{i_2}\ldots\varepsilon_{i_k}\Big]P_T x.$$

We recall a so called Krenecker's lemma which states that if $\{a_i\}_{i=1}^{\infty}$ is a sequence of real numbers such that $\sum_{i=1}^{n} a_i\to a$ finite, and $b_n\uparrow\infty$, then

$$\frac{1}{b_n}\sum_{i=1}^{n} a_i b_i \to 0.$$

This lemma can be seen for example in Breiman [1], p. 51. Applying this result to our second expression of $R_n(P_T x)$ with $b_n = n$ and

$$a_i = \sum_{k=1}^{i}\sum_{i_2=k-1}^{i-1}\cdots\sum_{i_{k-1}=2}^{i_{k-2}-1}\sum_{i_k=1}^{i_{k-1}-1}\|\varepsilon_i\varepsilon_{i_2}\ldots\varepsilon_{i_k}P_T x\|,$$

we obtain that $R_n(P_T x)\to 0$. ■

As an example we observe that (iv) is fulfilled if

$$\bar\varepsilon = \sum_{i=1}^{\infty}\|T_i - T\| < \infty.$$

Indeed,

$$Q(x)\leqslant\sum_{k=1}^{\infty}\sum_{i_1=k}^{\infty}\sum_{i_2=k-1}^{\infty}\cdots\sum_{i_{k-1}=2}^{\infty}\sum_{i_k=1}^{\infty}\|\varepsilon_{i_1}\ldots\varepsilon_{i_k}x\|\leqslant\sum_{k=1}^{\infty}\bar\varepsilon_1\ldots\bar\varepsilon_k\|x\|<\infty$$

where

$$\bar\varepsilon_j = \sum_{k\geqslant j}\|\varepsilon_k\|.$$

References

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