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Reflective functions on p-adic fields

by

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1. Introduction. For a fixed rational prime p, let Q_p be the field of p-adic numbers, and let K be a finite extension field of Q_p. Let $v\colon K\to Z\cup\{\infty\}$ be the (normalized) p-adic valuation on K, $\mathcal{O}$ the ring of integral elements with respect to v, U the group of units of $\mathcal{O}$, π a fixed prime element (i.e., $v(\pi)=1$), $\bar{K}=\mathcal{O}/\pi\mathcal{O}$, $q=\#\bar{K}$, and let

$$|x|_K = q^{-v(x)} \tag{1.1}$$

for all $x\in K$.

Given a polynomial $h(X_1,\ldots,X_s)$ in s indeterminates ($s\geqslant 1$) and coefficients in $\mathcal{O}$, let $c_\nu(h)$ be the number of zeros of the reduction of h in the residue class ring $\mathcal{O}/\pi^\nu\mathcal{O}$ for each integer $\nu\geqslant 0$. The power series

$$P_h(t)=\sum_{\nu=0}^{\infty} c_\nu(h)(tq^{-s})^\nu, \tag{1.2}$$

which converges at least in the open disk $|t|<1$ is called the *Poincaré series of h*. It is conjectured (see [1], p. 47, problem 9 and also [8]) that $P_h(t)$ is a rational function of t. As one easily observes, the rationality of $P_h(t)$ means that there is a positive integer N_h such that all the numbers $c_\nu(h)$, $\nu\geqslant 0$, can be computed from $c_0(h), c_1(h), \ldots, c_{N_h}(h)$ by means of a linear recurrence.

This conjecture has recently been proved by J.-I. Igusa [5], [6] using Hironaka's deep results on the resolution of singularities of algebraic varieties in characteristic zero.

Our aim in this paper is to expand the relatively small class of polynomials h for which the rationality of $P_h(t)$ can be proved by elementary means. Our results include all the previous results known to us for special

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polynomials h. In addition, we verify the rationality of $P_h(t)$ for any polynomial h which has the property that no two distinct monomials of h contain a common indeterminate. In particular, we prove that $P_h(t)$ is rational for *diagonal* polynomials, i.e., for polynomials of the form

$$\lambda_1 X_1^{d_1} + \lambda_2 X_2^{d_2} + \ldots + \lambda_s X_s^{d_s} + c$$

where $d_1, d_2, \ldots, d_s$ are positive integers and $\lambda_1, \lambda_2, \ldots, \lambda_s$ and c belong to $\mathcal{O}$. Recently E. Stevenson has provided an elementary proof of this for forms, but along quite different lines, using characters.

Our methods apply more conveniently to the following somewhat sharper conjecture which we refer to as the "Q-conjecture".

Q: *The series $P_h(t) = N(t)/D(t)$ where $N(t)$ and $D(t)$ are polynomials in $Z[t]$ and $D(t)$ is a product of factors of the form $q^m - t^n$, m and n being strictly positive integers with $m \leqslant ns$.*

Elementary proofs of the Q-conjecture for special polynomials do exist in the literature ([7], [8], [9]). In particular, the conjecture has been established by elementary means for the following general classes of polynomials:

(A) h is any polynomial in one variable.

(B) h is a form in two variables.

(C) h is any monomial.

(D) The manifold $h(x_1, \ldots, x_s) = 0$ in K^s has no singular point which lies in $\mathcal{O}^s$.

Proofs can be found in the dissertation of Shuck [8]. Also, see Gusev [2] for another proof of the Q-conjecture for h of type (B) (and [3] for a partial result on polynomials in two variables).

Let dx be the Haar measure on the additive group of K normalized so that $\mathcal{O}$ has measure 1. Viewed as a map from $\mathcal{O}^s$ to $\mathcal{O}$, h has a Radon–Nikodym derivative (¹) θ_h in this measure. Since $\mathcal{O}$ has finite measure, $\theta_h \in L^1(\mathcal{O})$, which justifies the following computation:

$$c_\nu(h)q^{-\nu s} = \operatorname{meas}\big(h^{-1}(\pi^\nu \mathcal{O})\big) = \int_{\pi^\nu \mathcal{O}} \theta_h(x)\,dx$$

so that

$$(1.3)\qquad q^{-\nu s}c_\nu(h) - q^{-(\nu+1)s}c_{\nu+1}(h) = \int_{\pi^\nu U} \theta_h(x)\,dx = q^{-\nu}\int_U \theta_h(\pi^\nu u)\,du$$

for all $\nu \geqslant 0$. Shuck (*loc. cit.*) uses (1.3) together with the change of variables theorem for p-adic integrals to obtain his results. In this paper, we define

(¹) See the Appendix for the definition and some properties.

for every $k > 0$ the concept of *k-reflectivity* for functions in $L^1(\mathcal{O})$. (The presence of a linear recurrence among the coefficients of the Poincaré series of h and the analogy of the reflection of images between a parallel pair of mirrors suggested the name.) Then we prove.

THEOREM 1. *If θ_h is s-reflective, where s is the number of variables in h, then* Q *is valid for h.*

THEOREM 2. *If h is a polynomial in one variable with integral coefficients, then θ_h is* 1*-reflective.*

THEOREM 3. *If the Radon–Nikodym derivatives of all the polynomials h in s variables are s-reflective for some fixed $s \geqslant 0$, and if f is a form in $s+1$ variables, then θ_{f-c} is $(s+1)$-reflective for every $c \in \mathcal{O}$* (cf. [8], Theorem 3.6).

THEOREM 4. *Suppose h (resp. g) is a polynomial in s (resp. r) variables, and suppose that θ_h is s-reflective and that θ_g is r-reflective. Then for any constant $c \in \mathcal{O}$, the Radon–Nikodym derivatives of the polynomials $h(X_1, \ldots, X_s) + g(Y_1, \ldots, Y_r) - c$ and $h(X_1, \ldots, X_s)g(Y_1, \ldots, Y_r) - c$ in $s+r$ variables each are both $(s+r)$-reflective.*

These theorems are proved below in § 5 after a preliminary development of the properties of "reflective" functions in §§ 2, 3 and 4. The validity of the Q-conjecture for the classes (A), (B), (C) and (D) listed above and for arbitrary diagonal polynomials follows as an immediate corollary of these theorems.

The Q-linear span of the series (1.2) for all polynomials with integral coefficients contains almost all rational functions of the type in Q above (see [7]). This result will not be needed in the sequel.

Note that throughout, Z denotes the ring of rational integers, Q the ordinary rational field and C the complex field.

2. Rational functions in $L^1(\mathcal{O})$. Throughout the remainder of the paper, whenever we refer to an "L^1-function" we mean a function in $L^1(\mathcal{O})$ with respect to the restriction to $\mathcal{O}$ of the measure dx introduced in § 1. We adopt the convention that the domain of any function f defined on $\mathcal{O}$ is automatically extended to K by defining $f(x) = 0$ for $x \in K \setminus \mathcal{O}$.

If φ and ψ are L^1-functions, then their *convolution* $\varphi * \psi$ is defined (as usual) by

$$(2.1)\qquad (\varphi * \psi)(z) = \int \varphi(x)\psi(z-x)\,dx$$

for all $z \in K$. Observe that $\varphi * \psi$ is also an L^1-function (i.e., it vanishes off $\mathcal{O}$). We also introduce the *multiplicative convolution* $\varphi *_m \psi$ of φ and ψ defined for $z \in K \setminus \{0\}$ by

$$(2.2)\qquad (\varphi *_m \psi)(z) = \int \varphi(x)\psi(zx^{-1})\frac{dx}{|x|_K}.$$

The integral in (2.2) exists because the integrand vanishes off the set

$$W(z) = \{x \in K \mid 0 \leqslant v(x) \leqslant v(z)\} \tag{2.3}$$

which is a finite union of multiplicative translates of U. We define $\varphi *_m \psi$ at $z = 0$ by continuity if possible; otherwise, we put $(\varphi *_m \psi)(0) = \infty$. We show in Proposition 2.3 below that $\varphi *_m \psi$ is also an L^1-function. (The reader should note that $dx/|x|_K$ is the Haar measure on the multiplicative group $K^* = K \setminus \{0\}$ which gives U the measure $1-(1/q)$.)

Given a and b in $\mathcal{O}$ and an L^1-function θ, we define

$$A_a\theta(z) = \theta(z+a), \tag{2.4}$$

$$M_b\theta(z) = \theta(bz) \tag{2.5}$$

for all $z \in \mathcal{O}$.

We note the standard translation formulas

$$A_a\varphi * \psi = \varphi * A_a\psi = A_a(\varphi * \psi) \quad (a \in \mathcal{O}), \tag{2.6}$$

$$M_b\varphi *_m \psi = \varphi *_m M_b\psi = M_b(\varphi *_m \psi) \quad (b \in U) \tag{2.7}$$

for all L^1-functions φ and ψ. We note also the following formula which is valid for all z such that $v(z) > 0$:

$$M_cA_c(\varphi *_m \psi)(z) = \sum_{\mu=0}^{v(c)} A_1(M_{\pi^\mu}\varphi *_m M_{c\pi^{-\mu}}\psi)(z). \tag{2.8}$$

This is clear since the operators M_cA_c and A_1M_c have the same effect, and c may be factored in $\mathcal{O}$ as $\pi^\mu c\pi^{-\mu}$ for $\mu = 0$ to $v(c)$. We will need this result in Theorem 4.4.

In trying to apply (1.3) to study the Q-conjecture, one is led to introduce the power series:

$$G_\theta(u, t) = \sum_{\nu=0}^{\infty} q^{-\nu}\theta(\pi^\nu u)t^\nu \tag{2.9}$$

associated to a given L^1-function θ and defined for almost all $u \in U$. Since

$$\int |\theta(x)|_K dx = \sum_{\nu=0}^{\infty} \int_{\pi^\nu U} |\theta(x)|_K dx = \sum_{\nu=0}^{\infty} q^{-\nu} \int_U |\theta(\pi^\nu u)|_K du,$$

we see by monotone convergence that the series (2.9) converges for t in the closed unit disk of the complex plane for almost all $u \in U$. We put

$$G_\theta^*(t) = \int_U G_\theta(u, t)\,du = \sum_{\nu=0}^{\infty} q^{-\nu}\Big[\int_U \theta(\pi^\nu u)\,du\Big]t^\nu. \tag{2.10}$$

Note that

$$G_\theta^*(1) = \int \theta(z)\,dz. \tag{2.11}$$

Proposition 2.1. *For a polynomial $h \in \mathcal{O}[X_1, \ldots, X_s]$*

$$P_h(t) = \big(tG_{\theta_h}^*(t) - 1\big)/(t-1). \tag{2.12}$$

Proof. The proposition follows from (1.3) by a simple calculation. ■

Definition 2.2. An L^1-function θ is called *rational* if there is an integer $d \geqslant 0$, functions $a_0(u), \ldots, a_d(u)$ $(u \in U)$ in $L^1(U)$ and a polynomial $D(t) \in \mathbf{Q}[t]$ such that

$$G_\theta(u, t) = \frac{a_0(u) + a_1(u)t + \ldots + a_d(u)t^d}{D(t)}. \tag{2.13}$$

As an example of a rational L^1-function, consider for a fixed integer $n \geqslant 1$ the function θ_n defined for $z \in \mathcal{O}$ as follows: $\theta_n(z) = 0$ if $n \nmid v(z)$, and $\theta_n(z) = q^{\nu(n-1)}N_n(u)/|n|_K$ if $z = \pi^{n\nu}u$, $u \in U$, where $N_n(u)$ is the number of nth roots of u in K. Then, as one easily checks

$$G_{\theta_n}(u, t) = \frac{qN_n(u)/|n|_K}{q - t^n}. \tag{2.14}$$

This function θ_n is the Radon–Nikodym derivative of the polynomial $h(X) = X^n$ viewed as a map from $\mathcal{O}$ to itself. (This assertion will be proved later in Theorem 2, using (2) of the Appendix.) Let us use (2.14) to compute $P_h(t)$. It follows from (2.14) that $G_{\theta_n}^*(t)$ is a constant multiple of $1/(q-t^n)$. Since θ_n is a derivative, its integral over $\mathcal{O}$ is 1, and so $G_{\theta_n}^*(1) = 1$ by (2.11). Thus, the constant multiplier is $q-1$. Plugging this series into (2.12), we find after a short calculation that

$$P_h(t) = \frac{q + t + t^2 + \ldots + t^{n-1}}{q - t^n} \tag{2.15}$$

for $h(X) = X^n$, a result which is in accordance with the Q-conjecture.

Proposition 2.3. *Let φ and ψ be L^1-functions, and let $\theta = \varphi *_m \psi$. Then θ is also an L^1-function, and*

$$G_\theta(u, t) = \int_U G_\varphi(\varepsilon, t)G_\psi(u\varepsilon^{-1}, t)\,d\varepsilon. \tag{2.16}$$

Further,

$$G_\theta^*(t) = G_\varphi^*(t)G_\psi^*(t). \tag{2.17}$$

Proof. Since the integrand in (2.2) is zero for x not in the set $W(z)$, we have for $\nu \geqslant 0$ and almost all $u \in U$

$$\theta(\pi^\nu u) = \sum_{\mu=0}^{\nu} \int_{\pi^\mu U} \varphi(x)\psi(\pi^\nu u x^{-1})\frac{dx}{|x|_K} = \int_U \Big[\sum_{\mu=0}^{\nu} \varphi(\pi^\mu \varepsilon)\psi(\pi^{\nu-\mu} u \varepsilon^{-1})\Big] d\varepsilon.$$

This shows that (2.16) holds in the sense of formal multiplication of power series. But since the integrand in (2.16) is the product of series which are convergent in the closed unit disk, it is also convergent there. Hence, by standard measure theory, $G_\theta(u,t)$ is absolutely convergent for $t=1$ and almost all $u \in U$. We see now by the calculation in (2.10) that the integral of $|\theta|_K$ is finite, and so θ is an L^1-function. Equation (2.17) follows from (2.16) by Fubini's Theorem. ∎

Corollary 2.4. *Let φ, ψ and θ be as in the above proposition. Then if φ and ψ are rational, so is θ.*

Suppose g (resp. h) is a polynomial with coefficients in $\mathcal{O}$ in r (resp. s) variables. Let the variables in g be distinct from those in h so that $g+h$ and gh are polynomials in $r+s$ variables. Then it is a simple exercise in the use of Fubini's Theorem to show that

$$\theta_{g+h} = \theta_g * \theta_h \tag{2.18}$$

and

$$\theta_{gh} = \theta_g *_m \theta_h. \tag{2.19}$$

We can conclude from (2.19), our example (2.14) and Proposition 2.3 that the Q-conjecture holds for any *monomial*.

We cannot conclude in general that the additive convolution of rational functions is rational. This motivates the introduction (see § 4) of the concept of reflectivity for L^1-functions, a stronger concept than rationality. We prove in § 4 that the additive and multiplicative convolutions of reflective functions are reflective.

3. Almost locally constant functions. An L^1-function φ is said to be *locally constant* at a point $z \in \mathcal{O}$ if φ is constant in some neighborhood of z. We define B_φ to be the set of points $z \in \mathcal{O}$ such that φ is not locally constant at z. We call the points in B_φ the *branch points* of φ.

Definition 3.1. Let φ be an L^1-function, and let r be a non-negative integer. We say that φ *belongs to class* $\mathscr{B}_r$ if B_φ is finite and contains exactly r points, i.e., if φ has exactly r branch points. Let $\mathscr{B} = \bigcup \mathscr{B}_r$ for $r \geqslant 0$. A function which belongs to $\mathscr{B}$ is said to be *almost locally constant.*

Let $h \in \mathcal{O}[X_1, \ldots, X_s]$. The *singular locus* S_h of h is the set of points in $\mathcal{O}^s$ at which the s partial derivatives of h vanish simultaneously. It is known (see [7] and [8]) that the Radon–Nikodym derivative θ_h is locally constant at every point of $\mathcal{O}\setminus h(S_h)$. (See the Appendix for a sketch of the proof.)

Proposition 3.2. *The derivative θ_h is almost locally constant.*

Proof. By the above remarks, it suffices to prove that $h(S_h)$ is finite. Let F be the algebraic closure of K, and let S_h^* be the algebraic variety consisting of the points in F^s where the partial derivatives of h vanish simultaneously. Clearly, it suffices to show that $h(S_h^*)$ is finite, an assertion which belongs to classical algebraic geometry. We will use the infamous Lefschetz Principle to prove it. By Lefschetz, we can take F to be the complex numbers. We can then appeal to the analytic result known as Sard's Theorem to conclude that $h(S_h^*)$ has Lebesgue measure zero. But since h is an algebraic morphism, $h(S_h^*)$ is either finite or else contains a non-empty Zariski open set. The latter possibility is excluded by the fact that a non-empty open set always has positive Lebesgue measure. ∎

The above proof is due to J. Fogarty to whom we extend our thanks. One can show by somewhat more lengthy but purely algebraic arguments that h is actually constant on the components of S_h^*. Proposition 3.2 will be used in this paper only when $s=1$, in which case it is trivial. We prove it in full generality because it helps put our results in § 4 in perspective.

We turn now to the main work of this section which is to show that convolutions of functions in $\mathscr{B}$ also belong to $\mathscr{B}$.

Theorem 3.3. *Suppose φ and ψ are L^1-functions belonging to $\mathscr{B}$. Then $\theta = \varphi * \psi$ also belongs to $\mathscr{B}$, and in fact*

$$B_\theta \subseteq B_\varphi + B_\psi. \tag{3.1}$$

Proof. Any function in $\mathscr{B}$ has r branch points for some integral $r \geqslant 0$. If $r \geqslant 2$, the function can be represented as a sum of r functions in $\mathscr{B}_1$. Since $*$ is distributive over addition, we may assume without loss of generality that φ and ψ belong to $\mathscr{B}_0 \cup \mathscr{B}_1$. Let us fix $z_0 \in \mathcal{O}$. Then for $w \in \mathcal{O}$,

$$\theta(z_0+w) = \int \varphi(x)\psi(z_0+w-x)\,dx. \tag{3.2}$$

Suppose firstly that one of φ and ψ, say ψ, belongs to $\mathscr{B}_0$. Then there is a positive integer m such that $\psi(x_1) = \psi(x_2)$ whenever $x_1 \equiv x_2 \pmod{\pi^m}$. We observe then that $\theta(z_0+w)$ is constant as w varies in $\pi^m\mathcal{O}$. Thus, $\theta \in \mathscr{B}_0$ and so (3.1) holds.

Suppose secondly that both φ and ψ belong to $\mathscr{B}_1$. By additive translation, we may assume that the single branch point of φ is zero. Let b be the branch point of ψ. We have to show that θ is locally constant at z_0 when $z_0 \neq b$. Choose a positive integer r so large that φ is constant on

$E = z_0 - b + \pi^r \mathcal{O}$. Next choose $m \geq r$ so large that $\psi(x_1) = \psi(x_2)$ whenever $x_1 \equiv x_2 \pmod{\pi^m}$ but $x_1 \not\equiv b \pmod{\pi^r}$. We will show that $\theta(z_0+w)$ is constant for $w \in \pi^m \mathcal{O}$.

Let $A = \mathcal{O} \setminus E$. Since $\psi(z_0+w-x)$ is independent of w when $w \in \pi^m \mathcal{O}$ and $x \in A$, it suffices by (3.2) to prove that

$$\int_E \varphi(x)\psi(z_0+w-x)\,dx$$

is independent of $w \in \pi^m \mathcal{O}$. But this is clear because φ is constant on E and $w+E = E$ for $w \in \pi^m \mathcal{O}$. Thus, θ can have no branch point other than b, and (3.1) holds. ∎

We wish to prove a similar theorem for the multiplicative convolution $*_m$. For this, we need to introduce the subgroups $U^{(r)} = 1+\pi^r \mathcal{O}$ $(r \geq 1)$ of U. It is well-known that these subgroups form a base for the filter of neighborhoods at 1 in U.

If $A \subseteq \mathcal{O}$, then we put $A^* = A \setminus \{0\}$.

THEOREM 3.4. *Suppose φ and ψ are L^1-functions belonging to $\mathcal{B}$. Then $\theta = \varphi *_m \psi$ also belongs to $\mathcal{B}$, and in fact*

$$B_\theta^* \subseteq B_\varphi^* B_\psi^*. \tag{3.3}$$

Proof. As in the proof of Theorem 3.3, we can assume that both φ and ψ belong to $\mathcal{B}_0 \cup \mathcal{B}_1$. Let us fix $z_0 = \pi^\nu u_0$ in $\mathcal{O}^*$ where $u_0 \in U$. Then by (2.2) and (2.3), we have for any $w \in U$

$$\theta(z_0 w) = \sum_{\mu+\gamma=\nu} \int_U \varphi(\pi^\mu \varepsilon)\psi(\pi^\gamma u_0 w \varepsilon^{-1})\,d\varepsilon. \tag{3.4}$$

Suppose that $b \in B_\varphi^*$ and $b' \in B_\psi^*$ imply that $v(bb') \neq \nu$. Consider a term of the summation in (3.4) corresponding to given values μ and γ with $\mu+\gamma = \nu$. By assumption, either φ is locally constant at every point of $\pi^\mu U$ or else ψ is locally constant at every point of $\pi^\gamma U$. We assume the latter possibility without loss of generality. Then there is a positive integer $r = r_\gamma$ such that $\psi(x_1) = \psi(x_2)$ whenever $x_1, x_2 \in \pi^\gamma U$ and $x_1 x_2^{-1} \in U^{(r)}$. We observe that the summand corresponding to μ and γ is constant as w varies in $U^{(r)}$. Taking the largest such r for all the summands, we conclude that θ is constant in the neighborhood $z_0 U^{(r)}$ of z_0. In particular, if either of B_φ^* or B_ψ^* is empty, then so is B_θ^* and (3.3) holds.

We have left to consider the case in which φ has the single branch point $\pi^{\mu_0}\varepsilon_0'$ and ψ has the single branch point $\pi^{\gamma_0}\varepsilon_0$ where $\varepsilon_0', \varepsilon_0 \in U$ and $\mu_0+\gamma_0 = \nu$. By multiplicative translation, we may assume that $\varepsilon_0' = 1$. We have to show that θ is locally constant at z_0 when $z_0 \neq \pi^\nu \varepsilon_0$. By the method used in the first part of this proof, it suffices to prove that the summand in (3.4) corresponding to $\mu = \mu_0$ and $\gamma = \gamma_0$ is constant for w belonging to a sufficiently small neighborhood of 1. Choose m so large that φ is constant on $\pi^{\mu_0} E$ where E is $u_0 \varepsilon_0^{-1} U^{(m)}$, and put $A = U \setminus E$. Next choose r so large that $r \geq m$ and $\psi(x_1) = \psi(x_2)$ whenever $x_1, x_2 \in \pi^{\gamma_0} U$ and $x_1 x_2^{-1} \in U^{(r)}$ but $x_1 \notin \pi^{\gamma_0} \varepsilon_0 U^{(m)}$.

Since $\psi(\pi^{\gamma_0} u_0 w \varepsilon^{-1})$ is independent of w for $w \in U^{(r)}$ and $\varepsilon \in A$, it suffices to prove that

$$\int_E \varphi(\pi^{\mu_0}\varepsilon)\psi(\pi^{\gamma_0} u_0 w \varepsilon^{-1})\,d\varepsilon$$

is independent of $w \in U^{(r)}$, but this is clear because φ is constant on E and $wE = E$ for $w \in U^{(r)}$. Thus, θ can have no branch point other than $\pi^\nu \varepsilon_0$, and (3.3) holds. ∎

4. Reflective functions in $L^1(\mathcal{O})$. Let $h \in \mathcal{O}[X_1, \ldots, X_s]$, and let $c \in \mathcal{O}$. It is then a simple consequence of the additive invariance of the measure dx that

$$\theta_{h-c}(z) = \theta_h(z+c) \tag{4.1}$$

for almost all $z \in \mathcal{O}$.

DEFINITION 4.1. An L^1-function θ is said to be *reflective* if it has the following two properties:

(R1) θ is almost locally constant.

(R2) For every $c \in \mathcal{O}$, $A_c\theta$ is rational with denominator a product of terms of the form $q^m - t^n$ for suitable strictly positive integers m and n. (Of course, this denominator may vary with c.)

Given a reflective function θ, let S be the set of all those complex numbers which occur as a pole of one of the rational functions $G_{A_c\theta}(t)$, $c \in \mathcal{O}$. Since θ has only finitely many branch points, S is a finite set. Let A (resp. B) be the minimum (resp. maximum) ordinary absolute value of the numbers S. Then we put

$$\begin{cases} \operatorname{lbd}(\theta) = \log_q A, \\ \operatorname{ubd}(\theta) = \log_q B. \end{cases} \tag{4.2}$$

Clearly, we always have $\operatorname{lbd}(\theta) > 0$. For $k > 0$, a reflective function θ is called *k-reflective* if $\operatorname{ubd}(\theta) \leq k$.

Theorem 1 of the introduction is a formal consequence of these definitions and (2.12).

We conjecture that the Radon–Nikodym derivative θ_h of any polynomial h in s variables over $\mathcal{O}$ is s-reflective. By Proposition 3.2 and (4.1) above, this conjecture is equivalent to the assertion that each θ_h is rational with denominator a product of factors of the form $q^m - t^n$, $m \leq ns$, $m > 0$, and $n > 0$.

Let $\mathcal{R}$ be the set of reflective L^1-functions. It is clear that $\mathcal{R}$ is a C-linear subspace of $L^1(\mathcal{O})$ which is closed under the additive translation

(2.4) and the multiplicative translation (2.5). We show in this section that $\mathscr{R}$ is also closed under additive and multiplicative convolution. We first introduce a set of simple functions, the *basic* reflective functions, which generate $\mathscr{R}$ as a C-vector space.

Let E be a non-trivial continuous character of the additive group of K which is trivial on $\mathcal{O}$. If $r \geq 0$ is an integer, then we define

$$E_r(z) = E(z/\pi^r)$$

for all $z \in K$. It is well known that the characters of the compact additive group $\mathcal{O}$ are the functions of the form

$$z \mapsto E_r(bz) \quad (z \in \mathcal{O}) \tag{4.3}$$

for $r \geq 0$ and $b \in \mathcal{O}$, where b is determined modulo π^r, provided E is non-trivial on $\pi^{-r}U$. Note that these characters all belong to $\mathscr{B}_0$.

A sequence of complex numbers c_ν, $\nu \geq 0$, is called a *reflective coefficient* if the power series

$$H(t) = \sum_{\nu=0}^{\infty} q^{-\nu} c_\nu t^\nu \tag{4.4}$$

is a rational function with denominator of the type mentioned in (R2) of Definition 4.1. The power series converges for t in the closed unit disk of the complex plane since $H(t)$ can have no pole in that disk.

Given a reflective coefficient c_ν, $\nu \geq 0$, and the character (4.3) of $\mathcal{O}$, we define a function θ on $\mathcal{O}$ as follows: If $z = \pi^\nu u$ ($\nu \geq 0$, $u \in U$) is a non-zero element of $\mathcal{O}$, then

$$\theta(z) = c_\nu E_r(bu). \tag{4.5}$$

We define $\theta(0)$ by continuity if possible; otherwise, we put $\theta(0) = \infty$. It is clear that θ is $L^1(\mathcal{O})$ by the calculation following (2.9). Also, θ is almost locally constant with 0 as its only possible branch point. We see that (R2) holds for θ since

$$G_\theta(u, t) = H(t) E_r(bu)$$

and since θ is locally constant at every point $c \in \mathcal{O}$, $c \neq 0$. Thus, θ is reflective. Any additive translate $A_a\theta$, $a \in \mathcal{O}$, is also reflective. We say that $A_a\theta$ is a *basic reflective function centered at* a.

PROPOSITION 4.2. *Let θ be a reflective function. Let Γ be the C-linear space of reflective functions γ such that* $\operatorname{lbd}(\theta) \leq \operatorname{lbd}(\gamma)$ *and* $\operatorname{ubd}(\gamma) \leq \operatorname{ubd}(\theta)$. *Then θ is a C-linear combination of basic reflective functions belonging to Γ.*

Proof. For the purpose of this proof, a function belonging to Γ is said to be Γ-reflective. Since the denominator of $G_{A_c\theta}(u, t)$ depends only on the local behavior of θ near c, we can write θ as a sum of Γ-reflective functions which belong to $\mathscr{B}_0 \cup \mathscr{B}_1$. Thus, we can assume without loss of generality that θ has at most one branch point, and by additive translation we can even assume that the branch point is zero (if there is one).

For such a θ, we have the representation (2.13) for $G_\theta(u, t)$, valid for almost all $u \in U$. Multiplying by the denominator $D(t)$ and equating coefficients, we find that each of the functions $a_i(u)$, $0 \leq i \leq d$, in (2.13) is a finite linear combination of the functions $\theta(\pi^\nu u)$, $\nu \geq 0$, on U. Each function $a_i(u)$, $0 \leq i \leq d$, is therefore locally constant on U since zero is the only possible branch point of θ. We can now conclude from (2.13) that for $\nu \geq 0$ and $u \in U$, $\theta(\pi^\nu u)$ is a sum of functions of the form $c_\nu a(u)$, where c_ν, $\nu \geq 0$, is a Γ-reflective coefficient and $a(z)$, $z \in \mathcal{O}$, belongs to $\mathscr{B}_0$. (The domain of any function on U is extended to $\mathcal{O}$ by setting it equal to zero at the points in $\mathcal{O} \setminus U$.) Now, it is a simple consequence of the Fourier Inversion Theorem for the compact additive group $\mathcal{O}$ that any function in $\mathscr{B}_0$ is a finite linear combination of characters of $\mathcal{O}$. ∎

Our proofs that the additive and multiplicative convolutions of reflective functions are reflective use certain operations which produce new reflective coefficients out of old ones. We now introduce these operations. For convenience, we drop the quantifier "$\nu \geq 0$" when writing down a reflective coefficient with index "ν". No confusion should arise. When we speak of the "poles" of a reflective coefficient c_ν, it is understood that we refer to the poles of the rational function $H(t)$ associated to c_ν by (4.4).

Translation: For a fixed integer $r \geq 0$, c_ν is a reflective coefficient if and only if $c_{r+\nu}$ is a reflective coefficient. Further, c_ν and $c_{r+\nu}$ have the same poles.

Lower summation: If c_ν is a reflective coefficient, then $d_\nu = \sum_{\mu=0}^{\nu} c_\mu$ is also a reflective coefficient. Further, c_ν and d_ν have the same poles except that d_ν may have an additional pole at $t = q$.

Upper summation: If c_ν is a reflective coefficient, then $d_\nu = \sum_{\mu > \nu} q^{\nu-\mu} c_\mu$ is also a reflective coefficient. Further, d_ν has the same poles as does c_ν.

Product: If c_ν and d_ν are reflective coefficients, then $q^{-\nu} c_\nu d_\nu$ is also a reflective coefficient. Each pole of $q^{-\nu} c_\nu d_\nu$ is the product of a pole of c_ν and a pole of d_ν.

The reader can easily verify the assertions made for the first three operations listed above. For the product, let $H(t)$ (resp. $H'(t)$) be the rational function associated to c_ν (resp. d_ν) by (4.4). We observe from the standard integral representation for the Hadamard product of two power series (see [4], p. 84) that the function $H''(t)$ associated to $q^{-\nu} c_\nu d_\nu$ by (4.4) is rational. We see also from this same integral that the poles of $H''(t)$ are among the products $\alpha\beta$, where α is a pole of $H(t)$ and β is a pole of $H'(t)$. Further, the order of $\alpha\beta$ as a pole of $H''(t)$ is less than or equal to the sum of the orders of α and β as poles of $H(t)$ and $H'(t)$ respectively. Thus, the denominator of $H''(t)$ divides the product of the factors $q^{m+m'}-t^{n+n'}$ where q^m-t^n is a factor of the denominator of $H(t)$ and $q^{m'}-t^{n'}$ is a factor of the denominator of $H'(t)$. ■

THEOREM 4.3. *Let φ and ψ be reflective functions in $L^1(\mathcal{O})$. Let Γ be the $\boldsymbol{C}$-linear space of reflective functions γ such that*

$$\begin{aligned} &\operatorname{ubd}(\gamma) \leqslant \max\{1, \operatorname{ubd}(\varphi)+\operatorname{ubd}(\psi)\}, \\ &\operatorname{lbd}(\gamma) \geqslant \min\{1, \operatorname{lbd}(\varphi)+\operatorname{lbd}(\psi)\}. \end{aligned} \tag{4.6}$$

Then $\theta = \varphi * \psi$ is a reflective function belonging to Γ.

Proof. By Proposition 4.2, we may assume without loss of generality that both φ and ψ are basic reflective functions. By additive translation, we may further assume that both φ and ψ are centered at zero. Therefore, we have $\varphi(z) = c_\nu E_r(au)$ and $\psi(z) = d_\nu E_r(bu)$ where $z = \pi^\nu u$, c_ν and d_ν are reflective coefficients, $r \geqslant 1$, and $a, b \in \mathcal{O}$. We will exhibit $\theta(z) = \theta(\pi^\nu u)$ as a $\boldsymbol{C}$-linear combination of basic reflective functions belonging to Γ.

From (2.1) and the decomposition

$$\mathcal{O} = \bigcup_{\mu=0}^{\infty} \pi^\mu U, \tag{4.7}$$

we find that

$$\theta(z) = \sum_{\mu=0}^{\infty} q^{-\mu} \int_U \varphi(\pi^\mu \varepsilon)\psi(\pi^\nu u - \pi^\mu \varepsilon)\,d\varepsilon. \tag{4.8}$$

Breaking the summation in (4.8) into the parts where $\mu < \nu$, $\mu = \nu$ and $\mu > \nu$, we obtain the representation

$$\theta(z) = cS_1(z) + S_2(z) + q^{-\nu}\psi(z)S_3(z) \tag{4.9}$$

where

$$\begin{aligned} S_1(z) &= \sum_{\mu=0}^{\nu-1} q^{-\mu} c_\mu d_\mu E_r(b\pi^{\nu-\mu}u), \\ S_2(z) &= q^{-\nu} c_\nu \int_U E_r(a\varepsilon)\psi\big(\pi^\nu(u-\varepsilon)\big)\,d\varepsilon, \\ S_3(z) &= \sum_{\mu>\nu} q^{\nu-\mu} c_\mu \int_U E_r(a\varepsilon)E_r(-b\pi^{\mu-\nu}\varepsilon)\,d\varepsilon \end{aligned} \tag{4.10}$$

and where

$$c = \int_U E_r\big((a-b)\varepsilon\big)\,d\varepsilon. \tag{4.11}$$

Let us adopt the conventions that a vacuous summation equals zero and that $c_\nu = d_\nu = 0$ whenever $\nu < 0$. Then $S_1(z)$ can be rewritten as

$$\sum_{\mu=0}^{\nu-r} q^{-\mu} c_\mu d_\mu + \sum_{\delta=1}^{r-1} q^\delta (q^{-\nu} c_{\nu-\delta} d_{\nu-\delta}) E_{r-\delta}(bu).$$

The first summation above is derived from c_ν and d_ν by the operations of *product*, *lower summation* and *translation* applied in that order. It is therefore a reflective coefficient belonging to Γ. The summands in the second summation above are derived in part from c_ν and d_ν by the operation of *translation* followed by *product*. It is therefore a linear combination of basic reflective functions belonging to Γ.

We turn now to $S_3(z)$, which can be rewritten as

$$\Big[\sum_{\mu=\nu+r}^{\infty} q^{\nu-\mu} c_\mu\Big] \int_U E_r(a\varepsilon)\,d\varepsilon + \sum_{\delta=1}^{r-1} q^{-\delta} c_{\nu+\delta} \Big[\int_U E_r(a\varepsilon)E_{r-\delta}(-b\varepsilon)\,d\varepsilon\Big].$$

The summation on the left above is obtained from c_ν by the *upper summation* operation followed by *translation*. It is therefore a reflective coefficient having the same poles as does c_ν. The summation on the right is a linear combination of reflective coefficients having the same poles as does c_ν by *translation*. Thus, by the *product* operation, in (4.9) $q^{-\nu}\psi(z)S_3(z)$ is a sum of basic reflective functions belonging to Γ.

Finally, we investigate $S_2(z)$. We have

$$S_2(z) = [q^{-\nu} c_\nu d_\nu E_r(bu)]I + q^{-\nu} c_\nu \int_{u+\pi\mathcal{O}} E_r(a\varepsilon)\psi\big(\pi^\nu(u-\varepsilon)\big)\,d\varepsilon \tag{4.12}$$

where I is an integral which is independent of z. The first summand on the right-hand side of (4.12) belongs to Γ by the *product* operation. The integral in the second summand equals

$$E_r(au) \int_{\pi\mathcal{O}} E_r(aw)\psi(-\pi^\nu w)\,dw.$$

It suffices to show that the integral above is a reflective coefficient having the same poles as does d_ν. In fact, it equals

$$\sum_{\mu=1}^{\infty} q^{-\mu} \int_U E_r(a\pi^\mu \varepsilon)\psi(-\pi^{\nu+\mu}\varepsilon)\,d\varepsilon = \sum_{\tau>\nu} q^{\nu-\tau} d_\tau \int_U E_r(a\pi^{\tau-\nu}\varepsilon)E_r(-b\varepsilon)\,d\varepsilon$$

which by *upper summation* and *translation* is seen to be a linear combination of reflective coefficients having the same poles as does d_ν. ■

THEOREM 4.4. *Let φ and ψ be reflective functions in $L^1(\mathcal{O})$. Let Γ be the space of reflective functions γ for which (4.6) holds. Then $\theta = \varphi *_m \psi$ is a reflective function belonging to Γ.*

Proof. Let us say just for the purpose of this proof that an L^1-function γ belongs to Γ at $c\in\mathcal{O}$ if $A_c\gamma$ is rational and the poles β of $G_{A_c\gamma}(t)$ satisfy $C\leqslant \log_q|\beta|\leqslant D$, where C is the min and D is the max appearing in (4.6). Now θ is almost locally constant by Theorem 3.4. Therefore, it suffices to show that θ belongs to Γ at every point $c\in\mathcal{O}$. For $c=0$, this is clear from Proposition 2.3. For $c\neq 0$, we proceed as follows. First, it is clear that $A_c\theta$ and $M_cA_c\theta$ are rational or not together and, if rational, the corresponding rational functions (2.13) have the same denominator. Therefore, by (2.8), we may assume $c=1$ without loss of generality. As we see below, $A_1\theta$ is a combination of basic reflective functions, hence rational.

In proving that θ belongs to Γ at 1, we may assume that both φ and ψ are basic reflective functions by Proposition 4.2. If $z=1$ is not a branch point of θ, then θ belongs trivially to Γ at 1. Otherwise, we see from (3.3) and the multiplicative translation rule (2.7) that we can assume that both φ and ψ are centered at 1. We may assume in other words that $\varphi(1+z)=c_\nu E_r(au)$ and $\psi(1+z)=d_\nu E_r(bu)$, where $z=\pi^\nu u$, c_ν and d_ν are reflective coefficients, $r\geqslant 1$, and $a,b\in\mathcal{O}$. We will show under these assumptions that on a sufficiently small neighborhood of 1, $A_1\theta(z)$ is a C-linear combination of basic reflective functions belonging to Γ.

For $\nu\geqslant 1$, we have, since φ and ψ are centered at 1,

$$A_1\theta(z)=\int_U\varphi(x)\psi\big((1+z)x^{-1}\big)dx=I+\int_{1+\pi\mathcal{O}}\varphi(x^{-1})\psi\big((1+z)x\big)\,dx$$

where I is independent of ν. From the decomposition (4.7), the second integral above equals

$$\sum_{\mu=1}^{\infty}q^{-\mu}\int_U\varphi(1+\pi^\mu\varepsilon'_\mu)\psi\big((1+\pi^\nu u)(1+\pi^\mu\varepsilon)\big)d\varepsilon$$

$$=\sum_{\mu=1}^{\infty}q^{-\mu}c_\mu\int_U E_r(a\varepsilon'_\mu)\psi(1+\pi^\nu u+\pi^\mu\varepsilon+\pi^{\nu+\mu}u\varepsilon)\,d\varepsilon,$$

where $1+\pi^\mu\varepsilon'_\mu=(1+\pi^\mu\varepsilon)^{-1}$. The proof now proceeds as does the proof of Theorem 4.3 with only minor differences. One has to use the fact that since $\varepsilon'_\mu\equiv-\varepsilon(\mathrm{mod}\,\pi^{\mu+1})$, we have $E_r(a\varepsilon'_\mu)=E_r(-a\varepsilon)$ for μ sufficiently large. One has also to keep in mind that the reasoning need be valid only for sufficiently large ν. We omit the details since they are so similar to those in the previous proof. ■

5. Applications to the Q-conjecture. In this section we give proofs of Theorems 1 to 4 of the Introduction. Theorems 1, 2, and 4 were proven in slightly different form in [7]. Shuck [8] gives more direct proofs of Theorems 2 (Th. 3.1) and 3 (Th. 3.6) as they apply to the original Borevich–Shafarevich conjecture.

Proof of Theorem 1. This is a straightforward consequence of the Definition 4.1 and the formula (2.12). ■

Proof of Theorem 2. It suffices by Proposition 3.2 and (4.1) to prove that each polynomial $h(X)\in\mathcal{O}[X]$ has a rational derivative θ_h with denominator a product of factors of the form $q-t^n$. We will prove this assertion, making use of the formula ([7], [8]) (see Appendix (2))

$$\theta_h(z)=\sum_{x\in h^{-1}(z)}|h'(x)|_K^{-1}\tag{5.1}$$

for almost all $z\in\mathcal{O}$.

Suppose first that $h(X)=X^n\big(\varepsilon+\pi^e g(X)\big)$ for $n\geqslant 1$, $\varepsilon\in U$, $e\geqslant 2v(n)+1$ and $g(X)\in\mathcal{O}[X]$. Then for $z\neq 0$, one sees easily by Hensel's Lemma that the roots x of $h(X)-z$ correspond one-to-one with those of εX^n-z. It is clear further that any such root x satisfies $|x|_K^n=|z|_K$ and $|h'(x)|_K=|nx^{n-1}|_K=|n|_K|z|_K^{1-1/n}$. Thus, we see from (5.1) that $\theta_h(z)=\theta_n(z)$, where θ_n is the function introduced in Section 2 above. By (2.14), θ_h is rational with the required denominator.

We now consider an arbitrary polynomial $h(X)\in\mathcal{O}[X]$. We cover each point $a\in\mathcal{O}$ with an open ball P_a as follows:

1) If $h(a)\neq 0$, then P_a is chosen so that $|h(x)|_K$ is bounded away from 0 for $x\in P_a$.

2) If $h(a)=0$, let $h(x)=b_n(x-a)^n+\ldots+b_d(x-a)^d$ be the Taylor expansion of $h(x)$ at a, $n\geqslant 1$, $b_n\neq 0$. Let $\mu=v(b_n)$, and put $P_a=a+\pi^\nu\mathcal{O}$ where $\nu=2v(n)+1+\mu$.

Since $\mathcal{O}$ is compact, there is a finite disjoint subcover $\mathscr{P}$ of the cover $\{P_a\}$, and we have

$$\theta_h(z)=\sum_{P\in\mathscr{P}}\theta_h(z;P)\tag{5.2}$$

where $\theta_h(z;P)$ is the Radon–Nikodym derivative of the restriction of h to P. Thus, it suffices to show that each $\theta_h(z;P)$, $P\in\mathscr{P}$, is rational with denominator of the form $q-t^n$. If $P=P_a$ is a ball of type 1 above, then $\theta_h(z;P)=0$ for all z with $v(z)$ sufficiently large. Therefore, $G_\theta(u,t)$ for $\theta(z)=\theta_h(z;P)$ is a polynomial in t and thus certainly is rational with the required denominator. If $P=P_a$ is a ball of type 2 above, then under the change of variable $x\to a+\pi^\nu x$ we find after a short calculation that

$$\theta_h(z;P)=q^{(n-1)\nu+\mu}\theta_{h^*}(\pi^{-n\nu-\mu}z),$$

where $h^*(X)=\pi^{-n\nu-\mu}h(a+\pi^\nu X)$ is a polynomial of the form considered in the second paragraph of this proof. Thus, $\theta_h(z;P)$ is rational with denominator $q-t^n$. ■

Proof of Theorem 3. By a *polydisk* in $\mathcal{O}^r$ we mean a set D which is the r-fold product of open balls in $\mathcal{O}$. Since an open ball in $\mathcal{O}$ has the form $z_0+\pi^{\nu_0}\mathcal{O}$ for $z_0 \in \mathcal{O}$ and $\nu_0 \geqslant 0$, each polydisk D is the image of $\mathcal{O}^r$ under a simple affine transformation T_D with coefficients from $\mathcal{O}$. Let $h \in \mathcal{O}[X_1, \ldots, X_r]$ and let $\theta_h(z; D)$ be the Radon–Nikodym derivative of h viewed as a map from D to $\mathcal{O}$. Then a simple calculation shows that

$$\theta_h(z; D) = |\det T_D|_K \theta_{hT_D}(z) \tag{5.3}$$

for all $z \in \mathcal{O}$.

Now let f be a form in $\mathcal{O}[X_1, \ldots, X_{s+1}]$ of degree d. Let E denote (generically) any subset of $\mathcal{O}^{s+1}$ which is a product of m copies of U and n copies of $\mathcal{O}$ taken in any order, where $m \geqslant 1$, $n \geqslant 0$ and $m+n = s+1$. Since each such E is a finite union of polydisks, we can utilize (5.3) in computing the derivative $\theta_f(z; E)$ of f viewed as a map from E to $\mathcal{O}$. We will show that each $\theta_f(z; E)$ is $(s+1)$-reflective. To that end, we assume, without loss of generality, that $E = U \times B$, where $B \subseteq \mathcal{O}^s$ is invariant under multiplication by units in U. We have then for a given L^1-function g,

$$\int_E g\big(f(X_1, \ldots, X_{s+1})\big)\,dX_1 \ldots dX_{s+1} = \int_E g\big(X_1^d f(1, X_2, \ldots, X_{s+1})\big)\,dX_1 \ldots dX_{s+1}$$

under the change of variables $X_1 \to X_1$ and $X_i \to X_1 X_i$ for $2 \leqslant i \leqslant s+1$. Therefore, f and the polynomial $f^*(X_1, \ldots, X_{s+1}) = X_1^d f(1, X_2, \ldots, X_{s+1})$ have the same derivative on E. If D is a polydisk in $\mathcal{O}^{s+1}$, then the polynomial f^*T_D remains expressible as a product in which one variable is separated from the others. We conclude from (2.19), Theorem 4.4 and (5.3) that $\theta_{f^*}(z; D)$ is $(s+1)$-reflective. Since E is a finite disjoint union of polydisks, $\theta_{f^*}(z; E)$ and hence also $\theta_f(z; E)$ is $(s+1)$-reflective.

Let D_0 be the polydisk in $\mathcal{O}^{s+1}$ which is the product of $s+1$ copies of $\pi\mathcal{O}$, and let $F_0 = \mathcal{O}^{s+1} \setminus D_0$. One observes by a simple combinatorial argument that the characteristic function of the subset F_0 of $\mathcal{O}^{s+1}$ is a Z-linear combination of characteristic functions of the sets E. Therefore, $\theta_f(z; F_0) = \theta_f(z) - \theta_f(z; D_0)$ is a Z-linear combination of the derivatives $\theta_f(z; E)$ and hence is $(s+1)$-reflective.

We next compute $\theta_f(z; D_0)$ in terms of $\theta_f(z)$. We have for $g \in L^1(\mathcal{O})$

$$\int_{D_0} g\big(f(X_1, \ldots, X_{s+1})\big)\,dX_1 \ldots dX_{s+1} = q^{-s-1} \int_{\mathcal{O}^s} g\big(\pi^d f(X_1, \ldots, X_{s+1})\big)\,dX_1 \ldots dX_{s+1}$$

under the change of variables $X_i \to \pi X_i$ for $1 \leqslant i \leqslant s+1$. Thus, $\theta_f(z; D_0) = q^{d-s-1}\theta_f(\pi^{-d}z)$, it being understood as usual that $\theta_f(x) = 0$ if $x \in K \setminus \mathcal{O}$.

We see now that the function

$$\varphi(z) = \theta_f(z) - q^{d-s-1}\theta_f(\pi^{-d}z) \tag{5.4}$$

is $(s+1)$-reflective.

From (5.4), we conclude for $c \in \mathcal{O}$, $c \neq 0$, $\nu > v(c)$ and $u \in U$ that

$$\theta_f(\pi^\nu u + c) = \sum_{\mu=0}^{[v(c)/d]} q^{\mu(d-s-1)} \varphi(\pi^{\nu-\mu d} u + \pi^{-\mu d} c), \tag{5.5}$$

where $[\cdot]$ denotes the greatest integer function.

Thus, the function $\theta_f(z+c)$ is certainly rational for every $c \neq 0$; furthermore, the denominator of the associated rational function (2.9) has poles which are to be found among the poles of the rational functions associated by (2.9) to the translates of φ by the values $\pi^{-\lambda}c$, $0 \leqslant \lambda \leqslant v(c)$. We learn also from (5.5) that the branch points of θ_f are a subset of $B_\varphi \cup \{0\}$. Therefore, in order to complete the proof, it remains only to show that $G_\theta(u, t)$ is rational with poles all less than or equal to q^{s+1} in absolute value. But from (5.4), $(1 - q^{-(s+1)}t^d)G_{\theta_f}(u, t) = G_\varphi(u, t)$. ∎

Proof of Theorem 4. Again by (4.1) it is sufficient to prove this for $c = 0$. Clearly (2.18), (2.19) and then Theorems 4.3, 4.4 imply the result. ∎

APPENDIX. THE RADON–NIKODYM DERIVATIVE

(For more details, the reader should consult [7] or [8].)

When h is viewed as a map from $\mathcal{O}^s$ to $\mathcal{O}$ it has a Radon–Nikodym derivative θ_h defined almost everywhere on $h(\mathcal{O}^s)$ by

$$\int_{\mathcal{O}^s} f\big(h(X)\big)\,dX = \int_{h(\mathcal{O}^s)} f(z)\theta_h(z)\,dz \tag{1}$$

for an integrable function f on $h(\mathcal{O}^s)$.

The p-adic Change of Variable Theorem (whose proof was sketched by A. Weil [10]) can be used to prove the existence of θ_h and provide a formula as the integral of the absolute value of an $(s-1)$-form over a p-adic manifold. We define ω to be the form

$$(\partial h/\partial X_i)^{-1} dX_1 \wedge \ldots \wedge dX_{i-1} \wedge dX_{i+1} \wedge \ldots \wedge dX_s$$

provided that the partial derivative is non-zero. It is an exercise in differential geometry to show that ω on $M_z = \{X \in \mathcal{O}^s\colon h(X) = z\}$ does not depend on i except for sign.

THEOREM. *For all $z \notin h(S_h)$,*

$$\theta_h(z) = \int_{M_z} |\omega|_K.$$

The function θ_h is continuous at all such z's and locally constant.

COROLLARY. *It is immediate that, if* $s = 1$,

(2) $$\theta_h(z) = \sum_{h(a)=z} |h'(a)|_K^{-1}$$

for z *such that* $h'(a) \neq 0$.

Proof of theorem. Let $T \subset \mathcal{O}^s$ be an open and closed set containing S_h and consider h on the compact open set $\mathcal{O}^s \setminus T = W$. Partition W into a finite number of disjoint open sets on each of which at least one partial is non-zero so $W = \bigcup_\lambda W_\lambda$ and let i_λ be the minimum value of i such that $\partial h/\partial X_i \neq 0$ on W_λ. Let H_λ be defined on W_λ by $H_\lambda(X) = Y$ where $Y_i = X_i$ except $Y_{i_\lambda} = h(X)$ so the Jacobian J_H of H on W_λ is $\partial h/\partial X_i \neq 0$.

By the Implicit Function Theorem the analytic transformation H_λ is bijective on V_λ where $\{V_\lambda\}$ further partitions W into finitely many disjoint open sets and (suppressing the λ's) has a local inverse H^{-1}, also an analytic transformation mapping Y in $H(V)$ to $X = H^{-1}(Y)$ where $X_j = Y_j$ for $j \neq i$ and $X_i = \varphi(Y)$ (i.e., for $j = i_\lambda$).

For any L^1 function F on $\mathcal{O}^s$, the Change of Variable Theorem gives

$$\int_V F\big(H(X)\big)\,dX = \int_{H(V)} F(Y)\big|\det J_H\big(H^{-1}(Y)\big)\big|_K^{-1}\,dY.$$

Now

$$\int_W f\big(h(X)\big)\,dX = \int_W \sum_V \chi_V(X) f\big(h(X)\big)\,dX$$

where $\{\chi_V\}$ is a partition of unity on $\{V\}$ and f is in $L^1\big(h(\mathcal{O}^s)\big)$. Let $F = f\circ\pi_j$ where $\pi_j\colon K^s \to K$ is the projection of the jth component of K^s onto K so that $f\big(h(X)\big) = F\big(H(X)\big)$.

Thus

$$\int_W f\big(h(X)\big)\,dX = \sum_V \int_{H(V)} F(Y)\big|\det J_H\big(H^{-1}(Y)\big)\big|_K^{-1}\,dY.$$

Now by Fubini's Theorem we can write the right side as

$$\sum_V \int_{h(V)} f(Y_i)\,dY_i \int_{V'} \left|\frac{\partial h}{\partial X_i}\big(H^{-1}(Y)\big)\right|_K^{-1} dY_1 \ldots dY_{i-1}dY_{i+1}\ldots dY_s$$

where V' is the projection of V (i.e., V_λ) into K^{s-1} which omits the ith component. Since the sum is finite, (1) implies that

$$\theta_h^{(T)}(z) = \sum_V \int_{V'} \left|\frac{\partial h}{\partial X_i}\big(H^{-1}(Y)\big)\right|_K^{-1} dY_1 \ldots dY_{i-1}dY_{i+1}\ldots dY_s$$

for almost all z in $h(W)$ and h is viewed as a map $W \to \mathcal{O}$.

Now, for z in $h(W)$ the integral $\int|\omega|_K$, where integration is over $M_z \cap W$, can be evaluated to obtain the same result. This establishes the formula above for almost all z in $h(W)$. Since the integral is well-defined and continuous (see below) for all $z \notin h(T)$ it gives a continuous Radon–Nikodym derivative for such z. Let $\theta_h^{(T)}(z) = \int|\omega|_K$ where the integration is over $M_z \cap W$ for any $z \notin h(S_h)$ and consider

(*) $$\int_W f\big(h(X)\big)\,dX = \int_{h(W)} \theta_h^{(T)}(z) f(z)\,dz$$

for a positive continuous function f on the compact set $h(\mathcal{O}^s)$. The filter of open sets T containing S_h has a countable base. The numbers corresponding on the left side above, form a bounded upward directed net of positive numbers.

So, by the dominated convergence theorem

$$\lim_{T\downarrow S_h} \int_{h(W)} \theta_h^{(T)}(z) f(z)\,dz$$

exists.

The restriction of f to positive continuous functions can be eliminated and the result extended to all f continuous with compact support, and hence to all integrable f, i.e., $f\circ h \in L^1(\mathcal{O}^s)$. Thus by (*),

$$\lim_{T\downarrow S_h} \theta_h^{(T)}(z)$$

is, in fact, the Radon–Nikodym derivative for all $z \notin h(S_h)$ and is given by the formula $\lim_{T\downarrow S_h} \int|\omega|_K$ where integration is over $M_z \cap W$.

For $z \notin h(S_h)$ $\int_{M_z} |\omega|_K$ is the result.

The explicit representation of $\theta_h(z)$ as

$$\sum_\lambda \int_{K^{s-1}} \left|\frac{\partial h}{\partial X_i}\big(Y_1, \ldots, Y_{i-1}, \varphi(Y), Y_{i+1}, \ldots, Y_s\big)\right|_K^{-1} \chi_V(Y)\,dY_1 \ldots \ldots dY_{i-1}dY_{i+1}\ldots dY_s$$

where $Y_i = z$ gives the parametrization on each V of $M_z \cap W$. It is clear that θ_h is locally constant there since $|\partial h/\partial X_i(Y')|_K$ is.

In particular then, $\theta_h(z)$ is continuous for $z \notin h(S_h)$ and in $L^1\big(h(\mathcal{O}^s)\big)$ since for $f = \chi_{\mathcal{O}^s}$,

$$\int_{h(\mathcal{O}^s)} \theta_h(z)\,dz = \int_{\mathcal{O}^s} dX = 1. \blacksquare$$

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Über die Verteilung der Primzahlen in Folgen der Form $[f(n+x)]$, II

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Es bezeichne $\mathscr{F}_{\text{pol}}$ die Menge aller verallgemeinerten Polynomfunktionen

$$f(y) = ay^k + \sum_{i=1}^{m} a_i y^{k_i}$$

mit positiven Leitkoeffizienten a, reellen a_i und monoton fallenden Exponenten $k > k_1 > \ldots > k_m \geqslant 0$. In [2] war die harmonische Dichte $D_{\text{har}}^{\text{har}}(P, f)(x)$ der Menge P aller Primzahlen in der Folge $[f(n+x)]_{n\in N}$ definiert worden als der (nicht immer existierende) Limes

$$D_{\text{har}}^{\text{har}}(P, f)(x) = \lim_{N\to\infty} \frac{\sum\limits_{1\leqslant n\leqslant N, [f(n+x)]\in P} 1/n}{\sum\limits_{1\leqslant n\leqslant N, n\in P} 1/n}.$$

Diese Dichte hat die Eigenschaft:

Satz. *Für jede Funktion f aus $\mathscr{F}_{\text{pol}}$ vom Grad $k > 12/5$ und für fast alle x (im Sinne des Lebesgue-Maßes) aus dem Intervall $0 \leqslant x < 1$ ist*

$$D_{\text{har}}^{\text{har}}(P, f)(x) = 1/k.$$

Diese Behauptung wurde als Satz 11 (ii) — und zwar für alle f aus $\mathscr{F}_{\text{pol}}$ vom Grad $k > 2$ — schon in [2], § 7, bewiesen, dort aber unter der Voraussetzung, daß der Primzahlsatz in der Form

$$\text{(PZS 2)} \qquad \pi(y) = \operatorname{li} y + O(\sqrt{y}\log y)$$

gültig ist, und die Richtigkeit dieser Abschätzung ist äquivalent zur Richtigkeit der Riemannschen Vermutung.

Satz 11 (i) in [2], § 7, machte eine ähnliche Dichteaussage für exponentiell wachsende Funktionen f, und zum Beweis dieser Aussage reichte der schwächere (und bewiesene) Primzahlsatz

$$\text{(PZS 1)} \qquad \pi(y) = \operatorname{li} y + O(y\exp(-\log^{1/2} y)).$$