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Integral inequalities with weights for the Hardy maximal function*

by

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Abstract. Necessary and sufficient conditions are obtained in order that inequalities of the form

$$\int_{R^n} \Phi\big((Mf)(x)\big) w(x)\,dx \leqslant C \int_{R^n} \Phi\big(|f(x)|\big) w(x)\,dx$$

hold, where Mf is the Hardy maximal function of f and Φ is an appropriate Young's function. This result gives similar inequalities for the usual singular integral operators.

1. Our aim is to study weighted integral inequalities involving the maximal function operator M defined for Lebesgue-measurable f on R^n by

$$(Mf)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy, \qquad x \in R^n;$$

as is always the case below, Q is a nondegenerate cube with sides parallel to the axes. More specifically, we extend to the context of Orlicz classes the result of B. Muckenhoupt, [4], for Lebesgue classes:

$$\int_{R^n} [(Mf)(x)]^p w(x)\,dx \leqslant C \int_{R^n} |f(x)|^p w(x)\,dx,$$

p fixed, $1 < p < \infty$, and C independent of Lebesgue-measurable f, if and only if $w(x)$ is in the class A_p of those weight functions for which

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}\,dx\right)^{p-1} \leqslant K,$$

for all cubes Q.

The integral inequalities of interest to us are of the form

$$\int_{R^n} \Phi\big((Mf)(x)\big) w(x)\,dx \leqslant C \int_{R^n} \Phi\big(|f(x)|\big) w(x)\,dx.$$

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The Young's functions $\Phi(t)$ involved is given by

$$\Phi(t)=\int_0^t \varphi(u)\,du, \quad t>0,$$

where $\varphi(u)$ is a nondecreasing function defined for $u>0$ with $\varphi(0^+)=0$. We require that $\Phi(t)$ satisfies the Δ_2 condition

$$\Phi(2t)\leqslant B\Phi(t), \quad t>0,$$

which is equivalent to the more general property

$$\Phi(At)\leqslant B\Phi(t), \quad t>0$$

(with possibly different B). It is also important that the Young's function $\Psi(t)=\int_0^t \varphi^{-1}(u)\,du$, complementary to $\Phi(t)$, obey the Δ_2 condition. (Here $\varphi^{-1}(u)=\sup\{s:\ \varphi(s)\leqslant u\}$.) These restrictions ensure that $\lim\limits_{t\to 0+}\varphi(t)=\lim\limits_{t\to 0+}\varphi^{-1}(t)=0$ and $\lim\limits_{t\to\infty}\varphi(t)=\lim\limits_{t\to\infty}\varphi^{-1}(t)=\infty$, and hence that these functions are equivalent to strictly monotonic ones. We will make use of the following properties of $\Phi(t)$ without explicit reference:

(i) $\Phi(t)$ is essentially equal to $t\varphi(t)$,

(ii) $t\leqslant \Phi^{-1}(t)\Psi^{-1}(t)\leqslant 2t$.

The Orlicz space $L_\Phi=L_\Phi(w)$, $w(x)$ positive and locally-integrable on R^n, consists of all Lebesgue-measurable functions on R^n for which there is a $K>0$ such that

$$\int_{R^n}\Phi\big(|f(x)|/K\big)w(x)\,dx\leqslant 1.$$

The norm of f in L_Φ is the infimum over all such K. Under our restrictions on $\Phi(t)$ and $\Psi(t)$, the spaces L_Φ and L_Ψ are mutually dual and, in particular, are reflexive.

Matuszewska and Orlicz, [3], have associated a pair of indices with a given L_Φ. A generalization of these, or rather their reciprocals, has been given in the more general context of rearrangement invariant spaces in Boyd [1]. There, the upper and lower indices α and β are defined by

$$\alpha=\inf_{0<s<1}-\frac{\ln h(s)}{\ln s}=\lim_{s\to 0+}-\frac{\ln h(s)}{\ln s}$$

and

$$\beta=\sup_{1<s<\infty}-\frac{\ln h(s)}{\ln s}=\lim_{s\to\infty}-\frac{\ln h(s)}{\ln s},$$

where, for Orlicz spaces,

$$h(s)=\sup_{t>0}\frac{\Phi^{-1}(t)}{\Phi^{-1}(st)}.$$

We refer to [1] for a complete discussion of their properties, some of which will be introduced below as needed. We just mention that for the $\Phi(t)$ we consider, $0<\beta\leqslant\alpha<1$; that in the case of Lebesgue spaces, L_p, when $\Phi(t)=t^p$, one has $\alpha=\beta=p^{-1}$.

We now state our main result.

THEOREM 1. *Let $w(x)$ be a positive, locally-integrable function on R^n and let $\Phi(t)=\int_0^t\varphi(u)\,du$ be a Young's function which, together with it complementary function $\Psi(t)$, satisfies the Δ_2 condition. Then, in order that the inequality*

$$\text{(1)}\qquad \int_{R^n}\Phi\big((Mf)(x)\big)w(x)\,dx\leqslant C\int_{R^n}\Phi\big(|f(x)|\big)w(x)\,dx$$

be valid for C independent of f, it is necessary and sufficient that either one of the following holds:

(2) *$w(x)$ is in the class A_Φ; that is,*

$$\Big(\frac{1}{|Q|}\int_Q \varepsilon w(x)\,dx\Big)\varphi\Big(\frac{1}{|Q|}\int_Q\varphi^{-1}\big(1/\varepsilon w(x)\big)\,dx\Big)\leqslant K$$

for all cubes Q and all $\varepsilon>0$;

(3) *$w(x)$ is in the class A_p, where p^{-1} is the upper index of L_Φ; that is*

$$p^{-1}=\lim_{s\to 0+}-\frac{\ln h(s)}{\ln s},\qquad h(s)=\sup_{t>0}\frac{\Phi^{-1}(t)}{\Phi^{-1}(st)}.$$

In § 2 we show that (2) is necessary for (1), in § 3 that (2) implies (3), which in turn is sufficient for (1).

Finally, arguments similar to those of [2], Theorem III, show that, given $w(x)\in A_\Phi$, integral inequalities of the form (1) hold for the usual singular integral operators. Indeed, for the Hilbert transformation, the condition $w(x)\in A_\Phi$ is also necessary.

2. It will be enough to obtain the condition of (2) with $\varepsilon=1$, provided that K is seen to depend only on C. To begin, we claim there is a constant C_1 so that for all cubes Q and all $\varepsilon>0$

$$\text{(4)}\qquad \|\chi_Q\|_\varepsilon\|\chi_Q/\varepsilon w\|'_\varepsilon\leqslant C_1|Q|.$$

Here $\|\ \|_\varepsilon$ denotes the norm in $L_\Phi(\varepsilon w)$; $\|\ \|'_\varepsilon$ the norm in $L_\Psi(\varepsilon w)$. Firstly, our assumptions on $w(x)$ ensure that $0<B<\infty$, where $B=\|\chi_Q/\varepsilon w\|'_\varepsilon$.

For, $B = 0$ implies that the $L_1(\varepsilon w)$ norm of $\chi_Q/\varepsilon w$ is zero, which means, in turn, that $|Q| = 0$, making Q a degenerate cube. Again, $B = \infty$ requires the existence of a nonnegative function f in $L_\Phi(\varepsilon w)$ on Q with $\int_Q f(x)\,dx = \infty$. This forces $Mf \equiv \infty$ on Q, which isn't consistent with (1) if Q is nondegenerate.

Next, the converse of Hölder's inequality allows us to choose a nonnegative function f, supported on Q, so that $\|f\|_\varepsilon = 1$ and $\int_Q f(x)\,dx = \|\chi_Q/\varepsilon w\|'_\varepsilon$. Then, for $x \in Q$,

$$(Mf)(x) \geqslant (\|\chi_Q/\varepsilon w\|'_\varepsilon/|Q|)\,\chi_Q(x),$$

and so

$$\int_Q \Phi(\|\chi_Q/\varepsilon w\|'_\varepsilon/|Q|)\,\varepsilon w(x)\,dx \leqslant C\int_Q \Phi(f(x))\,\varepsilon w(x)\,dx = C;$$

that is,

$$\Phi(\|\chi_Q/\varepsilon w\|'_\varepsilon/|Q|)\,\varepsilon w(Q) \leqslant C.$$

On taking $C_1 = h(C^{-1})$, (4) follows.

Now, from the definition of $\|\chi_Q/\varepsilon w\|'_\varepsilon$ and (4),

$$(C_2\|\chi_Q\|_\varepsilon/|Q|)\int_Q \varphi^{-1}\big(C_2\|\chi_Q\|_\varepsilon/|Q|\,\varepsilon w(x)\big)\,dx \leqslant B_1,$$

where $t\varphi^{-1}(t) \leqslant B_1\Psi(t)$ for all $t > 0$ and $C_2 = C_1^{-1}$. Let $\varepsilon > 0$ satisfy $C_2\|\chi_Q\|_\varepsilon/\varepsilon|Q| = 1$. Such an ε exists since the left hand side of the equation is a continuous function of ε which tends to infinity as $\varepsilon \to 0_+$ and to zero as $\varepsilon \to \infty$. Indeed, since $\|\chi_Q\|_\varepsilon = 1/\Phi^{-1}(1/\varepsilon w(Q))$,

$$C_2\|\chi_Q\|_\varepsilon/\varepsilon|Q| = C_2\big[\varepsilon|Q|\Phi^{-1}\big(1/\varepsilon w(Q)\big)\big]^{-1}$$

is essentially equal to $C_2 w(Q)\Psi^{-1}(1/\varepsilon w(Q))/|Q|$, which means the desired ε is essentially equal to $[w(Q)\Psi(|Q|C_1/w(Q))]^{-1}$. We thus have, for some B_2 comparable to B_1,

$$\int_Q \varphi^{-1}\big(1/w(x)\big)\,dx \leqslant B_2\big[w(Q)\Psi\big(|Q|C_1/w(Q)\big)\big] \leqslant B_2C_1|Q|\,\varphi^{-1}\big(|Q|C_1/w(Q)\big)$$

yielding (2) with $K = B_3B_2^{-1}$, where B_3 corresponds to $A = 2B_2C_1$ in the generalized Δ_2 condition for $\Phi(t)$.

3. In this section we prove that (2) implies (3) and that (3) suffices for (1). The former is a consequence of the three results proved below and the following interpolation criterion due to Stein and Weiss [6].

Suppose T is a sublinear operator defined for functions χ_E, E a subset of R^n of finite Lebesgue measure, and that $w(x)$ is a nonnegative, locally-integrable function on R^n. Suppose, further, T is simultaneously of restricted weak-types (p_1, p_1) and (p_2, p_2), $1 < p_1 < p_2 < \infty$, with respect to $w(x)$:

$$\int_{\{T\chi_E > \lambda\}} w(x)\,dx \leqslant Cw(E)\lambda^{-p_i}, \quad i = 1, 2,$$

with C independent of the set E and the positive number λ. Then T is bounded from $L_p(w)$ to itself, provided $p_1 < p < p_2$.

The first of the following results seems to be of some independent interest, particularly as it relates to the A_∞ condition; see [2] and [5].

PROPOSITION 1. *For $w(x)$ a positive, locally-integrable function on R^n, the restricted weak-type (p, p) inequality*

$$\int_{\{M\chi_E > \lambda\}} w(x)\,dx \leqslant Cw(E)\lambda^{-p}, \quad 1 \leqslant p < \infty, \tag{5}$$

with C independent of the Lebesgue-measurable set E and the positive number λ, is equivalent to the existence of a positive constant K such that for all cubes Q and all Lebesgue-measurable $E \subset Q$

$$|E|/|Q| \leqslant K[w(E)/w(Q)]^{1/p}. \tag{6}$$

Proof. Condition (6) is an immediate consequence of (5) and the fact that

$$M\chi_E \geqslant |E|/|Q|\chi_Q.$$

Assume that (6) holds for $w(x)$. Then

$$M\chi_E \leqslant K[M_w\chi_E]^{1/p},$$

where the maximal function operator M_w is given by

$$(M_wf)(x) = \sup_{x\in Q}(1/w(Q))\int_Q |f(y)|\,w(y)\,dy.$$

Clearly, $w(x)$ satisfies the doubling condition

$$w(Q^*) \leqslant Cw(Q),$$

where Q^* is the double of Q. Thus, as pointed out in [2], Lemma 1, M_w is of weak-type (1,1) with respect to $w(x)$. The inequality (5) is now seen to hold with $C = C_1K^p$, C_1 being a weak-type (1,1) bound for M_w.

LEMMA 1.[1] *Let Φ and p be as in Theorem 1. Then $w(x) \in A_\Phi$ implies $w(x) \in A_r$ whenever $r > p$.*

Proof. Given Proposition 1 and the interpolation criterion stated above, it is enough to show (6).

Let Q be a cube and let E be a Lebesgue-measurable subset of Q. We have, successively, by Hölder's inequality and (2), that $|E|/|Q|$ is

[1] We wish to thank J.-O. Strömberg for pointing out an error in the original proof of this result.

bounded above by

$$\|\chi_E\|_\varepsilon \|\chi_E/\varepsilon w\|'_\varepsilon/|Q| \leqslant C\,\Phi^{-1}\big(w(E)/\varepsilon w(Q)w(E)\big)/\Phi^{-1}\big(1/\varepsilon w(E)\big).$$

The latter term, however, is less than $2C\big(w(E)/w(Q)\big)^{1/p}$. For, $h_\Phi(s) \geqslant s^{-1/p}$ when $0<s<1$. This means that for fixed $s<1$ there is a $t>0$ with $\Phi^{-1}(t)/\Phi^{-1}(st) > s^{-1/p}/2$ and so $\Phi^{-1}(st)/\Phi^{-1}(t) < 2s^{1/p}$. Taking $s = w(E)/w(Q)$ and $\varepsilon = 1/tw(Q)$ yields (6).

Lemma 2. *Let Φ and p be as in Theorem 1. For $\delta>0$ define the Young's unction Φ_δ by the equation*

$$\varphi_\delta^{-1}(t) = \big(\varphi^{-1}(t)\big)^{1+\delta}.$$

Then, the upper index of L_{Φ_δ} is greater than p^{-1}. Moreover, if $w(x)\in A_\Phi$, then $w(x)\in A_{\Phi_\delta}$ for all sufficiently small δ.

Proof. To prove the second assertion it will be enough to establish the condition of (2) for Φ_δ when $\varepsilon=1$, provided it is seen the C depends only on K.

Set $v(x) = \varphi^{-1}\big(1/w(x)\big)$. Then $w(x) = 1/\varphi\big(v(x)\big)$ and $w(x)\in A_\Phi$ implies

$$\Big((1/|Q|)\int_Q \big(1/\varphi\big(v(x)\big)\big)dx\Big)\,\varphi\big(v(Q)/|Q|\big) \leqslant K. \tag{7}$$

We show there exist $\alpha,\beta,>0$, independent of Q, so that for $E = \{x\in Q: v(x) > \alpha v(Q)/|Q|\}$ we have $|E|\geqslant\beta|Q|$. On the complement of E in Q, E^c, $v(x)\leqslant \alpha v(Q)/|Q|$ and so, using (7),

$$|E^c|/|Q|\varphi\big(\alpha v(Q)/|Q|\big) \leqslant K/\varphi\big(v(Q)/|Q|\big).$$

Therefore,

$$|E^c|/|Q| \leqslant K\Big(\varphi\big(\alpha v(Q)/|Q|\big)/\varphi\big(v(Q)/|Q|\big)\Big) \leqslant K\Big(\Phi\big(2\alpha v(Q)/|Q|\big)/\alpha\Phi\big(v(Q)/|Q|\big)\Big).$$

As established below, given a fixed $r<p$ there is an s_0, $0<s_0<1$, with

$$\Phi(st) \leqslant s_0^{-r}s^r\Phi(t),\quad t>0,\ 0<s<1. \tag{8}$$

This will mean $|E^c|/|Q| \leqslant Ks_0^{-r}2^r\alpha^{r-1} < 1/2$ for small α.

Arguing as in [2], Theorem IV, we have the "reverse Hölder inequality"

$$\Big((1/|Q|)\int_Q \varphi_\delta^{-1}\big(1/w(x)\big)dx\Big)^{1/(1+\delta)} \leqslant C_1(1/|Q|)\int_Q \varphi^{-1}\big(1/w(x)\big)dx,$$

for all sufficiently small δ. Thus,

$$\varphi_\delta\Big((1/|Q|)\int_Q \varphi_\delta^{-1}\big(1/w(x)\big)dx\Big) \leqslant \varphi\Big(C_1(1/|Q|)\int_Q \varphi^{-1}\big(1/w(x)\big)dx\Big)$$

and the latter, by the generalized Δ_2 condition for Φ and by (7), is no bigger than

$$C_2\varphi\Big((1/|Q|)\int_Q \varphi^{-1}\big(1/w(x)\big)dx\Big) \leqslant C|Q|/w(Q),$$

with $C = C_2K$. Hence $w(x)\in A_{\Phi_\delta}$ for all small δ.

To see (8), observe that there exists s_0, $0<s_0<1$, with $h_\Phi(s)\leqslant s^{-1/r}$ when $0<s<s_0^r$. Thus

$$\Phi^{-1}(t)/\Phi^{-1}(st) < s^{-1/r},\quad t>0,\ 0<s<s_0^r,$$

that is,

$$\Phi(st)\leqslant s^r\Phi(t),\quad t>0,\ 0<s<s_0.$$

For $s_0\leqslant s<1$, (8) follows from the fact that Φ increases.

Finally, we show the upper index of L_{Φ_δ} is greater than p^{-1}. By duality, it will be sufficient to prove the associate space, L_{Ψ_δ}, has lower index less than q^{-1}, $(p^{-1}+q^{-1}=1)$, the lower index of L_Ψ. Given $\varepsilon>0$, we have, for fixed, sufficiently large $s>1$,

$$s^{-1/q}\leqslant h_\Psi(s)\leqslant s^{-1/q+\varepsilon},$$

where $h_\Psi(s) = \sup_{t>0}\big(\Psi^{-1}(t)/\Psi^{-1}(st)\big)$. Thus,

$$\Psi^{-1}(t)/\Psi^{-1}(st)\leqslant s^{-1/q+\varepsilon} = s^{-a},$$

for all $t>0$ and

$$s^{-1/q}\leqslant \Psi^{-1}(t)/\Psi^{-1}(st)$$

for some $t>0$. With $t=\Psi(\tau)$ the former gives

$$\Psi(s^a\tau)/\Psi(\tau)\leqslant s.$$

From Ψ satisfying the Δ_2 condition we infer that $\Psi_\delta(t)$ essentially equals $\Psi(t)^{1+\delta}/t^\delta$, and so the last inequality reads

$$(s^a\tau)^\delta\Psi_\delta(s^a\tau)/\tau^\delta\Psi_\delta(\tau)\leqslant Ks^{1+\delta}.$$

Letting $T=\Psi_\delta(\tau)$ and $\sigma = s^{(1-a)\delta+1}$ yields

$$\Psi_\delta^{-1}(T)/\Psi_\delta^{-1}(\sigma T)\leqslant K_1\sigma^b,\quad b = (-q^{-1}+\varepsilon)/\big(1+\delta(p^{-1}+\varepsilon)\big),$$

for fixed large σ and all $T>0$. Similarly,

$$K_2\sigma^c\leqslant \Psi_\delta^{-1}(T)/\Psi_\delta^{-1}(\sigma T),\quad c = -q^{-1}/(1+\delta p^{-1}),$$

for fixed large σ and some $T>0$. This shows the lower index of L_{Ψ_δ} must equal $q^{-1}/(1+\delta p^{-1}) < q^{-1}$.

Finally, we establish the

Sufficiency of (3). From the well-known "openness" of the condition for membership in A_p, w belongs to A_{r_0} for some r_0 with $1 < r_0 < p$, where p^{-1} is the upper index of L_Φ. Then, for Lebesgue-measurable f,

$$\int_{R^n} \Phi\big((Mf)(x)\big) w(x)\,dx \leqslant C \int_0^\infty w(\{Mf > s\}) \Phi(s)\,(ds/s)$$

$$\leqslant C \int_0^\infty \big(\Phi(s)/s^{r_0}\big) \Big(\int_{|f(x)|>s} |f(x)|^{r_0} w(x)\,dx \Big) (ds/s).$$

Interchanging the order of integration gives

$$\int_{R^n} |f(x)|^{r_0} w(x) \Big(\int_0^{|f(x)|} \big(\Phi(s)/s^{r_0}\big) (ds/s) \Big)\,dx.$$

But, using (8) with, say, $r = (r_0+p)/2$, means this is dominated by a constant multiple of $\int_{R^n} \Phi\big(|f(x)|\big) w(x)\,dx$. This completes our proof.

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On the convergence of bilinear and quadratic forms in independent random variables

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Abstract. We consider bilinear and quadratic forms $\sum a_{ij} X_i Y_j$ and $\sum a_{ij} X_i X_j$ in independent random variables with expectations 0 and variances 1. Necessary and sufficient conditions for these forms to converge a.s. are given. When the X_i and Y_j are normal, we consider $X = (X_i)$ and $Y = (Y_j)$ as vectors in $\boldsymbol{R}^N$ and ask when $\sum a_{ij} X_i Y_j$ converges for X and Y in a subspace of $\boldsymbol{R}^N$ of measure 1 for the distribution law. This is proved to happen precisely when the a_{ij} define a nuclear operator on l^2. The natural extension of this theorem to trilinear forms is shown to be false. An analogous result for stochastic integrals is also given.

1. Introduction and statements of results.

In this paper all random variables and coefficients will be real-valued. However, our results extend to the complex-valued case with only small modifications. The probability measure is denoted by P.

We shall say that a set of random variables *stays away from* 0 if it contains no sequence tending to 0 in probability, i.e., if there is an $\varepsilon > 0$ such that $P(|X| \geqslant \varepsilon) \geqslant \varepsilon$ for all X in the set.

Linear forms $\sum a_i X_i$ have been considered by Hoffmann-Jørgensen [2], Th. 4.10. If the X_i are independent, stay away from 0, and satisfy $EX_i = 0$ and $EX_i^2 = 1$, then the condition $\sum a_i^2 < \infty$ is necessary and sufficient for $\sum a_i X_i$ to converge a.s. Notice that this conclusion holds if and only if the X_i stay away from 0, when the other assumptions are satisfied.

For bilinear forms, several kinds of convergence exist. Call (M_k, N_k) an *admissible sequence* if each M_k and N_k is a natural number or ∞, and M_k and N_k increase to ∞ with k, and both M_k and N_k are not ∞. A bilinear form $\sum' a_{ij} X_i Y_j$ is said to *converge* for such a sequence if $\sum_{i<M_k, j<N_k} a_{ij} X_i Y_j$ converges as $k \to \infty$. Hoffmann-Jørgensen's techniques [2], pp. 155–156, are easily modified to give the following result.

THEOREM 1. *Let X_i and Y_j, $i, j = 1, 2, \ldots,$ be independent, have expectation 0 and variance 1, and stay away from 0. Then the following are equivalent:*