## References


[1] R. Cairoli, *Une inégalité pour martingales à indices multiples et ses applications*, *Séminaire de probabilité IV*, Lecture Notes in Mathematics 124, 1–27, Springer-Verlag, 1970.

[2] J. Hoffmann-Jørgensen, *Integrability of seminorms, the 0-1 law and the affine kernel for product measures*, Studia Math. 61 (1977), 137–159.

[3] A.M. Mantero & A. Tonge, *The Schur multiplication in tensor algebras*, ibid. 68 (1980), 1–24.

[4] M. Schreiber, *Fermeture en probabilité des chaos de Wiener*, C.R. Acad. Sci. Paris, Sér. A 265 (1967), 859–861.

[5] P. Sjögren, *Almost sure convergence of bilinear and quadratic forms in independent random variables*, Uppsala University, Department of Mathematics, Report 13 (1977).

[6] — *Convergence of Riemann sums for stochastic integrals*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 56 (1981), 181–193.



DEPARTMENT OF MATHEMATICS
CHALMERS UNIVERSITY OF TECHNOLOGY
UNIVERSITY OF GÖTEBORG
S-412 96 GÖTEBORG, SWEDEN



*Received February 12, 1979* (1514)
*Revised version October 8, 1979*


**STUDIA MATHEMATICA, T. LXXI. (1982)**

# Colacunary sequences in $L$-spaces

by

D. J. ALDOUS (Cambridge) and D.H. FREMLIN (Colchester)

**Abstract.** A sequence $\langle x_n\rangle_{n\in N}$ in a normed space is 2-colacunary if there is a continuous linear map from $\operatorname{lin}\{x_n\}$ to $l^2$ taking each $x_n$ to the $n$th basic unit vector of $l^2$. Our main result is that any sequence in an $L$-space which is bounded but not totally bounded has a 2-colacunary subsequence.

**1.** DEFINITION. Let $E$ be a normed space. A sequence $\langle x_n\rangle_{n\in N}$ in $E$ is 2-*colacunary* if there is a $\delta > 0$ such that

$$\Big\|\sum_{i\leqslant n} a_i x_i\Big\| \geqslant \delta\Big(\sum_{i\leqslant n}|a_i|^2\Big)^{1/2}$$

for any finite sequence $a_0, \ldots, a_n$ of scalars.

For "lacunary" sequences see [7]. Of course one can define "$p$-colacunary" sequences similarly for any $p \in ]0, \infty]$; and the results of §§2 and 9 below will generalise. But the central results of this paper (§§4–6) are special to $p = 2$.

**2.** LEMMA. *Let $E$ be a normed space, $\langle x_n\rangle_{n\in N}$ a sequence in $E$.*

(a) *If $\langle x_n\rangle_{n\in N}$ is 2-colacunary and $x \in E$, there is an $m \in N$ such that $\langle x_n + x\rangle_{n\geqslant m}$ is 2-colacunary.*

(b) *If $\langle x_n\rangle_{n\in N}$ is 2-colacunary and $\langle y_n\rangle_{n\in N}$ is another sequence in $E$ such that $\sum_{n\in N}\|x_n - y_n\| < \infty$, then $\langle y_n\rangle_{n\geqslant m}$ is 2-colacunary for some $m \in N$.*

(c) *If $F$ is another normed space and $T\colon E\to F$ a continuous linear operator such that $\langle Tx_n\rangle_{n\in N}$ is 2-colacunary, then $\langle x_n\rangle_{n\in N}$ is 2-colacunary.*

Proof. (a) Let $\sigma > 0$ be such that

$$\Big\|\sum_{i\leqslant n} a_i x_i\Big\| \geqslant 2\sigma\Big(\sum_{i\leqslant n}|a_i|^2\Big)^{1/2} \quad \forall a_0, \ldots, a_n.$$

Suppose, if possible, that $\langle x_n + x\rangle_{n\geqslant m}$ is never 2-colacunary. Then in particular $\langle x_n + x\rangle_{n\in N}$ is not 2-colacunary, so there are $a_0, \ldots, a_k$ such that

$$\Big\|\sum_{i\leqslant k} a_i(x_i + x)\Big\| < \sigma\Big(\sum_{i\leqslant k}|a_i|^2\Big)^{1/2}.$$

Because $\|\sum_{i\leqslant k}\alpha_i x_i\| \geqslant 2\sigma(\sum_{i\leqslant k}|\alpha_i|^2)^{1/2}$, $\sum_{i\leqslant k}\alpha_i$ cannot be $0$; multiplying through by a scalar if necessary, we may suppose that $\sum_{i\leqslant k}\alpha_i = 1$. Next, $\langle x_n + x\rangle_{n\geqslant k+1}$ is supposed not to be 2-colacunary, so there must be $\alpha_{k+1},\ldots,\alpha_r$ such that

$$\Big\|\sum_{k+1\leqslant i\leqslant r}\alpha_i(x_i+x)\Big\| < \sigma\Big(\sum_{k+1\leqslant i\leqslant r}|\alpha_i|^2\Big)^{1/2},$$

$$\sum_{k+1\leqslant i\leqslant r}\alpha_i = -1.$$

But now

$$\Big\|\sum_{i\leqslant r}\alpha_i x_i\Big\| = \Big\|\sum_{i\leqslant r}\alpha_i(x_i+x)\Big\| < \sigma\Big(\sum_{i\leqslant k}|\alpha_i|^2\Big)^{1/2} + \sigma\Big(\sum_{k+1\leqslant i\leqslant r}|\alpha_i|^2\Big)^{1/2} \leqslant 2\sigma\Big(\sum_{i\leqslant r}|\alpha_i|^2\Big)^{1/2},$$

which is impossible.

(b) Again let $\sigma > 0$ be such that

$$\Big\|\sum_{i\leqslant n}\alpha_i x_i\Big\| \geqslant 2\sigma\Big(\sum_{i\leqslant n}|\alpha_i|^2\Big)^{1/2} \quad \forall \alpha_0,\ldots,\alpha_n.$$

Let $m$ be such that

$$\sum_{i\geqslant m}\|y_i - x_i\| \leqslant \sigma.$$

Then

$$\Big\|\sum_{i\leqslant n}\alpha_i y_{m+i}\Big\| \geqslant \Big\|\sum_{i\leqslant n}\alpha_i x_{m+i}\Big\| - \sigma\sup_{i\leqslant n}|\alpha_i| \geqslant 2\sigma\Big(\sum_{i\leqslant n}|\alpha_i|^2\Big)^{1/2} - \sigma\sup_{i\leqslant n}|\alpha_i| \geqslant \sigma\Big(\sum_{i\leqslant n}|\alpha_i|^2\Big)^{1/2} \quad \forall\alpha_0,\ldots,\alpha_n.$$

(c) is elementary.

**3. Martingale difference sequences.** The central argument of this paper uses methods from the theory of martingales. Recall that if $(X, \mathscr{F}, \mu)$ is a probability space, a *martingale difference sequence* (an m.d.s.) is a sequence $\langle x_n\rangle_{n\in\mathbf{N}}$ of integrable functions on $X$ such that, for some increasing sequence $\langle\mathscr{F}_n\rangle_{n\in\mathbf{N}}$ of $\sigma$-subalgebras of $\mathscr{F}$,

(i) $x_n$ is $\mathscr{F}_n$-measurable,

(ii) the conditional expectation of $x_n$ on $\mathscr{F}_{n-1}$, $\mathscr{E}(x_n|\mathscr{F}_{n-1})$, is zero, for every $n\in\mathbf{N}$ (we take $\mathscr{F}_{-1} = \{X, \emptyset\}$). In this case we say that $\langle x_n\rangle_{n\in\mathbf{N}}$ is *adapted* to $\langle\mathscr{F}_n\rangle_{n\in\mathbf{N}}$. Of course $\langle\sum_{i\leqslant n}x_i\rangle_{n\in\mathbf{N}}$ is now a martingale adapted to $\langle\mathscr{F}_n\rangle_{n\in\mathbf{N}}$.



The principal theorems we need are:

(a) If $\langle x_n\rangle_{n\in\mathbf{N}}$ is an m.d.s. such that $\sum_{n\in\mathbf{N}} x_n^{\boldsymbol{\cdot}}$ exists in $L^1(X)$, then $\sum_{n\in\mathbf{N}}x_n(t)$ exists for almost all $t$ ([12], Theorem IV-1-2. We write $x^{\boldsymbol{\cdot}}\in L^1$ for the equivalence class of an integrable function $x$.)

(b) If $\langle\mathscr{F}_n\rangle_{n\in\mathbf{N}}$ is an increasing sequence of $\sigma$-algebras of subsets of a set $X$, a *stopping time* adapted to $\langle\mathscr{F}_n\rangle_{n\in\mathbf{N}}$ is a function $\tau\colon X\to\mathbf{N}\cup\{\infty\}$ such that

$$\{t\colon \tau(t) = n\}\in\mathscr{F}_n \quad \forall n\in\mathbf{N}.$$

Now if $\langle x_n\rangle_{n\in\mathbf{N}}$ is an m.d.s. adapted to $\langle\mathscr{F}_n\rangle_{n\in\mathbf{N}}$, so is $\langle y_n\rangle_{n\in\mathbf{N}} = \langle x_n 1_{(\tau\geqslant n)}\rangle_{n\in\mathbf{N}}$, where $1_{(\tau\geqslant n)}$ is the characteristic function of $\{t\colon \tau(t)\geqslant n\}$. It is easy to see that $y_n$ represents $\mathscr{E}(x_n|\mathscr{F}_\tau)$, where $\mathscr{F}_\tau$ is the $\sigma$-algebra

$$\{E\colon E\in\mathscr{F},\ E\cap\{t\colon \tau(t) = m\}\in\mathscr{F}_m\ \forall m\in\mathbf{N}\};$$

so that $\|\sum_{i\leqslant n}\alpha_i y_i\|_1 \leqslant \|\sum_{i\leqslant n}\alpha_i x_i\|_1$ for all $\alpha_0,\ldots,\alpha_n$ ([12], Prop. I-2-12).

(c) We shall need to discuss uniformly integrable sequences. Recall that a $\|\ \|_1$-bounded set $A$ of integrable functions on a probability space $X$ is *uniformly integrable* if for every $\varepsilon > 0$ there is a $\sigma > 0$ such that

$$\int_E |x| \leqslant \varepsilon \quad \text{whenever} \quad x\in A \text{ and } \mu E\leqslant\sigma.$$

The set $\{x^{\boldsymbol{\cdot}}\colon x\in A\}$ is relatively compact for the weak topology $\mathfrak{T}_s(L^1, L^\infty)$ iff $A$ is uniformly integrable ([11], II-T23 or [5], 83F). Recall also that if $A$ is uniformly integrable, so is

$$\{y\colon \exists x\in A,\ \sigma\text{-algebra } \mathscr{F}' \text{ such that } y \text{ represents } \mathscr{E}(x|\mathscr{F}')\}$$

(see the argument following V-T19 in [11]).

**4.** Lemma. *Let $(X, \mathscr{F}, \mu)$ be a probability space and $\langle x_n\rangle_{n\in\mathbf{N}}$ a uniformly integrable m.d.s. on $X$ adapted to a sequence $\langle\mathscr{F}_n\rangle_{n\in\mathbf{N}}$ of $\sigma$-algebras, such that $\sigma = \inf_{n\in\mathbf{N}}\|x_n\|_1 > 0$. Suppose that $\langle\alpha_n\rangle_{n\in\mathbf{N}}$ is a sequence of scalars such that $\sum_{n\in\mathbf{N}}\alpha_n x_n^{\boldsymbol{\cdot}}$ exists in $L^1(X)$. Then $\sum_{n\in\mathbf{N}}|\alpha_n|^2 < \infty$.*

Proof. Let $\eta > 0$ be such that

$$\int_E |x_n| \leqslant \tfrac{1}{3}\sigma \quad \text{whenever} \quad n\in\mathbf{N},\ \mu E\leqslant\eta.$$

As $\sum_{n\in\mathbf{N}}\alpha_n x_n^{\boldsymbol{\cdot}}$ exists in $L^1$, $\sum_{n\in\mathbf{N}}\alpha_n x_n(t)$ exists for almost all $t$ (the point is that $\langle\alpha_n x_n\rangle_{n\in\mathbf{N}}$ is also an m.d.s.), and there is a $\lambda\geqslant 0$ such that

$$\mu\Big\{t\colon \exists n\in\mathbf{N}, \Big|\sum_{i\leqslant n}\alpha_i x_i(t)\Big| > \lambda\Big\} \leqslant \eta.$$

Let $\tau$ be the stopping time given by

$$\tau(t) = \inf\Big\{n:\ \Big|\sum_{i\leqslant n} a_i x_i(t)\Big| > \lambda\Big\}$$

(taking $\inf\emptyset = \infty$); then $\tau$ is adapted to $\langle\mathscr{F}_n\rangle_{n\in\boldsymbol{N}}$. Set $y_n = x_n 1_{(\tau\geqslant n)}$.

(a) If $\beta = \sup_{n\in\boldsymbol{N}, t\in X} |a_n x_n(t)| < \infty$, we see that

$$\Big|\sum_{i\leqslant n} a_i y_i(t)\Big| = \Big|\sum_{i\leqslant\min(n,\tau(t))} a_i x_i(t)\Big| \leqslant \beta+\lambda$$

for all $n\in\boldsymbol{N}$, $t\in X$. At the same time

$$\mu\{t:\ y_n(t)\neq x_n(t)\} \leqslant \mu\{t:\ \tau(t)<\infty\}\leqslant\eta,$$

so that

$$\int|y_n| \geqslant \int|x_n| - \tfrac{1}{3}\sigma \geqslant \tfrac{2}{3}\sigma \qquad \forall n\in\boldsymbol{N}.$$

It follows that

$$(\beta+\lambda)^2 \geqslant \int\Big|\sum_{i\leqslant n} a_i y_i\Big|^2 = \sum_{i\leqslant n}|a_i|^2\int|y_i|^2$$

(because $\langle y_i\rangle_{i\in\boldsymbol{N}}$ is an m.d.s.)

$$\geqslant \sum_{i\leqslant n}|a_i|^2\Big(\int|y_i|\Big)^2 \geqslant \tfrac{4}{9}\sigma^2\sum_{i\leqslant n}|a_i|^2$$

for every $n\in\boldsymbol{N}$, and $\sum_{i\in\boldsymbol{N}}|a_i|^2<\infty$.

(b) If $\beta = \infty$, set

$$z_n = x_n 1_{(\tau>n)}.$$

Of course $\langle z_n\rangle_{n\in\boldsymbol{N}}$ need not be an m.d.s. However, since $z_n(t) = 0$ unless $\tau(t)>n$, in which case $\big|\sum_{i<n} a_i x_i(t)\big|$ and $\big|\sum_{i\leqslant n} a_i x_i(t)\big|$ must both be $\leqslant\lambda$, we have $|a_n z_n(t)|\leqslant 2\lambda$ for all $t$. So we may find an $\mathscr{F}_{n-1}$-measurable representative $w_n$ of the conditional expectation $\mathscr{E}(z_n|\mathscr{F}_{n-1})$ such that $|a_n w_n(t)| \leqslant 2\lambda$ for all $t$. Now $\langle z_n - w_n\rangle_{n\in\boldsymbol{N}}$ is an m.d.s. adapted to $\langle\mathscr{F}_n\rangle_{n\in\boldsymbol{N}}$. We shall complete the proof by showing that $\langle z_n-w_n\rangle_{n\in\boldsymbol{N}}$ satisfies the conditions of part (a).

Consider $\sum a_n(y_n-z_n)$. We know that

$$\Big\|\sum_{m\leqslant i\leqslant n} a_i y_i\Big\|_1 \leqslant \Big\|\sum_{m\leqslant i\leqslant n} a_i x_i\Big\|_1$$

whenever $m\leqslant n$, so that $\sum_{n\in\boldsymbol{N}} a_n y_n^{\cdot}$ exists in $L^1$. At the same time, $\sum_{i\in\boldsymbol{N}} a_i z_i(t)$ exists at least for those $t$ for which $\sum_{i\in\boldsymbol{N}} a_i x_i(t)$ exists, which is almost everywhere, and $\big|\sum_{i\leqslant n} a_i z_i(t)\big|\leqslant\lambda$ for every $t$, $n$; so $\sum_{n\in\boldsymbol{N}} a_n z_n^{\cdot}$ exists in $L^1$. Thus $\sum_{n\in\boldsymbol{N}} a_n(y_n-z_n)^{\cdot}$ exists in $L^1$. But

$$y_n - z_n = x_n 1_{(\tau=n)},$$



so $\langle|y_n-z_n|\rangle_{n\in\boldsymbol{N}}$ is a disjoint sequence, and

$$\sum_{n\in\boldsymbol{N}}|a_n|\,\|y_n-z_n\|_1 = \Big\|\sum_{n\in\boldsymbol{N}} a_n(y_n-z_n)^{\cdot}\Big\|_1 < \infty.$$

As $w_n$ is a representative of $\mathscr{E}(z_n|\mathscr{F}_{n-1}) = \mathscr{E}(z_n-y_n|\mathscr{F}_{n-1})$, it follows that $\sum_{n\in\boldsymbol{N}}|a_n|\,\|w_n\|_1<\infty$, so that $\sum_{n\in\boldsymbol{N}} a_n w_n^{\cdot}$ and $\sum_{n\in\boldsymbol{N}} a_n(z_n-w_n)^{\cdot}$ exist in $L^1$.

Now:

(i) Since $|z_n|\leqslant|x_n|$ for every $n\in\boldsymbol{N}$ and $\{x_n:\ n\in\boldsymbol{N}\}$ is uniformly integrable, $\{z_n:\ n\in\boldsymbol{N}\}$, $\{w_n:\ n\in\boldsymbol{N}\}$ and $\{z_n-w_n:\ n\in\boldsymbol{N}\}$ are uniformly integrable.

(ii) Since

$$\int|y_n-z_n| = \int|x_n 1_{(\tau=n)}|\leqslant\sigma/3, \qquad \int|x_n-z_n| = \int|x_n 1_{(\tau\leqslant n)}|\leqslant\sigma/3$$

for every $n$ (because $\mu\{t:\ \tau(t)<\infty\}\leqslant\eta$), we have $\|w_n\|_1\leqslant\|z_n-y_n\|_1\leqslant\sigma/3$ and

$$\|z_n-w_n\|_1 \geqslant \|x_n\|_1 - \|x_n-z_n\|_1 - \|w_n\|_1 \geqslant \sigma/3 \qquad \forall n\in\boldsymbol{N}.$$

(iii) Finally,

$$\big|a_n(z_n(t)-w_n(t))\big| \leqslant |a_n z_n(t)| + |a_n w_n(t)| \leqslant 4\lambda$$

for every $t$, $n$.

It follows by part (a) that $\sum_{n\in\boldsymbol{N}}|a_n|^2<\infty$.

**5.** Proposition. *Let $X$ be a probability space and $\langle x^n\rangle_{n\in\boldsymbol{N}}$ a uniformly integrable* m.d.s. *on $X$ such that $\inf_{n\in\boldsymbol{N}}\|x_n\|_1>0$. Then $\langle x_n^{\cdot}\rangle_{n\in\boldsymbol{N}}$ is 2-colacunary in $L^1(X)$.*

Proof. Because $\langle x_n\rangle_{n\in\boldsymbol{N}}$ is an m.d.s., $\langle x_n^{\cdot}\rangle_{n\in\boldsymbol{N}}$ is a basis for the closed linear subspace $E$ of $L^1$ which it spans. Now by Lemma 4 we can define $T\colon E\to l^2$ by writing $T\big(\sum_{n\in\boldsymbol{N}} a_n x_n^{\cdot}\big) = \langle a_n\rangle_{n\in\boldsymbol{N}}$. It is easy to see that $T$ has closed graph, therefore is continuous; which is exactly the same thing as $\langle x_n^{\cdot}\rangle_{n\in\boldsymbol{N}}$ being 2-colacunary.

**6.** Theorem. *Let $E$ be an $L$-space and $\langle e_n\rangle_{n\in\boldsymbol{N}}$ a bounded sequence in $E$. Then either $\langle e_n\rangle_{n\in\boldsymbol{N}}$ has a convergent subsequence or $\langle e_n\rangle_{n\in\boldsymbol{N}}$ has a 2-colacunary subsequence.*

Proof. As the closed Riesz subspace of $E$ generated by $\{e_n:\ n\in\boldsymbol{N}\}$ is a separable $L$-space in its own right, it can be embedded in $L^1([0,1])$ ([8], § 15, Theorem 3 and § 14, Cor. to Theorem 9); accordingly, we may take $E = L^1([0,1])$.

Let us suppose that $\langle e_n\rangle_{n\in\boldsymbol{N}}$ has no convergent subsequence. We need to take two cases separately.

(a) $\langle e_n\rangle_{n\in N}$ has no weakly Cauchy subsequence. In this case it has a subsequence equivalent to the usual basis of $l^1$ ([15], Main Theorem), which is certainly 2-colacunary (in fact, 1-colacunary).

(b) $\langle e_n\rangle_{n\in N}$ has a weakly Cauchy subsequence. As $L^1$ is weakly sequentially complete, this subsequence has a limit $e$; to simplify notation, let us suppose that $\langle e_n\rangle_{n\in N}$ itself converges weakly to $e$. Now $\langle e_n-e\rangle_{n\in N}$ is convergent to 0 for $\mathfrak{T}_s(L^1, L^\infty)$ but not for the norm of $L^1$. There is therefore a subsequence $\langle e_{n(k)}-e\rangle_{k\in N}$ and an m.d.s. $\langle x_k\rangle_{k\in N}$ consisting of simple functions such that

(i) $\inf_{k\in N}\|e_{n(k)}-e\|_1>0$,

(ii) $\|x_k-(e_{n(k)}-e)\|_1\leqslant 2^{-k}\ \forall k\in N$

(see [6], Lemma A); the idea is to take $n(k+1)$ so large that $\|\mathscr{E}_k(e_{n(k+1)}-e)\|_1<2^{-k-1}$, where $\mathscr{E}_k$ is the conditional expectation on the finite subalgebra of sets determined by $x_0,\ldots,x_k$. Now $\langle x_k\rangle_{k\in N}$ is $\mathfrak{T}_s(L^1,L^\infty)$-convergent to 0, therefore uniformly integrable, and $\liminf_{k\to\infty}\|x_k\|_1>0$. It follows from Proposition 5 that $\langle x_k\rangle_{k\geqslant m}$ is 2-colacunary for some $m\in N$, so that $\langle e_{n(k)}-e\rangle_{k\geqslant m}$ is 2-colacunary for some $m\in N$ and $\langle e_{n(k)}\rangle_{k\geqslant m}$ is 2-colacunary for some $m\in N$; this is the required subsequence of $\langle e_n\rangle_{n\in N}$.

**7. Immediate corollaries.** (a) *Let $X$ be a probability space and $\langle x_n\rangle_{n\in N}$ a sequence of random variables, of finite expectation, such that for each $k\in N$ the joint distribution of $(x_{n(0)},\ldots,x_{n(k)})$ is the same whenever $n(0)<\ldots\ldots<n(k)$. Then $\langle x_n\rangle_{n\in N}$ is either constant or 2-colacunary in $L^1(X)$ (since any subsequence is isomorphic in the relevant sense to $\langle x_n\rangle_{n\in N}$).*

(This result has an alternative derivation. The special case in which the $x_n$ are independent in a fairly easy consequence of the classical three-series theorem ([1], Theorem 5.3.3). The general case follows from [2] and the well-known representation of exchangeable sequences as mixtures of independent identically distributed sequences.)

(b) *If $E$ is a normed space, $F$ an L-space, and $T\colon E\to F$ is a non-compact continuous linear operator, then there is a subspace $G$ of $E$ and a non-compact continuous linear operator $S\colon G\to l^2$.*

For there is a bounded sequence $\langle x_n\rangle_{n\in N}$ in $E$ such that $\langle Tx_n\rangle_{n\in N}$ has no convergent subsequence. By Theorem 6, $\langle Tx_n\rangle_{n\in N}$ has a 2-colacunary subsequence, and therefore $\langle x_n\rangle_{n\in N}$ has a 2-colacunary subsequence $\langle x_{n(k)}\rangle_{k\in N}$ say. Now we can take $G=\operatorname{lin}\{x_{n(k)}\colon k\in N\}$, $Sx_{n(k)}=k$th basic unit vector of $l^2$.

(c) It follows that every continuous linear operator from any subspace of $l^p$ to $L^1$ is compact if $p>2$; see [13], Theorem A2.

**8. Applications to Banach lattices.** (a) Recall that if $E$ is a Banach lattice, then $\mathfrak{T}_{|s|}(E, E')$ is the topology on $E$ with basic seminorms $x\mapsto f(|x|)$ where $f$ runs through $E'^{+}$ ([5], § 81). Now if $\langle x_n\rangle_{n\in N}$ is a sequence in $E$ which is convergent to 0 for $\mathfrak{T}_s(E,E')$ but not for $\mathfrak{T}_{|s|}(E,E')$, then $\langle x_n\rangle_{n\in N}$ has a 2-colacunary subsequence. (We apply Theorem 6 to $\langle x_n\rangle_{n\in N}$ in the L-space completion of $E/\{x\colon f(|x|)=0\}$, where $f\in E'^{+}$ is such that $\limsup_{n\to\infty} f(|x_n|)>0$.)

(b) A Banach lattice has the *$l^2$-decomposition property* if $\sum_{n\in N}\|x_n\|^2<\infty$ whenever $\langle x_n\rangle_{n\in N}$ is a disjoint order-bounded sequence in $E^+$. If $E$ is a Banach lattice with the $l^2$-decomposition property, and $\langle x_n\rangle_{n\in N}$ is a sequence in $E$ which is weakly convergent to 0 but not norm-convergent to 0, then $\langle x_n\rangle_{n\in N}$ has a 2-colacunary subsequence. (If $\langle x_n\rangle_{n\in N}$ is not convergent to 0 for $\mathfrak{T}_{|s|}(E,E')$, use part (a). Otherwise, noting that the norm of $E$ must be order-continuous ([4], Theorem 2.5), we can find a subsequence $\langle x_{n(k)}\rangle_{k\in N}$ and a disjoint sequence $\langle y_k\rangle_{k\in N}$ such that $\sum_{k\in N}\|x_{n(k)}-y_k\|<\infty$. The $l^2$-decomposition property is just what we need to show that disjoint sequences bounded away from 0 are 2-colacunary, so that the result follows from Lemma 2(b).)

**9. Spaces of cotype 2.** A normed space $E$ is of *cotype* 2 (see [10]) if there is a $\sigma>0$ such that, for every $x_0,\ldots,x_n\in E$,

$$\mu\Big\{t\colon \Big\|\sum_{i\leqslant n} r_i(t)x_i\Big\|>1\Big\}\leqslant\sigma\Rightarrow\sum_{i\leqslant n}\|x_i\|^2\leqslant 1,$$

where $\langle r_i\rangle_{i\in N}$ is the sequence of Rademacher functions on [0,1] (independently taking values $\pm1$ with equal probability). In any normed space, a sequence $\langle x_n\rangle_{n\in N}$ is *unconditional* if there is a constant $\gamma$ such that $\|\sum_{i\leqslant n}a_ix_i\|\leqslant\gamma\|\sum_{i\leqslant n}\beta_ix_i\|$ whenever $n\in N$ and $|a_i|\leqslant|\beta_i|$ for each $i\leqslant n$. Evidently an unconditional sequence, bounded away from 0, in a space of cotype 2 is 2-colacunary. We have the following simple result:

Proposition. *Let $E$ be a Banach space of cotype 2 with an unconditional basis $\langle b_i\rangle_{i\in N}$. Let $\langle e_n\rangle_{n\in N}$ be any bounded sequence in $E$. Then either $\langle e_n\rangle_{n\in N}$ has a convergent subsequence or it has a 2-colacunary subsequence.*

Proof. As $c_0(N)$ is not of cotype 2, it cannot be embedded in $E$, and $\langle b_i\rangle_{i\in N}$ is boundedly complete ([3], Theorem IV. 4.2). Taking a subsequence of $\langle e_n\rangle_{n\in N}$ if necessary, we may suppose that $\lim_{n\to\infty} f_i(e_n)$ exists for every $i\in N$, where $\langle f_i\rangle_{i\in N}$ is the $w^*$-basis of $E'$ dual to $\langle b_i\rangle_{i\in N}$. Now there is an $e\in E$ such that $f_i(e)=\lim_{n\to\infty}f_i(e_n)$ for every $i\in N$. Either $\langle e_n\rangle_{n\in N}$ has a subsequence converging to $e$, or it has a subsequence $\langle e_{n(k)}\rangle_{k\in N}$ such that $\inf_{k\in N}\|e_{n(k)}-e\|>0$ and $\langle e_{n(k)}-e\rangle_{k\in N}$ is unconditional ([9], Proposition 1.a.12). But in this case $\langle e_{n(k)}-e\rangle_{k\in N}$ is 2-colacunary, so that $\langle e_n\rangle_{n\in N}$ has a 2-colacunary subsequence.

Remark. $L^1([0,1])$ does not have an unconditional basis, so, even though it is of cotype 2 ([10], Lemma 1.1), we cannot deduce Theorem 6 from this proposition. We doubt that the proposition remains true without the hypothesis that $E$ has an unconditional basis, but we do not have a counter-example.

**10. Concluding remarks.** We should like to thank the referee for several suggestions concerning the proofs in this paper. L. Dor and H.P. Rosenthal have given a sharper version of Proposition 5 with a different proof based on martingale inequalities.

### References


[1] K. L. Chung, *A course in probability theory*, Academic Press, 1974.
[2] D. Dacunha-Castelle, *Indiscernability and exchangeability in $L^p$ spaces*, Proc. Seminar on Random Series, Convex Sets and Geometry of Banach Spaces, Aarhus 1974.
[3] M. M. Day, *Normed linear spaces*, Springer-Verlag, 1962.
[4] P. Dodds, *Indices for Banach lattices*, Proc. Netherlands Acad. Sci. (A) 80 (1977), 73–86.
[5] D. H. Fremlin, *Topological Riesz spaces and measure theory*, Cambridge U.P., 1974.
[6] V. F. Gaposhkin, *Convergence and limit theorems for sequences of random variables*, Theor. Probability Appl. 17 (1972), 379–399.
[7] M. I. Kadec & A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces $L_p$*, Studia Math. 21 (1962), 161–176.
[8] H. E. Lacey, *The isometric theory of classical Banach spaces*, Springer-Verlag, 1974.
[9] J. Lindenstrauss & L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, 1977.
[10] B. Maurey & G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géometriques des espaces de Banach*, Studia Math. 58 (1976), 45–90.
[11] P. A. Meyer, *Probability and potentials*, Blaisdell, 1966.
[12] J. Neveu, *Discrete-parameter martingales*, North-Holland, 1975.
[13] H. P. Rosenthal, *On quasi-complemented subspaces of Banach spaces with an appendix on compactness of operators from $L^p(\mu)$ to $L^r(\nu)$*, J. Functional Anal. 4 (1969), 176–214.
[14] — *On subspaces of $L^p$*, Ann. Math. 102 (1973), 344–373.
[15] — *A characterization of Banach spaces containing $l^1$*, Proc. Nat. Acad. Sci. U.S.A. 71 (1974), 2411–2413.


STATISTICAL LABORATORY
UNIVERSITY OF CAMBRIDGE
CAMBRIDGE

MATHEMATICS DEPARTMENT
UNIVERSITY OF ESSEX
COLCHESTER

*Received July 17, 1979*
*Revised version October 17, 1979* (1446)

---

STUDIA MATHEMATICA, T. LXXI. (1982)

# Inégalités à poids pour le projecteur de Bergman dans la boule unité de $C^n$

par

DAVID BÉKOLLÉ (Orléans)

**Resumé.** Dans la boule unité de $C^n$ munie de la mesure $d\mu_a(\zeta) = (1-|\zeta|^2)^{a-1}d\mu(\zeta)$, où $a > 0$ et $\mu$ est la mesure de Lebesgue, nous caractérisons les mesures boréliennes positives $\Omega$ pour lesquelles le projecteur de Bergman

1° s'étend en un opérateur continu de $L^p(d\Omega)$ dans lui-même, si $1 < p < \infty$;

2° s'étend en un opérateur faiblement continu sur $L^1(d\Omega)$.

**§ I. Introduction.** $\boldsymbol{D} = \{z \in C^n\colon |z| < 1\}$ est la boule unité de $C^n$; $d\mu_a(\zeta) = (1-|\zeta|^2)^{a-1}d\mu(\zeta)$, où $a > 0$ et $\mu$ est la mesure de Lebesgue sur $C^n = \boldsymbol{R}^{2n}$. Nous désignons par $L^p(d\mu_a)$ les espaces de Lebesgue relatifs à $\mu_a$, $1 \leqslant p \leqslant \infty$.

La projection de Bergman $T_a f$ d'une fonction $f \in L^2(d\mu_a)$ sur le sous-espace de $L^2(d\mu_a)$ formé par les fonctions holomorphes est donnée, à une constante ne dépendant que de $a$ et $n$ près par

$$T_a f(z) = \int_D \frac{f(\zeta)}{(1-z\cdot\bar\zeta)^{n+a}}\, d\mu(\zeta),$$

où $z\cdot\bar\zeta = z_1\cdot\bar\zeta_1 + \ldots + z_n\cdot\bar\zeta_n$, quand $z = \{z_1, z_2, \ldots, z_n\}$ et $\zeta = (\zeta_1, \ldots, \zeta_n)$.

Il est bien connu que l'opérateur $T_a$ s'étend en un opérateur continu de $L^p(d\mu_a)$ dans lui-même si $1 < p < \infty$, et est faiblement continu sur $L^1(d\mu_a)$ (E.M. Stein [13], F. Forelli et W. Rudin [8]). Ceci se démontre de la façon suivante. Selon la théorie des intégrales singulières sur les espaces homogènes développée par R.R. Coifman et G. Weiss [5], si $\boldsymbol{D}$ est munie d'une pseudo-distance $d$ pour lequelle le triplet $(\boldsymbol{D}, d, \mu_a)$ constitue un espace homogène et qu'on note $K_a(z, \zeta) = 1/(1-z, \bar\zeta)^{n+a}$ le noyau du projecteur $T_a$, il suffit de démontrer que $K_a$ vérifie

S1: il existe trois constantes $\beta$, $C_1$, $C_2$ telles que

$$|K_a(z,\zeta) - K_a(z,\zeta^0)| \leqslant C_1[d(\zeta,\zeta^0)]^\beta/[d(z,\zeta^0)]^{n+a+\beta},$$

quels que soient $z, \zeta, \zeta^0$, vérifiant $d(z,\zeta^0) > C_2 d(\zeta,\zeta^0)$.

C'est le cas quand sur $\boldsymbol{D}$ on prend la pseudo-distance $d$ définie par

$$d(z,\zeta) = \big||z|-|\zeta|\big| + \big|1 - z\bar\zeta/|z|\,|\zeta|\big|,$$