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A remark on finite-dimensional P_λ-spaces

by

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Abstract. It is shown that finite-dimensional P_λ-spaces contain $l^\infty(m)$-subspaces, where m is proportional to the dimension of the P_λ-space.

Introduction. Let us recall that the Banach–Mazur distance $\Delta(E, F)$ of two normed linear spaces E and F of the same finite dimension is given by $\Delta(E,F) = \inf\{\|T\|\,\|T^{-1}\|;\ T\colon E\to F$ is a linear isomorphism$\}$.

We say that X is a *P_λ-space* provided for any Banach space Y in which X embeds isometrically, there exists a projection P from Y onto X with $\|P\| \leqslant \lambda$. As is well known, the space Y in the above definition may as well be replaced by the space l^∞. The characterization of P_λ-spaces is a rather old and still unsolved problem. One may hope for an affirmative solution to the following question, dealing with the finite-dimensional version of the problem:

Does there exist for all $\lambda < \infty$ some constant $c_\lambda < \infty$ such that $\Delta\big(E, l^\infty(d)\big) < c_\lambda$ holds, for any P_λ-space E of dimension d?

In [4], the existence is shown of a function $d(\lambda, m, \varepsilon)$ so that given a P_λ-space E, $\dim(E) \geqslant d(\lambda, m, \varepsilon)$, one can find a subspace F of E with $\dim(F) = m$ and $\Delta\big(F, l^\infty(m)\big) < 1+\varepsilon$.

Our purpose is to show the following fact which, taking into account a related observation of [4], will improve the above result.

THEOREM 1. *Given $\lambda < \infty$, one can find a constant $c = c_\lambda < \infty$ such that given a finite-dimensional P_λ-space E, there exists a subspace F of E satisfying $\dim(F) = m > c^{-1}\dim(E)$ and $\Delta\big(F, l^\infty(m)\big) \leqslant c$.*

Proof of the result. We recall that if $T\colon X\to Y$ is an operator between Banach spaces X and Y, then T is (p, q)-*absolutely summing* if there exists a constant $M < \infty$ such that

$$(*)\qquad \Big(\sum_{i=1}^{n} \|T(x_i)\|^p\Big)^{1/p} \leqslant M \sup_{\|x^*\|\leqslant 1} \Big(\sum_{i=1}^{n} |\langle x_i, x^*\rangle|^q\Big)^{1/q}$$

holds, whenever $(x_i)_{1\leqslant i\leqslant n}$ is a finite sequence of vectors in X.

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The infimum over all possible values of M is the (p,q)-summing norm and is denoted by $\pi_{p,q}(T)$.

If only sequences $(x_i)_{1\leqslant i\leqslant n}$ of fixed length n are considered, the infimum over all M for which (*) holds is called $\pi_{p,q}^{(n)}(T)$.

We will use here the following result due to Tomczak-Jaegermann [6].

LEMMA 1. *If T is a rank n operator, then $\pi_{2,2}(T)\leqslant 2\pi_{2,2}^{(n)}(T)$.*

We now show the following

LEMMA 2. *There exists a constant $\beta>0$ such that whenever E is an n-dimensional subspace of L^1 and P is a projection onto E, then $\pi_{2,2}^{(n)}(i^*) > \beta\|P\|^{-1}\sqrt{n}$, where $i\colon E\to L^1$ is the injection.*

Proof. If I denotes the identity on E, then $I=Pi$ and thus $I^* = i^*P^*$. Therefore, by Lemma 1,

$$\sqrt{n}=\pi_{2,2}(I^*)\leqslant \pi_{2,2}(i^*)\|P^*\|\leqslant 2\|P\|\pi_{2,2}^{(n)}(i^*).$$

This proves the lemma for $\beta=\frac{1}{2}$.

The next step is the proof of the following

LEMMA 3. *For all $\lambda<\infty$ there exists a constant $\gamma=\gamma_\lambda>0$ such that if E is a λ-complemented n-dimensional subspace of $L^1(\mu)$, then there are functions $f_1,\dots,f_n$ in E satisfying*

$$(**)\qquad \int|f_i|\,d\mu\leqslant 1\quad (1\leqslant i\leqslant n)\quad\text{and}\quad \int\max_{1\leqslant i\leqslant n}|f_i|\,d\mu\geqslant\gamma n.$$

Proof. Take $\gamma_\lambda=\beta^2\lambda^{-2}$, where β is as in Lemma 2. Under the hypothesis of Lemma 3, we deduce from Lemma 2 that $\pi_{2,2}^{(n)}(i^*)>\beta\lambda^{-1}\sqrt{n}$, which means that there exist n functions $\varphi_1,\dots,\varphi_n$ in $L^\infty(\mu)$ for which

$$\sum_{i=1}^{n}\|i^*(\varphi_i)\|^2>\beta^2\lambda^{-2}n\sup\sum_{i=1}^{n}|\langle\varphi_i,f\rangle|^2,$$

where the sup is taken over all $f\in L^1(\mu)$ with $\|f\|\leqslant 1$. Remark also that

$$\sup\sum_{i=1}^{n}|\langle\varphi_i,f\rangle|^2=\Big\|\sum_{i=1}^{n}|\varphi_i|^2\Big\|_\infty.$$

For $i=1,\dots,n$, let $f_i\in E$ satisfy $\|f_i\|_1\leqslant 1$ and $\|i^*(\varphi_i)\|=\langle\varphi_i,f_i\rangle$. We also can find scalars $a_1,\dots,a_n$ with $\sum_i|a_i|^2=1$ and

$$\sum_{i=1}^{n}\|i^*(\varphi_i)\|^2=\Big[\sum_i a_i\|i^*(\varphi_i)\|\Big]^2=\Big[\sum_i a_i\langle\varphi_i,f_i\rangle\Big]^2.$$

Now by the Schwartz inequality,

$$\sum_i a_i\langle\varphi_i,f_i\rangle=\int\sum_i a_i\varphi_i f_i\,d\mu\leqslant\Big\|\sqrt{\sum_i|\varphi|^2}\Big\|_\infty\int\sqrt{\sum|a_i|^2|f_i|^2}\,d\mu.$$

Therefore, combining the above inequalities,

$$\Big(\int\sqrt{\sum_i|a_i|^2|f_i|^2\,d\mu}\Big)^2>\beta^2\lambda^{-2}n.$$

On the other hand,

$$\sum_i|a_i|^2|f_i|^2\leqslant\max_i|f_i|\sum_i|a_i|^2|f_i|,$$

and hence, again applying the Schwartz inequality,

$$\int\sqrt{\sum_i|a_i|^2|f_i|^2}\,d\mu\leqslant\Big(\int\max_i|f_i|\,d\mu\Big)^{1/2}\Big(\sum_i|a_i|^2\int|f_i|d\mu\Big)^{1/2}$$
$$\leqslant\Big(\int\max_i|f_i|\,d\mu\Big)^{1/2}.$$

We conclude that $\int\max_i|f_i|\,d\mu\geqslant\beta^2\lambda^{-2}n=\gamma n$, as required. Our final objective is the following.

PROPOSITION 4. *For all $\gamma>0$, there exists a constant $\delta=\delta_\gamma>0$ such that if $f_1,\dots,f_n$ are functions in an $L^1(\mu)$-space satisfying (**) stated above, then there exist a subset $\{g_1,\dots,g_m\}$ of $\{f_1,\dots,f_n\}$ and a projection Q from $L^1(\mu)$ onto $F=[g_1,\dots,g_m]$, satisfying*

1. $m\geqslant\delta n$,
2. $\Delta(F,l^1(m))\leqslant\delta^{-1}$,
3. $\|Q\|\leqslant\delta^{-1}$.

This result is stated in [1] in a slightly different version and an independent proof is also due to G. Schechtman [5]. We proceed in two steps.

LEMMA 5. *If $f_1,\dots,f_n$ are functions in an $L^1(\mu)$-space satisfying (**), then there exist a subset $\{g_1,\dots,g_m\}$ of $\{f_1,\dots,f_n\}$ and disjoint μ-measurable sets A_j $(1\leqslant j\leqslant m)$, satisfying*

1. $m\geqslant\frac{1}{3}\gamma n$,
2. $\int_{A_j}|g_j|\,d\mu\geqslant\frac{1}{3}\gamma$ *for* $j=1,\dots,m$.

Proof. Consider the following convex cone in $l^1(n)$

$$K=\Big\{(a_1,\dots,a_n);\ a_i\geqslant 0\ (1\leqslant i\leqslant n)\ \text{and}\ \int\max_i a_i|f_i|\leqslant\tfrac{1}{3}\gamma\sum_i a_i\Big\}.$$

Remark that for $(a_1,\dots,a_n)\in K$

$$\sum_i|a_i-1|\geqslant\max\Big\{\Big|\sum_i a_i-n\Big|,\ \gamma n-\tfrac{1}{3}\gamma\sum_i a_i\Big\}\geqslant\tfrac{1}{3}\gamma n.$$

A separation argument yields us therefore some $(b_1,\dots,b_n)$ in the unit ball of $l^\infty(n)$, satisfying

$$\sum_i b_i\geqslant\sum_i a_ib_i+\tfrac{1}{3}\gamma n,\qquad\text{whenever}\qquad(a_1,\dots,a_n)\in K.$$

Define $J = \{i = 1, \ldots, n;\ b_i > 0\}$, for which $\operatorname{card}(J) \geqslant \frac{1}{8}\gamma n$. Take $\{g_1, \ldots, g_m\} = \{f_i\colon i \in I\}$. Since the only member of K supported by J is the 0-vector, we find that

$$\int \max_j a_j |g_j| \geqslant \tfrac{1}{8}\gamma \sum_j a_j \quad \text{for all positive scalars } a_1, \ldots, a_m.$$

We now apply a result of Dor (see [2], Proposition 2.2) in order to obtain disjoint μ-measurable sets $A_1, \ldots, A_m$ such that

$$\int_{A_j} |g_j|\, d\mu \geqslant \tfrac{1}{8}\gamma \quad \text{for} \quad j = 1, \ldots, m.$$

Proof of Proposition 4. By Lemma 5, it suffices to prove Proposition 4 replacing (**) by the following stronger property:

(***) *There exist disjoint μ-measurable sets $A_1, \ldots, A_n$ so that*

$$\int_{A_i} |f_i|\, d\mu \geqslant \gamma.$$

We now claim that there exists a subset D of $\{1, \ldots, n\}$ so that

(i) $\operatorname{card}(D) \geqslant \frac{1}{32}\gamma n$,

(ii) $\int_{A_i'} |f_i|\, d\mu < \frac{1}{2}\gamma$ for each $i \in D$, where $A_i' = \bigcup_{\substack{j \in D\\ j \neq i}} A_j$.

In order to show that, consider a system ι_i $(1 \leqslant i \leqslant n)$ of independent $\{0, 1\}$-valued functions of mean $\frac{1}{8}\gamma$. Define further the following functions:

$$\iota(t) = \sum_i \iota_i(t),$$

$$\xi_i(t, \omega) = \sum_{j \neq i} \iota_j(t) A_j(\omega) \quad \text{for} \quad i = 1, \ldots, n,$$

$$\psi(t, \omega) = \sum_i \iota_i(t)\, \xi_i(t, \omega) |f_i|(\omega).$$

A straightforward verification shows then that

$$\int \iota(t)\, dt = \tfrac{1}{8}\gamma n \quad \text{and} \quad \iint \psi(t, \omega)\, dt\, \mu(d\omega) \leqslant \tfrac{1}{64}\gamma^2 n.$$

Therefore, one can find some t such that

$$\iota(t) \geqslant \tfrac{1}{16}\gamma n \quad \text{and} \quad \iota(t) \geqslant \frac{4}{\gamma} \int \psi(t, \omega)\, \mu(d\omega).$$

If we take $J = \{i = 1, \ldots, n;\ \iota_i(t) = 1\}$ and $B_i = \bigcup_{j \in J, j \neq i} A_j$, for each $i \in J$ we get

$$\operatorname{card}(J) \geqslant \tfrac{1}{16}\gamma n \quad \text{and} \quad \operatorname{card}(J) \geqslant \frac{4}{\gamma} \sum_{i \in J} \int_{B_i} |f_i|\, d\mu.$$

Let $D = \{i \in J;\ \int_{B_i} |f_i|\, d\mu < \frac{1}{2}\gamma\}$. Then clearly $\operatorname{card}(J \setminus D) \leqslant \frac{1}{2}\operatorname{card}(J)$ and hence $\operatorname{card}(D) \geqslant \frac{1}{2}\operatorname{card}(J) \geqslant \frac{1}{32}\gamma n$. This proves the claim.

Take $\{g_1, \ldots, g_m\} = \{f_i;\ i \in D\}$. It remains to show that F is a complemented $l^1(m)$-isomorph. The argument is contained in [2], but we repeat it here for the sake of completeness. Let $(e_i)_{i \in D}$ be the $l^1(m)$ unit-vector basis and define the operator

$$U\colon L^1(\mu) \to l^1(m) \quad \text{by} \quad U(f) = \sum_{i \in D} \Big(\int_{A_i} f \operatorname{sgn} f_i \Big) e_i.$$

Obviously, $\|U\| \leqslant 1$. If we take $x_i = U(f_i)$, then

$$\Big\| \sum_{i \in D} a_i x_i \Big\| = \sum_{j \in D} \Big| \sum_{i \in D} a_i x_i(j) \Big| \geqslant \sum_{j \in D} |a_j|\, |x_j(j)| - \sum_{i, j \in D, i \neq j} |a_i|\, |x_i(j)|$$

$$\geqslant \sum_{j \in D} |a_j| \int_{A_j} |f_j| - \sum_{i \in D} |a_i| \int_{A_i'} |f_i| \geqslant \tfrac{1}{2}\gamma \sum_{i \in D} |a_i|.$$

Thus if $V\colon l^1(m) \to l^1(m)$ is given by $V(x_i) = e_i$, then $\|V\| \leqslant 2/\gamma$. Finally, define $W\colon l^1(m) \to F$ by $W(e_i) = f_i$.

Since WVU is the identity on F, we find that $\Delta(F, l^1(m)) \leqslant \|U\|\, \|V\|\, \|W\| \leqslant 2/\gamma$. Moreover, $Q = WVU$ is a projection and $\|Q\| \leqslant 2/\gamma$, completing the proof of Proposition 4.

Proof of Theorem 1. Combining Lemma 3 and Proposition 4, we find for fixed $\lambda < \infty$ some $\delta = \delta_\lambda > 0$ such that:

If E is a λ-complemented n-dimensional subspace of $L^1(\mu)$, then E has a subspace F such that

(i) $\dim F \geqslant \delta \dim E$,

(ii) $\Delta(F, l^1(m)) \leqslant \delta^{-1}$, *where* $m = \dim F$,

(iii) F *is* δ^{-1}*-complemented in* $L^1(\mu)$.

The claim stated in Theorem 1 is then simply obtained by dualization.

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