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On the Banach–Mazur distance of finite-dimensional symmetric Banach spaces and the hypergeometric distribution

by

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Abstract. It is proved that for n-dimensional symmetric Banach spaces the inequality

$$d(E, l_n^\infty) < C(\log_2 4\lambda(E))^{7/2}\lambda(E)$$

is valid. Moreover, we give the order of $d(E, l_p^n)$, $p \in \{1, 2, \infty\}$, of a class of n-dimensional Banach spaces containing tensor products of symmetric Banach spaces.

In [7] it was asked whether there is a real-valued function f such that for the projection constant $\lambda(E)$ and the Banach–Mazur distance $d(E, l_n^\infty)$ of n-dimensional Banach spaces E we have

$$d(E, l_n^\infty) \leqslant f(\lambda(E)).$$

This question was answered for certain classes of spaces [7]. Nevertheless the problem is still open. Moreover, one may ask what such a function f looks like. Since $\lambda(E) \leqslant d(E, l_n^\infty)$ is valid for all finite-dimensional E, one may ask whether $\lambda(E)$ and $d(E, l_n^\infty)$ are the same up to a constant. In the first section we show that for symmetric spaces $d(E, l_n^\infty)$ can be estimated by $C(\log_2 2\lambda(E))^{7/2}\lambda(E)$. We also give a formula for $\lambda(E)$ and $d(E, l_n^\infty)$, using the geometry of the unit ball of E.

In the second section we estimate the Banach–Mazur distance of symmetric Banach spaces to l_n^2. In fact, the result is more general, so that it covers also tensor products of symmetric spaces. Finally, we give applications of the results in the second section. We give the order of the Banach–Mazur distance of $l_n^r \otimes_s l_n^r$ to $l_{n^2}^p$ for $1 \leqslant r \leqslant \infty$ and $p \in \{1, 2, \infty\}$.

Not all the estimations we present in the third section are new. Some can be found in [3]. Professor T. Figiel has shown (in an unpublished paper) that $\lambda(E \otimes_s F) = \lambda(E)\lambda(F)$. We would like to thank Dr. N. Tomczak-Jaegermann and Professor A. Pełczyński for discussions and suggestions concerning this paper.

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0. Preliminaries. Most of our notation is standard and can be found in [8]. The Banach–Mazur or isomorphic distance of two Banach spaces E and F is defined by

$$d(E,F) = \inf\{\|J\|\,\|J^{-1}\| \mid J \in L(E,F),\ J \text{ is isomorphism}\}.$$

If there is no isomorphism, we set $d(E,F)=\infty$. The p-factoring norm of an operator $A \in L(E,F)$ where E and F are finite-dimensional is given by

$$\gamma_p(A) = \inf\{\|B\|\,\|C\| \mid A = BC,\ C \in L(E, l^p),\ B \in L(l^p, F)\}.$$

For A being the identity we denote $\gamma_p(\mathrm{id}_E)$ by $\gamma_p(E)$. $\gamma_\infty(E)$ is also called the projection constant $\lambda(E)$.

The 1-absolutely summing norm of an operator $A \in L(E,F)$ is given by the infimum of all $C \in \mathbf{R}$ such that for all sequences $\{x_i\}_{i=1}^n$ of vectors of E we have

$$\sum_{i=1}^n \|A(x_i)\| \leqslant C \sup_{\|x^*\|=1} \sum_{i=1}^n |\langle x_i, x^*\rangle|.$$

The norm is denoted by $\pi_1(A)$. Let G be a subset of $H = \{\varepsilon = (\varepsilon_i)_{i=1}^n \mid \varepsilon_i = \pm 1\}$. We say that a basis $\{e_i\}_{i=1}^n$ of a Banach space E is *G-unconditional with constant C* if

$$\Big\|\sum_{i=1}^n a_i e_i\Big\| \leqslant C \Big\|\sum_{i=1}^n \varepsilon_i a_i e_i\Big\| \quad \text{for all } a \in \mathbf{R}^n \text{ and } \varepsilon \in G.$$

Moreover, let D be a subset of the set P of all permutations of the set $\{1, \ldots, n\}$. We say that a basis is *G, D-symmetric with constant C* if

$$\Big\|\sum_{i=1}^n a_i e_i\Big\| \leqslant C \Big\|\sum_{i=1}^n \varepsilon_i a_{\pi(i)} e_i\Big\| \quad \text{for all } a \in \mathbf{R}^n,\ \pi \in D \text{ and } \varepsilon \in G.$$

We say that a basis is *unconditional* (*symmetric*) *with constant C* if $G = H$ ($G = H$, $D = P$). We say that a basis is *unconditional* (*symmetric*) if it is normalized and unconditional (symmetric) with constant 1. By $\{e_i^*\}_{i=1}^n$ we denote the dual basis of $\{e_i\}_{i=1}^n$.

The ε-tensor product for finite-dimensional spaces is $E \otimes_\varepsilon F = L(E^*, F)$ and the π-tensor product $(E^* \otimes_\pi F)^* = E \otimes F$. We identify tensors and matrices in a natural way. We need especially the Walsh matrices

$$W_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad W_{n+1} = \begin{bmatrix} W_n & W_n \\ W_n & -W_n \end{bmatrix} \quad \text{for} \quad n = 1, 2, \ldots$$

1. Estimations of the projection constant and the Banach–Mazur distance.

PROPOSITION 1. *Suppose that $\{e_i\}_{i=1}^n$ is a symmetric basis of a Banach space E and the extreme points of the dual unit ball are of the form $x = c_x \sum_{i=1}^n x_i e_i^*$ where $c_x \in \mathbf{R}$ and $x_i \in \{0, -1, 1\}$. Then*

$$\lambda(E) \leqslant \min_{\sum_{k=1}^n a_k = 1} \max_{j \leqslant n} (n/j) b_j \sum_{k=1}^n a_k b_k^{-1} \sum_{l=1}^k \sqrt{l}\, \frac{\binom{j}{l}\binom{n-j}{n-l}}{\binom{n}{k}} = \sqrt{2}\,\lambda(E),$$

where $a_j \geqslant 0$ and $b_j = \|\sum_{i=1}^j e_i\|$, $j = 1, \ldots, n$.

Proof. We first prove the right-hand side inequality. Let M be the set of extreme points of the dual unit ball of E. We consider the embedding $I \in L(E, l^\infty(M))$, $I(x) = \sum_{i=1}^n x_i g_i$, where $g_i = (z_i)_{z\in M}$, $i = 1, \ldots, n$. We estimate the norm of the projection

$$P(y) = \sum_{i=1}^n \langle f_i, y\rangle g_i$$

with

$$\delta_{ij} = \langle f_i, g_j\rangle = \sum_{z\in M} f_i^z z_j.$$

We have

$$\begin{aligned}\|P\| &= \max_{\|y\|_\infty=1} \Big\|\sum_{i=1}^n \langle f_i, y\rangle g_i\Big\|_\infty \\ &= \max_{\|y\|_\infty=1} \max_{z\in M} \Big|\sum_{i=1}^n \langle f_i, y\rangle z_i\Big| \\ &= \max_{z\in M} \sum_{w\in M} \Big|\sum_{i=1}^n z_i f_i^w\Big|.\end{aligned}$$

Because of the unconditionality of the basis $\{e_i^*\}_{i=1}^n$ and the Khintchine inequality [5] or [11] we get

$$\|P\| \geqslant (1/\sqrt{2}) \max_{z\in M} \sum_{w\in M} \Big(\sum_{i=1}^n |z_i f_i^w|^2\Big)^{1/2}$$

with

$$\sum_{z\in M} f_i^z z_i = 1 \quad \text{for} \quad i = 1, \ldots, n.$$

Obviously, we may assume that $\sum_{z\in M}|f_i^z z_i| = 1$ for $i = 1, \ldots, n$ and we may assume that $f_i^z = 0$ if $z_i = 0$. Moreover, we may restrict ourselves to considering only the set $\tilde{M}$ of extreme points with non-negative coordinates $z_i \geqslant 0$, $i = 1, \ldots, n$. Therefore

$$(1)\qquad \|P\| \geqslant (1/\sqrt{2}) \max_{z\in\tilde{M}} \sum_{w\in\tilde{M}} \Big(\sum_{i=1}^n |z_i f_i^w|^2\Big)^{1/2}$$

with

$$(2)\qquad f_i^w = 0 \quad \text{if} \quad w_i = 0 \quad \text{for} \quad i = 1, \ldots, n \text{ and } w \in \tilde{M}$$

and

$$n = \sum_{w\in\tilde{M}} \sum_{i=1}^n |w_i f_i^w|.$$

By denoting the number of non-zero coordinates of w by k_w we get

$$(3)\qquad n = \sum_{w\in\tilde{M}} k_w^{-1} b_{k_w} \sum_{i=1}^n |f_i^w|.$$

Now we group certain vectors. For $w \in \tilde{M}$ with exactly k_w positive coordinates and $n-k_w$ zero coordinates there is a set of $\binom{n}{k_w}$ permutations G_w such that for all $\pi_1, \pi_2 \in G_w$ we have $w_{\pi_1}(i) \neq w_{\pi_2}(i)$ for at least one i, $1 \leqslant i \leqslant n$. For every $\pi \in G_w$ there is a set of permutations D_π leaving the vector $(w_{\pi(i)})_{i=1}^n$ invariant. This set has $k_w!(n-k_w)!$ elements. So we have, because of symmetry, by averaging over all permutations

$$\|P\| \geqslant (1/\sqrt{2}) \max_{z\in\tilde{M}} \frac{1}{n!} \sum_{w\in\tilde{M}} \sum_{\pi\in G_w} \sum_{\eta\in D_\pi} \Big(\sum_{i=1}^n |z_i f_{\eta\circ\pi(i)}^w|^2\Big)^{1/2}.$$

Applying the triangle inequality, we get

$$\|P\| \geqslant (1/\sqrt{2}) \max_{z\in\tilde{M}} \frac{1}{n!} \sum_{w\in\tilde{M}} \sum_{\pi\in G_w} \Big(\sum_{i=1}^n \Big| z_i \sum_{\eta\in D_\pi} |f_{\eta\circ\pi(i)}^w|\Big|^2\Big)^{1/2}.$$

This means in view of (2) that we average $(f_{\pi(i)}^w)_{i=1}^n$ exactly over those coordinates that might be non-zero. Introducing the function $\Theta\colon \boldsymbol{R} \to \{0, 1\}$, $\Theta(0) = 0$ and $\Theta(a) = 1$ if $a \neq 0$, we get

$$\sum_{\eta\in D_\pi} |f_{\eta\circ\pi(i)}^w| = k_w!(n-k_w)!(1/k_w) \sum_{j=1}^n |f_j^w| \Theta(w_{\pi(i)}).$$

Denoting

$$a_w = (nk_w)^{-1} b_{k_w} \sum_{j=1}^n |f_j^w|,$$

we get

$$\|P\| \geqslant (1/\sqrt{2}) n \max_{z\in\tilde{M}} \sum_{w\in\tilde{M}} \binom{n}{k_w}^{-1} b_{k_w}^{-1} |a_w| \sum_{\pi\in G_w} \Big(\sum_{i=1}^n |z_i \Theta(w_{\pi(i)})|^2\Big)^{1/2}.$$

Condition (3) changes into

$$(4)\qquad \sum_{w\in\tilde{M}} |a_w| = 1.$$

Grouping once again $\tilde{M}_k = \{w \mid w \in \tilde{M}$ and w has exactly k non-zero coordinates$\}$ and denoting

$$a_k = \sum_{w\in\tilde{M}_k} |a_w|,$$

we get for (4) the condition

$$(5)\qquad \sum_{k=1}^n |a_k| = 1.$$

Moreover, we have

$$\|P\| \geqslant (1/\sqrt{2}) n \max_{z\in\tilde{M}} \sum_{k=1}^n \sum_{w\in\tilde{M}_k} \binom{n}{k}^{-1} b_k^{-1} |a_w| \sum_{\pi\in G_w} \Big(\sum_{i=1}^n |z_i \Theta(w_{\pi(i)})|^2\Big)^{1/2}.$$

Suppose z has exactly j non-zero coordinates. Let us denote $A_z = \{m \mid 1 \leqslant m \leqslant n, z_m \neq 0\}$ and for every l, $1 \leqslant l \leqslant k$, let $B_l = \{B \mid B \subset \{1, \ldots, n\}$, $\operatorname{card}(B) = k$, $\operatorname{card}(B\cap A_z) = l\}$. Thus we have

$$\binom{n}{k}^{-1} \sum_{\pi\in G_w} \Big(\sum_{i=1}^n |z_i \Theta(w_{\pi(i)})|^2\Big)^{1/2}$$
$$= (1/b_j^*) \binom{n}{k}^{-1} \sum_{l=1}^k \operatorname{card}(B_l)\sqrt{l} = (1/b_j^*) \sum_{l=1}^k \sqrt{l}\, \frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}},$$

where $b_j^* = \big\|\sum_{i=1}^j e_i^*\big\|$. Therefore we get

$$\|P\| \geqslant (1/\sqrt{2}) \max_{j\leqslant n} (n/j) b_j \sum_{k=1}^n |b_k^{-1} a_k| \sum_{l=1}^k \sqrt{l}\, \frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}},$$

which gives together with (5) the right-hand side estimation. We prove the left-hand side inequality. We have for symmetric Banach spaces [2]

$$n = \lambda(E)\pi_1(E).$$

We estimate $\pi_1(E)$ from below. We choose sets of vectors $K_k = \{x \mid \|x\| = 1$, x has exactly k coordinates different from zero, the absolute values of the non-zero coordinates are the same$\}$. For every $\{a_k\}_{k=1}^n$ such that $\sum_{k=1}^n |a_k| = 1$ we have

$$\pi_1(E) \geqslant \Big(\sum_{k=1}^n \sum_{x\in K_k} \Big\| \binom{n}{k}^{-1} 2^{-k} a_k x\Big\|\Big)\Big(\max_{\|x^*\|=1} \sum_{k=1}^n \sum_{x\in K_k} \Big|\Big\langle \binom{n}{k}^{-1} 2^{-k} a_k x, x^*\Big\rangle\Big|\Big)^{-1}$$

$$= \Big(\max_{\|x^*\|=1} \sum_{k=1}^n \sum_{x\in K_k} |a_k| \binom{n}{k}^{-1} 2^{-k} |\langle x, x^*\rangle|\Big)^{-1}.$$

On the other hand, denoting $\tilde{K}_k = \{x \mid x \in K_k,\ x_i \geqslant 0$ for $i = 1, \ldots, n\}$ and applying the Khintchine inequality, we have

$$\max_{\|x^*\|=1} \sum_{k=1}^n |a_k| \binom{n}{k}^{-1} 2^{-k} \sum_{x\in K_k} |\langle x, x^*\rangle|$$

$$\leqslant \max_{\|x^*\|=1} \sum_{k=1}^n |a_k| \binom{n}{k}^{-1} \sum_{x\in \tilde{K}_k} \Big(\sum_{i=1}^n |x_i x_i^*|^2\Big)^{1/2}$$

$$\leqslant \max_{j\leqslant n} b_j^{*-1} \sum_{k=1}^n |a_k| b_k^{-1} \sum_{l=1}^k \sqrt{l}\, \frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}}. \ \blacksquare$$

Proposition 2. *There is an absolute constant $C > 0$ such that*

(i) *for $jk \leqslant n$ and $1 \leqslant j,\ k \leqslant n$*

$$C(jk/n) \leqslant \sum_{l=1}^k \sqrt{l}\, \frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} \leqslant jk/n,$$

(ii) *for $jk \geqslant n$ and $1 \leqslant j,\ k \leqslant n$*

$$C\sqrt{jk/n} \leqslant \sum_{l=1}^k \sqrt{l}\, \frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} \leqslant \sqrt{jk/n}.$$

Proof. We need the following lemmas:

Lemma 3. *Let $n-j \geqslant k$ and $1 \leqslant j,\ k \leqslant n$. Then*

$$\binom{n-j}{k}\Big/\binom{n}{k} \leqslant \big(1-(k-1)/n\big)^j.$$

Proof.

$$\binom{n-j}{k}\Big/\binom{n}{k} = \frac{(n-j)(n-j-1)\ldots(n-j-k+1)}{n(n-1)\ldots(n-k+1)}$$

$$= (1-j/n)\big(1-j/(n-1)\big)\ldots\big(1-j/(n-k+1)\big).$$

Because of $1+r \leqslant e^r$ for all $r \in \mathbf{R}$ we get

$$\binom{n-j}{k}\Big/\binom{n}{k} \leqslant \exp(-j/n)\exp\big(-j/(n-1)\big)\ldots\exp\big(-j/(n-k+1)\big)$$

$$= \exp\Big(-j\sum_{l=0}^{k-1} 1/(n-l)\Big).$$

Thus we get

$$\binom{n-j}{k}\Big/\binom{n}{k} \leqslant \exp\big(-j(\ln(n)-\ln(n-k+1))\big) = \big((n-k+1)/n\big)^j. \ \blacksquare$$

We also use the following elementary fact:

Lemma 4. *For every $C > 1$ there exists an $m \in \mathbf{N}$ such that for all $s \in \mathbf{N}$ and $r \in \mathbf{R}$ with $s/r \leqslant 1/m$ we have*

$$(1-1/r)^s \leqslant 1-(1/C)(s/r).$$

In order to prove the left-hand side inequality of (i) we choose $d = m$ in Lemma 4 so big that the estimation holds for $C = 2$. Suppose first that $jk \leqslant n/d$. Then we have

$$\sum_{l=1}^k \sqrt{l}\, \frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} \geqslant \sum_{l=1}^k \frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} = 1 - \binom{n-j}{k}\Big/\binom{n}{k}.$$

By Lemmas 3 and 4 we get

$$\sum_{l=1}^k \sqrt{l}\, \frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} \geqslant (1/2)\big(j(k-1)/n\big).$$

If $n/d < jk \leqslant n$ is valid, we choose $\tilde{j}, \tilde{k}$ with $\tilde{j} \leqslant j$, $\tilde{k} \leqslant k$ and $n/2d \leqslant \tilde{j}\tilde{k}$. And we observe

$$\sum_{l=1}^k \sqrt{l}\, \frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} \geqslant \sum_{l=1}^{\tilde{k}} \sqrt{l}\, \frac{\binom{\tilde{j}}{l}\binom{n-\tilde{j}}{\tilde{k}-l}}{\binom{n}{\tilde{k}}} \geqslant (1/2)(\tilde{j}\tilde{k}/n) \geqslant (1/4d)(jk/n).$$

In order to prove the right-hand side inequalities of (i) and (ii) we use the fact that the expectation of the hypergeometric distribution is jk/n [9].

$$\sum_{l=1}^{k}\sqrt{l}\,\frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} \leqslant \sum_{l=1}^{k} l\,\frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} = jk/n.$$

And by Hölder's inequality

$$\sum_{l=1}^{k}\sqrt{l}\,\frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} \leqslant \left(\sum_{l=1}^{k} l\,\frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}}\right)^{1/2}\left(\sum_{l=0}^{k}\frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}}\right)^{1/2} = \sqrt{jk/n}.$$

It remains to prove the left-hand side inequality of (ii). Again, it is enough to prove it for $k,\ j \leqslant n/4$ and $jk \geqslant dn$ for a real number $d > 1$ that will be specified later. We need the following lemma:

LEMMA 5. *For every $\varepsilon > 0$, $a \geqslant 1$ there is a constant $C > 1$ such that for every $r \in \mathbf{R}$ with $r \geqslant 1$ we have*

$$\sum_{l\leqslant r/C}(1/l!)(ar)^l \leqslant (r/C+1)(1+\varepsilon)^r.$$

Proof. Of course, C can be chosen so big that for every $l \leqslant r/C$,

$$(1/l!)(ar)^l \leqslant (ear/l)^l = \big((ae)^{l/r}(r/l)^{l/r}\big)^r \leqslant (1+\varepsilon)^r.$$

Therefore

$$\sum_{l\leqslant r/C}(1/l!)(ar)^l \leqslant (1+r/C)(1+\varepsilon)^r. \blacksquare$$

Because of $k, j \leqslant n/4$ we have

$$\frac{\binom{j}{l+1}\binom{n-j}{k-l-1}}{\binom{j}{l}\binom{n-j}{k-l}} = \frac{(j-l)(k-l)}{(l+1)(n-j-k+l+1)} \leqslant 2\frac{jk}{(l+1)n}.$$

Hence

$$\binom{j}{l}\binom{n-j}{k-l} \leqslant \binom{n-j}{k}(1/l!)(2jk/n)^l,$$

and by applying Lemma 5 for $\varepsilon = 1/2$ and $a = 2$ we get a constant C such that for all $jk/n \geqslant 1$ we have

$$\sum_{l\leqslant(1/C)(jk/n)}\frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} \leqslant \frac{\binom{n-j}{k}}{\binom{n}{k}}\sum_{l\leqslant(1/C)(jk/n)}(1/l!)(2jk/n)^l$$

$$\leqslant \frac{\binom{n-j}{k}}{\binom{n}{k}}\big(1+(1/C)(jk/n)\big)(3/2)^{jk/n}.$$

And by Lemma 3 and $1+r \leqslant e^r$ the last quantity is not greater than

$$e\big(1+(1/C)(jk/n)\big)(2e/3)^{-jk/n}.$$

We choose d big enough to have

$$e\big(1+(1/C)(jk/n)\big)(2e/3)^{-jk/n} \leqslant 1/2.$$

With this we eventually get

$$\sum_{l=0}^{k}\sqrt{l}\,\frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}} \geqslant \sqrt{(1/C)(jk/n)}\left(1-\sum_{l\leqslant(1/C)(jk/n)}\frac{\binom{j}{l}\binom{n-j}{k-l}}{\binom{n}{k}}\right) \geqslant (1/2\sqrt{C})\sqrt{jk/n}. \blacksquare$$

The following theorem is obtained from Propositions 1 and 2:

THEOREM 6. *Suppose that $\{e_i\}_{i=1}^n$ is a symmetric basis of Banach space E and the extreme points of the dual unit ball are of the form $x = c_x\sum_{i=1}^n x_i e_i^*$ where $c_x \in \mathbf{R}$ and $x_i \in \{0, -1, 1\}$. Then*

$$\lambda(E) \leqslant \min_{\sum_{k=1}^n a_k=1}\ \max_{j\leqslant n} b_j\Big(\sum_{k\leqslant n/j}a_k b_k^* + \sum_{k>n/j}\sqrt{n/jk}\,a_k b_k^*\Big) \leqslant C\lambda(E),$$

where C is an absolute constant, $a_k \geqslant 0$, $b_k = \|\sum_{i=1}^k e_i\|$ and $b_k^* = \|\sum_{i=1}^k e_i^*\|$ for $k = 1, \ldots, n$.

COROLLARY 7. *Suppose that $\{e_i\}_{i=1}^n$ is a symmetric basis of a Banach space E and the extreme points of the dual unit ball are of the form $x = c_x\sum_{i=1}^n x_i e_i^*$ where $c_x \in \mathbf{R}$ and $x_i \in \{0, -1, 1\}$. Then*

$$\lambda(E) \geqslant (1/C)\Big(\log_2\Big(2\Big\|\sum_{i=1}^n e_i\Big\|\Big)\Big)^{-1}\min_{k\leqslant n}\max_{j\geqslant n/k}\sqrt{n/kj}\,\Big\|\sum_{i=1}^j e_i\Big\|\,\Big\|\sum_{i=1}^k e_i^*\Big\|$$

where C is an absolute constant.

Proof. We define $N_m = \{k \mid 2^{m-1} \leqslant b_k < 2^m\}$, $1 \leqslant m \leqslant t \leqslant \log_2 2b_n$, and for given $a_i \geqslant 0$, $i = 1, \ldots, n$ we define

$$a_m^* = \sum_{k\in N_m} a_k,$$

$$\tilde{a}_k = \begin{cases} a_m^* & \text{if } k = \min\{j \mid j \in N_m\},\\ 0 & \text{elsewhere.}\end{cases}$$

First, we prove that for all $a_i \geqslant 0$, $i = 1, \ldots, n$, we have

$$2\Big(\sum_{k\leqslant n/j} a_k(k/b_k) + \sum_{k>n/j} \sqrt{nk/j}\,(1/b_k)\,a_k\Big) \geqslant \sum_{k\leqslant n/j} \tilde{a}_k(k/b_k) + \sum_{k>n/j} \sqrt{nk/j}\,(1/b_k)\,\tilde{a}_k.$$

We introduce the notation $M^1 = \{m \mid \text{for all } k \in N_m \text{ we have } k \leqslant n/j\}$ and $M^2 = \{m \mid \text{for all } k \in N_m \text{ we have } k > n/j\}$. Thus there is exactly one m_0 so that $m_0 \notin M^1$ and $m_0 \notin M^2$. So we get

$$\sum_{k\leqslant n/j} a_k(k/b_k) + \sum_{k>n/j} \sqrt{nk/j}\,(1/b_k)\,a_k$$
$$= \sum_{m\in M^1}\sum_{k\in N_m} a_k(k/b_k) + \sum_{\substack{k\in N_{m_0}\\ k\leqslant n/j}} a_k(k/b_k) + \sum_{\substack{k\in N_{m_0}\\ k>n/j}} \sqrt{nk/j}\,(1/b_k)\,a_k +$$
$$+ \sum_{m\in M^2}\sum_{k\in N_m} \sqrt{nk/j}\,(1/b_k)\,a_k.$$

Considering the definition of N_m we get for the last expression

$$\sum_{m\in M^1}\sum_{k\in N_m} 2^{-m} a_k k + \sum_{\substack{k\in N_{m_0}\\ k\leqslant n/j}} 2^{-m_0} k a_k + \sum_{\substack{k\in N_{m_0}\\ k>n/j}} \sqrt{nk/j}\,2^{-m_0} a_k +$$
$$+ \sum_{m\in M^2}\sum_{k\in N_m} \sqrt{nk/j}\,2^{-m} a_k \geqslant \sum_{m\in M^1}\sum_{k\in N_m} 2^{-m}\tilde{a}_k k + \sum_{\substack{k\in N_{m_0}\\ k\leqslant n/j}} 2^{-m_0} k\tilde{a}_k +$$
$$+ \sum_{\substack{k\in N_{m_0}\\ k>n/j}} \sqrt{nk/j}\,2^{-m_0}\tilde{a}_k + \sum_{m\in M^2}\sum_{k\in N_m} \sqrt{nk/j}\,2^{-m}\tilde{a}_k.$$

Using the definition of N_m again, we get eventually

$$2\Big(\sum_{k\leqslant n/j} a_k(k/b_k) + \sum_{k>n/j} \sqrt{nk/j}\,(1/b_k)\,a_k\Big) \geqslant \sum_{k\leqslant n/j} \tilde{a}_k(k/b_k) + \sum_{k>n/j} \sqrt{nk/j}\,(1/b_k)\tilde{a}_k.$$

Applying now Theorem 6, we get that we have for a certain $\tilde{a}$

$$2C\,\lambda(E) \geqslant \max_{j\leqslant n} b_j\Big(\sum_{k\leqslant n/j} \tilde{a}_k b_k^* + \sum_{k>n/j} \sqrt{n/jk}\,\tilde{a}_k b_k^*\Big).$$

On the other hand, there are at most t numbers $\tilde{a}_k$, $t \leqslant \log_2 2b_n$, different from zero and $\sum_{k=1}^{n} \tilde{a}_k = 1$. Thus there is at least one k_0 so that

$$\tilde{a}_{k_0} \geqslant (2\log_2 2b_n)^{-1}.$$

So we get

$$2C\,\lambda(E) \geqslant (2\log_2 2b_n)^{-1} \max_{j\leqslant n} b_j \begin{cases} b_{k_0}^* & \text{if} \quad k_0 \leqslant n/j, \\ \sqrt{n/k_0 j} & \text{if} \quad k_0 > n/j. \end{cases}$$

Observing that $b_k b_j^* \leqslant b_{n/j} b_j^*$ for $jk \leqslant n$ and choosing a proper constant we finish the proof. ■

Proposition 8. *Suppose that $\{e_i\}_{i=1}^n$ is a symmetric basis of a Banach space E and the extreme points of the dual unit ball are of the form $x = c_x \sum_{i=1}^{n} x_i e_i^*$ where $c_x \in \boldsymbol{R}$ and $x_i \in \{0, -1, 1\}$. Then*

$$d(E, l_n^\infty) \leqslant C\Big(\log_2 2\Big\|\sum_{i=1}^{n} e_i\Big\|\Big)^{1/2} \min_{l\leqslant n} \max_{k\geqslant n/l} \sqrt{n/lk}\,\Big\|\sum_{i=1}^{k} e_i\Big\|\,\Big\|\sum_{i=1}^{l} e_i^*\Big\|.$$

Proof. Instead of $d(E, l_n^\infty)$ we consider $d(E^*, l_n^1)$. Admitting another constant, it is enough to take the minimum only over $l = 2^m$, $m \in \boldsymbol{N}$. First, suppose that $n/l \in \boldsymbol{N}$. Then we choose the mapping $U \in L(l_n^1, E^*)$ represented by the matrix

$$U = (1/b_l^*)\begin{bmatrix} W & & & 0 \\ & W & & \\ & & \ddots & \\ 0 & & & W \end{bmatrix},$$

where W is a Walsh matrix of rank l. We have $\|U\| = 1$ and

$$\|U^{-1}\| = \max_{\|x\|=1} \|U^{-1}(x)\|_1$$
$$= (b_l^*/l) \max_{k\leqslant n} \max_{\sum_{m=1}^{n/l} i_m = k} \max_{1\leqslant j_1^m,\ldots,j_{i_m}^m \leqslant l} \max_{\varepsilon_1^m = \pm 1} b_k^{*-1} \sum_{m=1}^{n/l} \sum_{r=1}^{l} \Big|\sum_{i=1}^{i_m} \varepsilon_i^m w_{j_i^m}(r)\Big|,$$

where $w_{j_i^m}(r)$ are the components of the Walsh matrix W. Because of the orthogonality of the Walsh matrix we get

$$\leqslant (b_l^*/l) \max_{k\leqslant n} \max_{\sum_{m=1}^{n/l} i_m = k} b_k^{*-1} l \sum_{m=1}^{n/l} \sqrt{i_m}$$

$$\leqslant \max_{k\leqslant n} b_l^*(b_k/k)\min\{k, \sqrt{nk/l}\} = \max_{k\geqslant n/l} b_l^* b_k \sqrt{n/kl}.$$

Suppose n/l is not a natural number. Then we choose the greatest natural number a that is smaller than n and $s/l \in \boldsymbol{N}$ and a matrix u of rank s as above. Moreover, we take the matrix

$$\tilde{u} = \begin{bmatrix} u & 0 \\ 0 & 0 \end{bmatrix}.$$
$$\qquad\;\; s \quad n-s$$

We also choose another matrix V. For $n-s$ there are unique numbers $a_i \in \{0, 1\}$ such that $n-s = \sum_{i=0}^{t} a_i 2^i$.

$$V = \begin{bmatrix} (1/b^*_{2^t})\, V_t & & & & & & 0 \\ & \ddots & & & & & \\ & & & (1/b^*_{2^i})\, V_i & & & \\ & & & & \ddots & & \\ 0 & & & & & & V_0 \end{bmatrix},$$

where V_i are Walsh matrices of rank 2^i that appear in the matrix V whenever $a_i = 1$. Now we take

$$v = \begin{array}{c} \begin{bmatrix} 0 & 0 \\ 0 & V \end{bmatrix} \\ \begin{matrix} s & n-s \end{matrix} \end{array}$$

and the operators $U_1, U_2 \in L(l^1_n, E)$ represented by $\tilde{u}$ and v. We consider $U_1 + U_2$ and get $\|U_1 + U_2\| = 1$ and

$$\|(U_1+U_2)^{-1}\| = \|U_1^{-1} + U_2^{-1}\| \leqslant \|U_1^{-1}\| + \|U_2^{-1}\|.$$

We computed $\|U_1^{-1}\|$ already in the first part of this proof. It remains to estimate $\|U_2^{-1}\|$.

$$\|U_2^{-1}\| \leqslant \max_{k\leqslant n} \max_{\sum_{m=1}^{t} i_m = k} \max_{1\leqslant j_1^m,\ldots,j_{i_m}^m \leqslant 2^m} \max_{\varepsilon_i^m = \pm 1} b_k^{*-1} \sum_{m=1}^{t} b^*_{2^m} 2^{-m} \sum_{r=1}^{2^m} \Big| \sum_{i=1}^{i_m} \varepsilon_i^m w_{j_i^m}(r) \Big|.$$

Because of the orthogonality of the Walsh matrix the last quantity is less than

$$\max_{k\leqslant n} \max_{\sum_{m=1}^{t} i_m = k} b_k^{*-1} \sum_{m=1}^{t} b^*_{2^m} \sqrt{i_m}.$$

Because of the Hölder inequality and $t \leqslant \log_2 l$ we get for the last quantity

$$\max_{k\leqslant n} b_k^{*-1} b_l^* (\log_2 l)^{1/2} \sqrt{k}.$$

It is left to prove that for such an l where

$$\max_{k\geqslant n/l} \sqrt{n/kl}\, b_k b_l^*$$

takes the minimum we have $l \leqslant b_n^4$. Indeed, we would have

$$\begin{aligned} \max_{k\leqslant n} \frac{b_k}{\sqrt{k}} b_l^* (\log_2 l)^{1/2} &\leqslant (4\log_2 b_n)^{1/2} \max_{k\leqslant n} \frac{b_k}{\sqrt{k}} b_l^* \\ &\leqslant (4\log_2 b_n)^{1/2} \max\Big\{ \max_{k<n/l} \frac{b_k}{\sqrt{k}} b_l^*, \max_{k\geqslant n/l} \frac{b_k}{\sqrt{k}} b_l^* \Big\} \\ &\leqslant (4\log_2 b_n)^{1/2} \max\Big\{ \max_{k<n/l} b_k b_l^*, \max_{k\geqslant n/l} \sqrt{\frac{n}{kl}}\, b_k b_l^* \Big\}. \end{aligned}$$

Choosing a proper constant C, we get

$$\max_{k\leqslant n} \frac{b_k}{\sqrt{k}} b_l^* (\log_2 l)^{1/2} \leqslant C (\log_2 b_n)^{1/2} \max_{k\geqslant n/l} \sqrt{n/kl}\, b_k b_l^*.$$

Thus it is left to show $l \leqslant b_n^4$. Obviously, we may assume that $2 \leqslant b_n$. Thus we get by the triangle inequality

$$\Big\| \sum_{i=1}^{l} e_i^* \Big\| \geqslant \frac{l}{2n} \Big\| \sum_{i=1}^{n} e_i^* \Big\|$$

or if $l > b_n^4$,

$$b_l^* \geqslant \frac{l}{2n} b_n^* \geqslant \frac{1}{2n} b_n^4 b_n^* = \frac{1}{2} b_n^3 \geqslant b_n^2.$$

By this we get

$$\max_{k\geqslant n/l} \sqrt{n/kl}\, b_k b_l^* \geqslant \frac{1}{\sqrt{2}} b_l^* \geqslant \sqrt{2}\, b_n.$$

Since the minimum is less than $b_n = \big\|\sum_{i=1}^{n} e_i\big\|$, this shows that $l \leqslant b_n^4$. ∎

THEOREM 9. *Suppose E is an n-dimensional Banach space with a symmetric basis. Then*

$$\lambda(E) \leqslant d(E, l^\infty_n) \leqslant C\big(\log_2 4\lambda(E)\big)^{7/2} \lambda(E),$$

where C is an absolute constant.

For the proof we need a generalization of a lemma in [12].

LEMMA 10. *Suppose E is an n-dimensional Banach space with a symmetric basis. Then there exists another Banach space $\tilde{E}$ with a symmetric* *basis $\{e_i\}_{i=1}^n$ such that the extreme points of the dual unit ball are of the form $x = c_x \sum_{i=1}^n e_i^*$, where $c_x \in \mathbf{R}$ and $x_i \in \{0, -1, 1\}$ and*

$$d(E, \tilde{E}) \leqslant C\Big(\log_2 2\Big\|\sum_{i=1}^n e_i\Big\|\Big),$$

where C is an absolute constant.

The proof of the lemma is based on the simple observation that the norm of a vector $\sum_{i=1}^n a_i e_i$ with norm 1 does not change much if we set zero all coordinates less than $2\|\sum_{i=1}^n e_i\|$. The rest of the vector is "sliced" into $\log(2\|\sum_{i=1}^n e_i\|)$ pieces.

Proof of Theorem 9. Suppose the dual unit ball fulfills the hypothesis of Corollary 7 and Proposition 8. Then

$$d(E, l_n^\infty) \leqslant C_1\Big(\log_2 2\Big\|\sum_{i=1}^n e_i\Big\|\Big)^{1/2} \min_{l\leqslant n}\max_{k>n/l} \sqrt{n/kl}\,\Big\|\sum_{i=1}^k e_i\Big\|\,\Big\|\sum_{i=1}^l e_i^*\Big\|$$

$$\leqslant C_2\Big(\log_2 2\Big\|\sum_{i=1}^n e_i\Big\|\Big)^{3/2}\lambda(E).$$

Now we apply Corollary 3 of [10] saying that

$$\Big\|\sum_{i=1}^n e_i\Big\| \leqslant 2\lambda(E)^2.$$

Thus we get

$$d(E, l_n^\infty) \leqslant C_3(\log_2 4\lambda(E))^{3/2}\lambda(E).$$

If the dual unit ball does not fulfill the hypothesis of Corallary 7 we apply Lemma 10 and get because of $d(E, l_n^\infty) \leqslant d(E, \tilde{E})\, d(\tilde{E}, l_n^\infty)$ and $\lambda(\tilde{E}) \leqslant \lambda(E)\, d(E, \tilde{E})$ the right hand side inequality. ∎

2. Estimations of $\gamma_p(E)$ for $p \in (1, 2, \infty)$. In the following theorem $|G|$ denotes card(G).

THEOREM 11. *Suppose that there are sets G and D such that $\{e_i\}_{i=1}^n$ is a G, D-symmetric basis of the Banach space E with constant C and*

$$(6)\qquad (1/|D|)\sum_{\pi\in D}|a_{\pi(i)}| = (1/n)\sum_{i=1}^n |a_i| \quad \textit{for all } a \in \mathbf{R}^n,$$

$$(7)\qquad (1/\sqrt{|G|})\Big(\sum_{\varepsilon\in G}\Big|\sum_{i=1}^n \varepsilon_i a_i\Big|^2\Big)^{1/2} \underset{K}{\sim} \Big(\sum_{i=1}^n |a_i|^2\Big)^{1/2} \quad \textit{for all } a \in \mathbf{R}^n.$$

Let $I \in L(E, l_n^2)$ denote the isomorphism $I\big(\sum_{i=1}^n a_i e_i\big) = (a_i)_{i=1}^n$. Then

$$(8)\qquad d(E, l_n^2) \leqslant \|I\|\,\|I^{-1}\| \leqslant K^2C^2 d(E, l_n^2).$$

Proof. We consider isomorphisms $J \in L(E, l_n^2)$ with $J\big(\sum_{k=1}^n a_k f_k\big) = (a_k)_{k=1}^n$ where $\{f_k\}_{k=1}^n$ is a basis of E. We choose $x = \sum_{i=1}^n x_i e_i$ with $\|x\| = 1$ and $\|I(x)\| = \|I\|$. Then we have

$$\|J\| \geqslant (1/C)\max_{\substack{\varepsilon\in G\\ \pi\in D}}\Big\|J\Big(\sum_{i=1}^n \varepsilon_i x_{\pi(i)} e_i\Big)\Big\|$$

$$= (1/C)\max_{\substack{\varepsilon\in G\\ \pi\in D}}\Big(\sum_{k=1}^n\Big|\sum_{i=1}^n \varepsilon_i x_{\pi(i)}\langle e_i, f_k^*\rangle\Big|^2\Big)^{1/2}$$

$$\geqslant (1/C)\max_{\pi\in D}(1/\sqrt{|G|})\Big(\sum_{\varepsilon\in G}\sum_{k=1}^n\Big|\sum_{i=1}^n \varepsilon_i x_{\pi(i)}\langle e_i, f_k^*\rangle\Big|^2\Big)^{1/2}.$$

Because of (7) we get

$$\|J\| \geqslant (1/CK)\max_{\pi\in D}\Big(\sum_{i,k=1}^n |x_{\pi(i)}\langle e_i, f_k^*\rangle|^2\Big)^{1/2}.$$

Averaging in the same way over D, we get

$$\|J\| \geqslant (1/CK)(1/\sqrt{n})\Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2}\Big(\sum_{i,k=1}^n |\langle e_i, f_k^*\rangle|^2\Big)^{1/2}.$$

Because of $\|I(x)\| = \|I\|$ we have

$$(9)\qquad \|J\| \geqslant (1/CK)(1/\sqrt{n})\,\|I\|\Big(\sum_{i,k=1}^n |\langle e_i, f_k^*\rangle|^2\Big)^{1/2}.$$

On the other hand, we choose $x = \sum_{i=1}^n x_i e_i$ with $\|x\| = 1$ and $\|I(x)\|^{-1} = \|I^{-1}\|$. We have

$$\|J^{-1}\| \geqslant (1/C)\Big(\min_{\substack{\varepsilon\in G\\ \pi\in D}}\Big\|J\Big(\sum_{i=1}^n \varepsilon_i x_{\pi(i)} e_i\Big)\Big\|\Big)^{-1}$$

$$= (1/C)\Big(\min_{\substack{\varepsilon\in G\\ \pi\in D}}\Big(\sum_{k=1}^n\Big|\sum_{i=1}^n \varepsilon_i x_{\pi(i)}\langle e_i, f_k^*\rangle\Big|^2\Big)^{1/2}\Big)^{-1}.$$

Now we average in the same way as above over G and D and obtain

$$\|J^{-1}\| \geqslant (1/CK)\sqrt{n}\Big(\Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2}\Big(\sum_{i,k=1}^n |\langle e_i, f_k^*\rangle|^2\Big)^{1/2}\Big)^{-1}.$$

But because of $\|I(x)^{-1}\| = \|I^{-1}\|$ we get

$$\|J^{-1}\| \geqslant (1/CK)\sqrt{n}\,\|I^{-1}\|\Big(\sum_{i,k=1}^{n} |\langle e_i, f_k^*\rangle|^2\Big)^{-1/2}.$$

Together with (9) we get the right-hand side inequality of (8). ■

COROLLARY 12. *Let E and F be Banach spaces with symmetric bases $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^m$ with constant 1 and let $I \in L(E\otimes_a F, l_{nm}^2)$ be the map $I(\sum_{i,j=1}^{n,m} a_{ij} e_i\otimes f_j) = (a_{ij})_{i,j=1}^{n,m}$ where $\otimes_a$ denotes any tensor product. Then*

$$d(E\otimes_a F, l_{nm}^2) = \|I\|\,\|I^{-1}\|.$$

Proof. As basis we choose $\{e_i\otimes f_j\}_{i,j=1}^{n,m}$. For G we choose the set $\{(\varepsilon_i\eta_j)_{i,j=1}^{n,m} \mid \varepsilon_i = \pm 1, \eta_j = \pm 1\}$ and for D the set $\{\pi\circ\sigma \mid \pi$ changes rows, σ changes columns$\}$. It is easy to check that (6) is fulfilled. To verify (7) we apply the Khintchine inequality for $p = 2$ [5], [11] twice. We have $C = 1$ and $K = 1$. ■

THEOREM 13. *Suppose that there is a set G such that $\{e_i\}_{i=1}^n$ is a G-unconditional basis with constant C of a Banach space E and*

$$K(1/|G|)\sum_{\varepsilon\in G}\Big|\sum_{i=1}^{n}\varepsilon_i a_i\Big| \geqslant \Big(\sum_{i=1}^{n}|a_i|^2\Big)^{1/2} \quad \textit{for all } a\in \mathbf{R}^n. \tag{10}$$

Let $I \in L(E, l_n^2)$ denote the isomorphism $I(\sum_{i=1}^{n} a_i e_i) = (a_i)_{i=1}^n$. Then

$$\max_{\pm}\Big\|\sum_{i=1}^{n} \pm e_i\Big\|\,\|I^{-1}\|^{-1} \leqslant CK\,\gamma_\infty(E).$$

The proof is the same as the proof of Theorem 1 in [10].

COROLLARY 14. *Let E and F be Banach spaces with unconditional bases $\{e_i\}_{i=1}^n$ and $\{f_j\}_{j=1}^m$ and let $I \in L(E\otimes_a F, l_{nm}^2)$ be the isomorphism*

$$I\Big(\sum_{i,j=1}^{n,m} a_{ij} e_{ij}\otimes f_j\Big) = (a_{ij})_{i,j=1}^{n,m},$$

where $\otimes_a$ denotes any tensor product. Then

$$\max_{\pm}\Big\|\sum_{i,j=1}^{n,m} \pm e_i\otimes f_j\Big\|\,\|I^{-1}\|^{-1} \leqslant 2\gamma_\infty(E\otimes_a F).$$

Proof. As basis we choose $\{e_i\otimes f_j\}_{i,j=1}^{n,m}$ and for G the set $\{(\varepsilon_i\eta_j)_{i,j=1}^{n,m} \mid \varepsilon_i = \pm 1, \eta_j = \pm 1\}$. By applying the Khintchine inequality [5], [11] for $p = 1$ twice and the triangle inequality once we get (10) with $C = 1$ and $K = 2$. ■

3. Applications. As applications we want estimate the Banach–Mazur distance of ε- and π-tensor products of l_n^r, $1 \leqslant r \leqslant \infty$, to $l_{n^2}^p$ for $p = 1, 2, \infty$.

LEMMA 15. *Let W denote the Walsh matrix of rank n. We consider W as an element of $l_n^r\otimes_\varepsilon l_n^r$ and $W^k = (\underbrace{W, \ldots, W}_{k\text{-times}})$ as an element of $l_n^r\otimes_\varepsilon l_{k\cdot n}^r$. Then*

$$\|W\| \leqslant n^{2/r-1/2} \quad \textit{if} \quad 1 \leqslant r \leqslant 2, \tag{11}$$

$$\|W^k\| \leqslant (kn)^{1/r} \quad \textit{if} \quad 2 \leqslant r \leqslant \infty. \tag{12}$$

(11) and (12) for $k = 1$ can be proved by using Clarkson's inequalities [1]. For $k \neq 1$ we have $\|W^k\| \leqslant k^{1/r}\|W\|$.

The following estimations are due to Hardy and Littlewood [6].

PROPOSITION 16. *Let $A = (a_{ij})_{i,j=1}^{n,m} \in l_n^r\otimes_\varepsilon l_m^r$, $1/\alpha = 2/r - 1$ and $1/\beta = 1/r - 1/4$. Then we have*

(i) *for* $1 \leqslant r \leqslant 4/3$

$$\Big(\sum_{i,j=1}^{n,m}|a_{ij}|^\beta\Big)^{1/\beta} \leqslant K\|A\|,$$

(ii) *for* $4/3 \leqslant r \leqslant 2$

$$\Big(\sum_{i=1}^{n}\Big(\sum_{j=1}^{m}|a_{ij}|^2\Big)^{\alpha/2}\Big)^{1/\alpha} \leqslant K\|A\|,$$

where K is an absolute constant.

PROPOSITION 17. *Let $I \in L(l_n^r\otimes_\varepsilon l_m^r, l_{nm}^2)$ be an identity. Then we have*

$$\|I^{-1}\| = \begin{cases} (nm)^{1/r-1/2} & \textit{if} \quad 1 \leqslant r \leqslant 2,\\ 1 & \textit{if} \quad 2 \leqslant r \leqslant \infty \end{cases} \tag{13}$$

and

$$(1/K) f(n, m, r) \leqslant \|I\| \leqslant K f(n, m, r), \tag{14}$$

where

$$f(n, m, r) = \begin{cases} 1 & \textit{if} \quad 1 \leqslant r \leqslant 4/3,\\ (\min\{n, m\})^{3/2-2/r} & \textit{if} \quad 4/3 \leqslant r \leqslant 2,\\ (nm)^{1/2}(\max\{n, m\})^{-1/r} & \textit{if} \quad 2 \leqslant r \leqslant \infty \end{cases}$$

and K is an absolute constant.

Proof. By considering the vectors $(1, \ldots, 1)\otimes(1, \ldots, 1)$ and $(1, 0, \ldots, 0)\otimes(1, 0, \ldots, 0)$ it follows that $\|I^{-1}\|$ is greater than the right-hand side expression in (13). On the other hand, $\|A\| \leqslant (\sum_{i,j=1}^{n,m}|a_{ij}|^r)^{1/r}$, so that we get equality in (13).

Of course, we have $\|I\| \geqslant 1$. The other left-hand side inequalities follow from Lemma 15. The right-hand side inequalities for $1 \leqslant r \leqslant 2$ follow from Proposition 16. For $2 \leqslant r \leqslant \infty$ we have

$$\|A\| \geqslant \max_{i \leqslant n} \Big(\sum_{j=1}^{m} |a_{ij}|^r \Big)^{1/r} \geqslant n^{-1/2} m^{1/r-1/2} \Big(\sum_{i,j=1}^{n,m} |a_{ij}|^2 \Big)^{1/2}. \ \blacksquare$$

We shall need the following functions:

$$h_1(n, r) = \begin{cases} n^{5/2-2/r} & \text{if} \quad 1 \leqslant r \leqslant 4/3, \\ n & \text{if} \quad 4/3 \leqslant r \leqslant \infty, \end{cases}$$

$$h_2(n, r) = \begin{cases} n^{2/r-1} & \text{if} \quad 1 \leqslant r \leqslant 4/3, \\ n^{1/2} & \text{if} \quad 4/3 \leqslant r \leqslant 2, \\ n^{1/r} & \text{if} \quad 2 \leqslant r \leqslant \infty, \end{cases}$$

$$h_\infty(n, r) = \begin{cases} n & \text{if} \quad 1 \leqslant r \leqslant 2, \\ n^{2/r} & \text{if} \quad 2 \leqslant r \leqslant \infty. \end{cases}$$

THEOREM 18. *Let $p \in \{1, 2, \infty\}$. Then*

$$(1/K) h_p(n, r) \leqslant \gamma_p(l_n^r \otimes_\varepsilon l_n^r) \leqslant d(l_n^r \otimes_\varepsilon l_n^r, l_{n^2}^p) \leqslant K h_p(n, r),$$

where K is an absolute constant.

Proof. The case $p = 2$ follows from Corollary 12 and Proposition 17. The left-hand side inequality for $p = \infty$ follows from Corollary 14 and Proposition 17. In order to prove the right-hand side inequality for $2 \leqslant r \leqslant \infty$ we consider the identity $I \in L(l_n^r \otimes_\varepsilon l_n^r, l_{n^2}^\infty)$. In the case $1 \leqslant r \leqslant 2$ we choose a mapping $U \in L(l_n^r, l_n^\infty)$ such that $\|U\| \leqslant 1$ and $\|U^{-1}\| \leqslant C\sqrt{n}$ [4]. Then $\|U \otimes U\| \leqslant 1$ and $\|(U \otimes U)^{-1}\| = \|U^{-1} \otimes U^{-1}\| \leqslant C^2 n$. In order to prove the left-hand side inequality for $p = 1$ we use $\gamma_1(l_n^r \otimes_\varepsilon l_n^r) = \gamma_\infty(l_n^{r'} \otimes_\pi \otimes_\pi l_n^{r'})$ and apply Corollary 14. We find that

$$\|W\|_\pi \|I^{-1}\|^{-1} \leqslant 2\gamma_1(l_n^r \otimes_\varepsilon l_n^r)$$

is valid where $I \in L(l_n^{r'} \otimes_\pi l_n^{r'}, l_{n^2}^2)$ denotes an identity. But $(I^{-1})^t$ is an identity in $L(l_n^r \otimes_\varepsilon l_n^r, l_{n^2}^2)$. So we can apply Proposition 17. Now we prove the right-hand side estimation. If $1 \leqslant r \leqslant 4/3$, we consider the identity $I \in L(l_n^r \otimes_\varepsilon l_n^r, l_{n^2}^1)$ and apply Proposition 16. For the case $4/3 \leqslant r \leqslant \infty$ we need the following lemma:

LEMMA 19. *Suppose $\{e_i\}_{i=1}^m$ and $\{f_j\}_{j=1}^m$ are normalized bases in a Banach space E with*

$$(15) \qquad \sum_{i=1}^{m} \langle e_i, f_j^* \rangle \langle e_i, f_k^* \rangle = 0$$

if $j \neq k$ and

$$(16) \qquad \Big(\sum_{i=1}^{m} |\langle e_i, f_j^* \rangle|^2 \Big)^{1/2} \leqslant C \min_{\|y^*\|=1} \Big(\sum_{i=1}^{m} |\langle e_i, y^* \rangle|^2 \Big)^{1/2}$$

for all $j = 1, \ldots, m$. Then

$$(17) \qquad d(E, l_m^1) \leqslant C\sqrt{m}.$$

Proof. We consider the isomorphism $I \in L(E, l_m^1)$ with $I(\sum_{j=1}^m a_j f_j) = (a_j)_{j=1}^m$. We have $\|I^{-1}\| = 1$ and

$$\begin{aligned} \|I\| &= \max_{\|y\|=1} \sum_{j=1}^{m} |\langle y, f_j^* \rangle| \\ &= \max_{\|y\|=1} \sum_{j=1}^{m} \Big| \sum_{i=1}^{m} \langle y, e_i^* \rangle \langle e_i, f_j^* \rangle \Big| \\ &\leqslant \max_{\|y\|=1} \sqrt{m} \Big(\sum_{j=1}^{m} \Big| \sum_{i=1}^{m} \langle y, e_i^* \rangle \langle e_i, f_j^* \rangle \Big|^2 \Big)^{1/2}. \end{aligned}$$

Now we put

$$z_j = \Big(\sum_{i=1}^{m} |\langle e_i, f_j^* \rangle|^2 \Big)^{-1/2} (\langle e_i, f_j^* \rangle)_{i=1}^m = \lambda_j (\langle e_i, f_j^* \rangle)_{i=1}^m$$

and have because of (15)

$$\langle z_j, z_k \rangle = \delta_{jk} \quad \text{for} \quad j, k = 1, \ldots, m.$$

Therefore we get

$$\begin{aligned} \|I\| &\leqslant \sqrt{m} \max_{\|y\|=1} \Big\| \sum_{j=1}^{m} \Big(\sum_{i=1}^{m} \langle y, e_i^* \rangle \langle e_i, f_j^* \rangle \Big) z_j \Big\|_2 \\ &= \sqrt{m} \max_{\|y\|=1} \Big\| \sum_{j=1}^{m} \Big\langle \frac{1}{\lambda_j} z_j, (\langle y, e_i^* \rangle)_{i=1}^m \Big\rangle z_j \Big\|_2 \\ &\leqslant \sqrt{m} \max_{\|y\|=1} \max_{j \leqslant m} \frac{1}{|\lambda_j|} \Big\| \sum_{j=1}^{m} \langle z_j, (\langle y, e_i^* \rangle)_{i=1}^m \rangle z_j \Big\|_2 \\ &= \sqrt{m} \max_{j \leqslant m} \Big(\sum_{i=1}^{m} |\langle e_i, f_j^* \rangle|^2 \Big)^{1/2} \max_{\|y\|=1} \Big(\sum_{i=1}^{m} |\langle y, e_i^* \rangle|^2 \Big)^{1/2}. \end{aligned}$$

And because of (16) we get

$$\|I\| \leqslant C\sqrt{m} \min_{\|y^*\|=1} \Big(\sum_{i=1}^{m} |\langle e_i, y^* \rangle|^2 \Big)^{1/2} \max_{\|y\|=1} \Big(\sum_{i=1}^{m} |\langle y, e_i^* \rangle|^2 \Big)^{1/2} = C\sqrt{m}. \ \blacksquare$$

We prove that for all $n = 2^i$, $m = 2^j$, $i, j \in \mathbf{N}$ we have with a constant C

$$(18) \qquad d(l_n^r \otimes_\varepsilon l_m^r, l_{nm}^1) \leqslant C\sqrt{nm}.$$

Suppose we have $r \leqslant 2$ and $m = kn$, $k \in \mathbf{N}$. Then we put

$$W_{s,t} = \big(w(\alpha, s)w(\alpha, \beta)w(t, \beta)\big)_{\alpha,\beta=1}^{m} \quad \text{for} \quad 1 \leqslant s, t \leqslant m,$$

where $w(\alpha, \beta)$ denote the coordinates of the Walsh matrix W of rank n. Now we put

$$W_{s,t}^{l} = (0, \ldots, 0, \underset{\substack{\uparrow \\ l\text{th coordinate}}}{W_{s,t}}, 0, \ldots, 0) \quad \text{for} \quad 1 \leqslant l \leqslant k,$$

and applying Lemma 19 we choose as bases $\{e_i \otimes e_j\}_{i,j=1}^{n,m}$ and $\{\|W\|_s^{-1} W_{s,t}^{l}\}_{l,s,t}$ where $\{e_i\}_{i=1}^{n}$ and $\{e_j\}_{j=1}^{m}$ denote the unit vector bases of l_n^r and l_m^r. The dual basis of $\{\|W\|_s^{-1} W_{s,t}^{l}\}_{l,s,t}$ is $\{\|W\|_\pi^{-1} W_{s,t}^{l}\}_{l,s,t}$ and (15) is easily verified. We verify (16). The left-hand side expression equals $n\|W\|_\pi^{-1}$. By applying Lemma 15 we get

$$n\|W\|_\pi^{-1} \leqslant \|W\|_s(1/n) \leqslant n^{2/r-3/2}.$$

And the right-hand side expression equals $\|I\|^{-1}$ of Proposition 17. Now suppose $r \geqslant 2$ and $n = km$, $k = 2^l$, $l \in \mathbf{N}$. We take $W^k = \big(w^k(\alpha, \beta)\big)_{\alpha,\beta=1}^{n,m}$ as defined in Lemma 15 and a Walsh matrix $V = \big(v(\alpha, \beta)\big)_{\alpha,\beta=1}^{m}$. We put

$$U_{s,t} = \big(w(\alpha, s)w^k(\alpha, \beta)v(t, \beta)\big)_{\alpha,\beta=1}^{n,m}$$

and choose as bases in Lemma 19 $\{e_i \otimes e_j\}_{i,j=1}^{n,m}$ and $\{\|W^k\|_s^{-1} U_{s,t}\}_{s,t=1}^{n,m}$. Now we proceed as above. Thus we have (18). In order to get estimation (18) for arbitrary $n, m \in \mathbf{N}$ we apply an inductive argument used in [1] to prove a proposition (p. 28). First, we fix $n = 2^i$ and apply the inductive argument to m. Then, having (18) with another constant for arbitrary m, we fix m and apply the argument to n. ∎

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