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On the integrability of the ergodic maximal function

by

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Abstract. Let f be a nondecreasing integrable function on the Lebesgue unit interval $[0,1]$, and define

$$M(x) = \int_0^x f(t)\,dt + \int_{1-x}^1 f(t)\,dt, \qquad 0 \leqslant x \leqslant 1/2.$$

It is shown that there is a stationary ergodic sequence $f_1, f_2, \ldots$ of random variables which has integrable ergodic maximal function, and such that f_1 has the same distribution as f, if and only if

$$\int_0^{1/2} |M(x)/x|\,dx < \infty.$$

For nonnegative functions this characterization reduces to $L\log L$, a result of Ornstein. Related inequalities are proved.

1. Introduction. Let $U = \{e^{i\theta}\colon 0 \leqslant \theta < 2\pi\}$ be the unit circle in the complex plane, and make U a probability space by equipping its Borel sets with the measure m given by $dm = d\theta/2\pi$. If f is an integrable function on any probability space let $\hat{f}$ be the function on U which is equidistributed with f, and which is not increasing as θ goes from 0 to 2π. (Functions equal almost everywhere are identified.) Define $M_f(\theta) = M(\theta) = \int_{-\theta}^{\theta} \hat{f}(e^{i\varphi})\,d\varphi$, and let

$$H(f) = \int_0^{\pi} |M(\theta)/\theta|\,d\theta.$$

Clearly, if f and $-f$ are equidistributed then $H(f) = 0$, and it is not difficult to check that, if f is nonnegative, then $H(f)$ is finite if and only if $f \in L\log L$, that is $E|f|\log[\max(|f|, 1)] < \infty$.

Now let T be an invertible and ergodic measure preserving transformation of a probability space $(\Omega, \mathfrak{F}, P)$ onto itself (see [1] for definitions). Functions defined on Ω, and subsets of Ω, will always be $\mathfrak{F}$ measurable. If f is a function on Ω, the ergodic maximal function f^* of f

is defined by

$$f^* = \sup_{n\geqslant 1} (1/n)\Big|\sum_{k=0}^{n-1} T^k f\Big|,$$

where $T^k f = f(T^k)$. The following theorem is proved.

THEOREM 1.1. *There is a positive constant c, not depending on Ω, T, or f, such that*

$$Ef^* \geqslant cH(f). \tag{1.1}$$

Inequality (1.1) may be considered to be an extension of Ornstein's theorem, of [5], that if $f \geqslant 0$ then $Ef^* = \infty$ if $f \notin L\log L$.

It is also shown that there is an absolute positive constant C such that given any function f on Ω there is a function g on Ω which is equidistributed with f satisfying

$$Eg^* \leqslant C\big(H(f) + E|f|\big). \tag{1.2}$$

Of course $H(f) = H(g)$ and $E|f| = E|g|$. Inequalities (1.1) and (1.2) together imply that the distribution of f is the distribution of a function on Ω which has integrable ergodic maximal function if and only if $H(f) < \infty$.

It is well known (see [4] for a proof) that $f \in L\log L$ implies $Ef^* < \infty$, and it is easily shown that, if $f \notin L\log L$, there is a function h on Ω which is equidistributed with f such that $Eh^* = \infty$. A little more will be said about this at the end of Section 4. We note that for $p > 1$ there are absolute constants C_p such that $Ef^{*p} \leqslant C_p E|f|^p$. See [3], p. 469.

The quantity $H(f)$ was defined in [2], and it was shown there that $H(f) < \infty$ characterized the distributions of functions in the Hardy space $\operatorname{Re} H_1(U)$ as well as the distributions of functions in several spaces of martingales.

The proof of Theorem (1.1) first uses methods similar to Ornstein's in [5] to prove a distributional inequality for f^* and then employs arguments like those of [2].

The construction of the function g_1 satisfying (1.2) involves a decomposition of a function f satisfying $H(f) < \infty$ into a bounded function plus a countable number of functions f_i which have disjoint support and expectation 0, such that f_i takes on only two non-0 values and $\sum_{i=1}^{\infty} H(f_i) < \infty$. This decomposition could be used to give an alternative construction of the function analagous to g in [2].

The proof of Theorem 1.1 goes over almost without change to prove a theorem involving infinite measure spaces which will be stated here but not mentioned further. If f is an integrable function on an infinite measure space $(X, \mathcal{O}, \mu)$, let $\hat{f}$ be the function on $(-\infty, \infty)$ which is not negative and not decreasing on $(-\infty, 0)$, is not positive and not decreasing on $(0, \infty)$, and has the same distribution (with respect to Lebesgue measure) as f, and define $M(x) = \int_{-x}^{x} \hat{f}(y)\,dy$ and

$$H(f) = \int_0^\infty |M(x)/x|\,dx.$$

If T is an invertible measure preserving ergodic transformation of X into itself, then there exists an absolute positive constant k, such that if f is an integrable function on X then $\int_X f^*\,d\mu \geqslant kH(f)$. This immediately implies Ornstein's result ([5]) that if $f \geqslant 0$ and $\mu(f > 0) > 0$ then $\int_X f^*\,d\mu = \infty$. We have not been able to prove an analogue of (1.2).

2. Lower bounds for Ef^*. In this section the inequalities (1.1) will be proved. The symbols c, k, etc. will stand for positive absolute constants, and the same symbol will be used for different constants. Notation will be as in the introduction except that until further notice it will be assumed that $Ef = 0$. If $\lambda > 0$, let $\theta_\lambda = \sup\{\theta \in [0, 2\pi): \hat{f}(e^{i\theta}) \geqslant \lambda\}$. Note $\theta_\lambda < 2\pi$ since $Ef = 0$. Let φ_λ be the unique number in $(-2\pi, 0]$ satisfying

$$\int_{\varphi_\lambda}^{\theta_\lambda} f(e^{i\theta})\,dm = \lambda(\theta_\lambda - \varphi_\lambda)/2\pi,$$

define $\psi_\lambda(f) = \psi_\lambda = (\theta_\lambda - \varphi_\lambda)/2\pi$, and take $\psi_0 = P(f \neq 0)$. First the inequality

$$P(f^* \geqslant \lambda) \geqslant \psi_\lambda/2, \qquad \lambda > 0, \tag{2.1}$$

will be proved. The argument resembles Ornstein's in [5], and it will be apparent after the proof is completed that it is not always true that $P(f^* \geqslant \lambda) \geqslant C\psi_\lambda$ for any constant $C > 1/2$.

Define, for $x \in \Omega$,

$$S_n(x) = (1/n)\big(f(x) + f(T^{-1}x) + \ldots + f(T^{-(n-1)}x)\big), \qquad n \geqslant 1.$$

For $\lambda > 0$ and $k \geqslant 1$ let

$$A_{k,\lambda} = A_k = \{iS_i \geqslant i\lambda,\ 0 \leqslant i \leqslant k-1, \text{ and } kS_k < k\lambda\}.$$

We say the points in A_k *initiate a chain of length* k. Let

$$B_k = \bigcup_{j=0}^{k} T^{-j} A_k,$$

and define C_k by $C_k = A_k - \bigcup_{j=k+1}^{\infty} B_j$, that is, the points in C_k initiate a chain of length k and are not part of a longer chain. Then all the sets in the array

$$
(2.2)\qquad
\begin{array}{l}
C_1,\ T^{-1}C_1\\
C_2,\ T^{-1}C_2,\ T^{-2}C_2\\
\cdot\\
\cdot\\
\cdot\\
C_k,\ T^{-1}C_k,\ \ldots,\ T^{-k}C_k\\
\cdot\\
\cdot\\
\cdot
\end{array}
$$

are disjoint. To see this first note that $T^{-k}A_k$ and A_k are disjoint since $f \geqslant \lambda$ on A_k while $f(T^{-k}x) = kS_k(x)-(k-1)S_{k-1}(x) < \lambda$ if $x \in A_k$ so $f<\lambda$ on T^kA_k. Thus $T^{-k}C_k$ and C_k are disjoint. Secondly note that if $0 \leqslant i < j \leqslant k$, with either $i \neq 0$ or $j \neq k$, then

$$T^i(T^{-i}C_k \cap T^{-j}C_k) = C_k \cap T^{-(j-i)}C_k = \Gamma = \emptyset .$$

This follows since $S_1(x), S_2(x), \ldots, S_{k-1}(x)$ are all $\geqslant \lambda$ if $x \in C_k$, so that if $x \in \Gamma$ then

$$f(T^{-(j-i)}x)+f(T^{-[(j-i)+1]}x)+\ldots+f(T^{-k}x) \geqslant \big(k-(j-i)+1\big)\lambda,$$

implying

$$(k+1)S_{k+1}(x) = (j-i)S_{j-i}(x)+f(T^{-(j-i)}x)+\ldots+f(T^{-k}x) \geqslant (k+1)\lambda,$$

so the points in Γ initiate a chain of length $k+1$ and thus cannot be in C_k, so $\Gamma = \emptyset$ and so the sets $C_k, TC_k, \ldots, T^kC_k$ are disjoint.

Now let $m<n$. If $T^{-j}C_m \cap T^{-k}C_n \neq \emptyset$ for some $0 \leqslant k \leqslant n$, $0 \leqslant j \leqslant m$, $j \leqslant k$, then $C_m \cap T^{-(k-j)}C_n \neq \emptyset$, which cannot happen since the points in C_m do not belong to a chain of length n. To prove $T^{-j}C_m \cap T^{-k}C_n = \emptyset$ for $0 \leqslant k \leqslant n$, $0 \leqslant j \leqslant m$, $k \leqslant j$, use an argument similar to that in the preceeding paragraph to show that the points of $C_m \cap T^{-(k-j)}C_n$, a set contained in C_m, initiate a chain of length at least $m+1$, contradicting the definition of C_m. Thus all the sets in (2.2) are disjoint.

Now let $C = \bigcup_{i=1}^{\infty}\bigcup_{j=0}^{i-1} T^{-j}C_i$ and $K = \bigcup_{i=1}^{\infty}\bigcup_{j=0}^{i} T^{-j}C_i$. Then clearly

$$(2.3)\qquad P(C) \geqslant \tfrac{1}{2}P(K),$$

with equality if and only if $P(C_i) = 0$ for each $i>1$. Furthermore $\{f^* \geqslant \lambda\} \supset C$ implying

$$(2.4)\qquad P(f^* \geqslant \lambda) \geqslant P(C).$$

Now

$$\int\limits_{\bigcup_{k=0}^{i} T^{-k}C_i} f = \int\limits_{C_i} f + \int\limits_{C_i} T^{-1}f+\ldots+\int\limits_{C_i} T^{-i}f = \int\limits_{C_i} S_{i+1}(f)$$

$$\leqslant \lambda(i+1)P(C_i) = \lambda P\Big(\bigcup_{k=0}^{i} T^{-k}C_i\Big),$$

so that

$$(2.5)\qquad \int\limits_K f \leqslant \lambda P(K),$$

while clearly

$$(2.6)\qquad \{f \geqslant \lambda\} \supset \bigcup_{i=1}^{\infty} C_i \supset K.$$

Any set S satisfying (2.5) and (2.6) with $K = S$ must satisfy $P(S) \geqslant \psi_\lambda$. Since f and $\hat f$ have the same distribution, this statement is equivalent to the statement that if $\Gamma \supset U$ satisfies $\Gamma \supset \{e^{i\theta}\colon\ 0<\theta\leqslant\theta_\lambda\}$ and $\int_\Gamma \hat f\,dm \leqslant \lambda m(\Gamma)$ then $m(\Gamma) \geqslant m\{e^{i\theta}\colon\ \varphi_\lambda \leqslant \theta \leqslant \theta_\lambda\}$, which is not difficult to verify. Intuitively Γ uses the smallest possible values of $\hat f$ to cancel the large values of $\hat f$ on $\{e^{i\theta}\colon\ 0<\theta\leqslant\theta_\lambda\}$. Thus $P(K) \geqslant \psi_\lambda$, and this together with (2.3) establishes (2.1).

The rest of the proof of (1.1) is very similar to portions of the paper [2], and the next two paragraphs are reproduced, almost verbatum, for completeness.

Inequality (2.1) gives

$$Ef^* = \int_0^\infty P(f^* \geqslant \lambda)\,d\lambda \geqslant \int_0^\infty (\psi_\lambda/2)\,d\lambda.$$

Now let

$$a_n = \Big(2^n \int_{-b_n}^{b_n} \hat f(e^{i\theta})\,dm\Big)_+,$$

where $x_+ = \max(x, 0)$ and $b_n = \pi 2^{-n}$. If $a_n > 0$, $\psi_{a_n} \geqslant 2^{-(n+1)}$. This is clear if $\theta_{a_n} > b_n$, while when $0<\theta_{a_n}<b_n$, $\hat f(e^{i\theta}) < a_n$ if $\theta \in (\theta_{a_n}, b_n)$ so that

$$\int_{-b_n}^{\theta_{a_n}} \hat f(e^{i\theta})\,dm = \int_{-b_n}^{b_n} \hat f(e^{i\theta})\,dm - \int_{\theta_{a_n}}^{b_n} \hat f(e^{i\theta})\,dm$$

$$> 2^{-n}a_n - a_n(b_n-\theta_{a_n})(2\pi)^{-1} = a_n\big(\theta_{a_n}-(-b_n)\big)/2\pi,$$

implying $\varphi_{a_n} \leqslant -b_n$.

Now let $j_1 = 1$, let j_2 be the first $i>1$, if it exists, such that $a_i > 2a_1$, and in general, if a_{j_i} exists let j_{i+1} be the first $k > j_i$, if it exists, such that

$a_k > 2a_{j_i}$. If $\{j_i\colon i \in A\}$ denotes the finite or infinite collection of integers obtained in this manner, we have

$$\sum_{n=0}^{\infty} 2^{-(n+1)} a_n \leqslant 3 \sum_{i\in A} 2^{-(j_i+1)} a_{j_i}.$$

This, and the fact that ψ_λ is non-increasing, give

$$\begin{aligned} Ef^* &\geqslant \int_0^\infty \psi_\lambda\, d\lambda/2 \geqslant \sum_{i\in A} \int_{a_{j_i}/2}^{a_{j_i}} \psi_\lambda\, d\lambda/2 \geqslant c \sum_{i\in A} a_{j_i} 2^{-(j_i+1)} \\ &\geqslant c \sum_{n=0}^{\infty} a_n 2^{-(n+1)} = c \sum_{n=0}^{\infty} \Big(\int_{-b_n}^{b_n} \hat f(e^{i\theta})\, dm \Big)_+. \end{aligned} \tag{2.7}$$

Replacing f by $-f$ in this inequality gives

$$Ef^* \geqslant c \sum_{n=0}^{\infty} \Big(\int_{-b_n}^{b_n} \hat f(e^{i\theta})\, dm \Big)_-,$$

where $x_- = \max(-x, 0)$, and this together with the previous inequality yields

$$Ef^* \geqslant c \sum_{n=0}^{\infty} \Big| \int_{-b_n}^{b_n} \hat f(e^{i\theta})\, dm \Big| = cA(f). \tag{2.8}$$

It is not difficult to show that

$$A(f) \geqslant cH(f) - CE|f| \tag{2.9}$$

(see [2], Lemma 2.1), and this, together with (2.8) and the fact that $f^* \geqslant |f|$, gives (1.1), at least in the special case $Ef = 0$.

To extend (1.1) to the case where $Ef \neq 0$, we note that if $e = Ef$ and $g = f - e$ then, by the ergodic theorem,

$$f^* \geqslant \lim_{n\to\infty} |f + Tf + \ldots + T^{n-1} f|/n = |e| \text{ a.e.}$$

so that $g^* \leqslant 2f^*$ a.e., so we have

$$A(f) \leqslant C\big(A(g) + |e|\big) \leqslant CA(g) + CEf^*.$$

Since $Eg = 0$,

$$A(g) \leqslant CEg^*,$$

and so

$$A(f) \leqslant CEg^* + CEf^* \leqslant CEf^*,$$

which, together with (2.9), proves Theorem 1.1.

3. A decomposition. This section is devoted to proving the following theorem. We will call the set where a function is not 0 its *support*.

THEOREM 3.1. *Let f be an integrable function on a non-atomic probability space. Then f can be written $f = \bar g + \sum_{i=1}^{\infty} f_i$, where the f_i have disjoint supports and*

(i) $\|\bar g\|_\infty \leqslant E|f|$,

(ii) *each f_i takes on only two non-0 values and $Ef_i = 0$, and*

(iii) $\sum_{i=1}^{\infty} H(f_i) \leqslant C\big(H(f) + E|f|\big)$.

This decomposition will be used to construct the function g of (1.2). It can also be used to give an alternative construction of the analagous functions in [2], where a much more specialized argument was used, and its counterpart for infinite measure spaces can be similarly used in the infinite measure cases of [2].

First the following lemma is proved.

LEMMA 3.2. *There are absolute constants c and C such that, if $Ef = 0$,*

$$cH(f) - CE|f| \leqslant \int_0^\infty \psi_\lambda(f)\, d\lambda + \int_0^\infty \psi_\lambda(-f)\, d\lambda \leqslant C\big(H(f) + E|f|\big). \tag{3.1}$$

Proof. The left hand inequality is implicit in the proof of inequality (2.8). This inequality is only going to be used for functions f taking on two nonzero values, and can be verified in this case by direct computation.

To prove the right inequality we assume W.L.O.G. that $P(f = r) = 0$ for each real number r. Then $\psi_\lambda(f)$ is continuous. Now define λ_n by $\psi_{\lambda_n}(f) = \psi_{\lambda_n} = 2^{-n}$, $n \geqslant 1$, and define $\lambda_0 = 0$. Then recalling that $b_n = \pi 2^{-n}$, either

$$2^{-(n+2)} \lambda_n \leqslant \int_{b_{n+2}}^{b_{n+1}} \hat f\, dm \quad \text{if} \quad \theta_{\lambda_n} \geqslant b_{n+1},$$

or

$$\lambda_n 2^{-n} = \int_{-\theta_{\lambda_n}}^{\theta_{\lambda_n}} \hat f\, dm \leqslant \int_{-b_{n+1}}^{b_{n+1}} \hat f\, dm \quad \text{if} \quad \theta_{\lambda_n} \leqslant b_{n+1}.$$

Thus, since ψ_λ is decreasing as λ increases,

$$\begin{aligned} \int_0^\infty \psi_\lambda\, d\lambda &= \sum_{n=0}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} \psi_\lambda\, d\lambda \leqslant \sum_{n=0}^{\infty} (\lambda_{n+1} - \lambda_n) \psi_{\lambda_n} = \sum_{n=1}^{\infty} \lambda_n 2^{-n} \\ &\leqslant \sum_{n=1}^{\infty} \Big(\int_{-b_{n+1}}^{b_{n+1}} \hat f\, dm \Big)_+ + 4 \sum_{n=1}^{\infty} \int_{b_{n+2}}^{b_{n+1}} \hat f\, dm \\ &\leqslant \sum_{n=0}^{\infty} \Big(\int_{-b_n}^{b_n} \hat f\, dm \Big)_+ + 4\, E|f|, \end{aligned}$$

and this inequality together with the analagous one for $-f$ yields the right hand side of (2.1).

Next the functions f_i and $\bar{g}$ in the statement of Theorem 3.1 will be described. Evidently, it is sufficient to prove this theorem in the case $f=\hat{f}$, so for the remainder of the proof we will make this assumption. Let $\delta=\int|f|\,dm$, let $h=k\delta$ if $k\delta\leqslant f<(k+1)\delta$, $-\infty<k<\infty$, and let $\bar{g}=f-h+\int h\,dm$, $q=h-\int h\,dm$. Then $|\int h\,dm|\leqslant 2\delta$ so $|\bar{g}|\leqslant 3\delta$ and (i) is satisfied, while $\int q\,dm=0$, and, noting that both $h=\hat{h}$ and $q=\hat{q}$, it is not difficult to show

$$(3.2)\qquad \int|q|\,dm\leqslant 4\int|f|\,dm,\quad\text{ and }\quad H(q)\leqslant H(f)+6\pi E|f|.$$

Let $\eta=\sup\{\theta\colon q(e^{i\theta})>0\}$, and if $0<\theta\leqslant\eta$ let $y(\theta)$ be the number in $(\theta-2\pi,0)$ which satisfies $\int_{y(\theta)}^{\theta}q\,dm=0$. Let $x_1=\eta$, and if $x_i>0$ define $x_{i+1}=\inf\{\theta\leqslant x_i\colon q(e^{i\theta})$ is constant if $\theta\in(x_{i+1},x_i)$ and q is constant if $\theta\in(y(x_i),\,y(x_{i+1}))\}$. This gives either a finite or infinite collection of points x_i. If the collection is finite the last $x_i=0$ and if it is infinite $x_i\to 0$. For notational convenience assume there are an infinite number of x_i. We write

$$f_i=q(e^{i\theta})I\big(\theta\in(x_{i+1},x_i)\cup(y(x_i),y(x_{i+1}))\big).$$

Everything claimed for the f_i in Theorem 3.1 except (iii) is easily checked.

To verify (iii), we will establish the inequality

$$(3.3)\qquad \sum_{i=1}^{\infty}\psi_\lambda(f_i)\leqslant 4\psi_{\lambda/8}(q).$$

When this inequality, and its twin with $-f_i$ in place of f_i and $-q$ in place of q, are integrated from 0 to ∞ and then added, Lemma 3.2 and the inequalities (3.2) give (iii).

Let $f_N,f_{N+1},\ldots$ be those f_i such that the positive value taken on by f_i is $\geqslant\lambda$. Then $\sum_{i=1}^{\infty}\psi_\lambda(f_i)=\sum_{i=N}^{\infty}\psi_\lambda(f_i)$. For each $i\geqslant N$ let the set $D_i\subset\{f_i\neq 0\}$ satisfy $D_i\supset\{f_i>0\}$ and $\int_{D_i}f_i\,dm=\lambda m(D_i)$, so that $m(D_i)=\psi_\lambda(f_i)$. Note that $\bigcup_{i=N}^{\infty}D_i$ contains the interval $(0,\theta_\lambda(q))$. Suppose, temporarily, that there exists an integer M such that

$$(3.4)\qquad \sum_{i=N}^{\infty}m(D_i)/4<\sum_{i=M}^{\infty}m(f_i\neq 0)<3\sum_{i=N}^{\infty}m(D_i)/4.$$

Then $M>N$ and

$$\int\limits_{(\bigcup_{i=N}^{\infty}D_i)\cup(\bigcup_{i=M}^{\infty}\{f_i\neq 0\})}q\,dm=\sum_{i=N}^{M-1}\int_{D_i}q\,dm+\sum_{i=M}^{\infty}\int_{\{f_i\neq 0\}}q\,dm$$
$$=\sum_{i=N}^{M-1}\int_{D_i}q\,dm=\sum_{i=N}^{M-1}\lambda m(D_i)$$
$$\geqslant\lambda\sum_{i=N}^{\infty}m(D_i)/4\geqslant\lambda\Big(\sum_{i=N}^{\infty}m(D_i)+\sum_{i=M}^{\infty}m\{f_i\neq 0\}\Big)\Big/8,$$

the next to last inequality following from the right side of (3.4) and the fact that $D_i\subset\{f_i\neq 0\}$.

Now if $A_\lambda=\bigcup_{i=M}^{\infty}\{f_i\neq 0\}\cup\bigcup_{i=N}^{\infty}\{f_i>0\}$, then A_λ is a single arc, and the average of q on this arc is no less than the average of q on $(\bigcup_{i=N}^{\infty}D_i)\cup\cup(\bigcup_{i=M}^{\infty}\{f_i\neq 0\})$, so that

$$\int_{A_\lambda}q\,dm\geqslant(\lambda/8)m(A_\lambda),$$

implying

$$\psi_{(\lambda/8)}(q)\geqslant m(A_\lambda)\geqslant\sum_{i=M}^{\infty}m\{f_i\neq 0\}$$
$$\geqslant(1/4)\sum_{i=N}^{\infty}m(D_i)=(1/4)\sum_{i=N}^{\infty}\psi_\lambda(f_i).$$

This inequality was proved under the assumption that M existed such that (3.4) holds. If it does not, let Γ satisfy

$$\sum_{i=N}^{\infty}m(D_i)/4\geqslant\sum_{i=\Gamma-1}^{\infty}m(f_i\neq 0)$$

and

$$3\sum_{i=N}^{\infty}m(D_i)/4\leqslant\sum_{i=\Gamma}^{\infty}m(f_i\neq 0).$$

Let J be so large that $m(f_\Gamma\neq 0)/J<\sum_{i=N}^{\infty}m(D_i)/2$ and write

$$f_\Gamma=f_{\Gamma,1}+f_{\Gamma,2}+\ldots+f_{\Gamma,J}$$

where

$$f_{\Gamma,k}=qI(x_{\Gamma,k-1},x_{\Gamma,k})+qI\big(y(x_{\Gamma,k}),y(x_{\Gamma,k-1})\big),$$

where the points $x_{\Gamma+1}=x_{\Gamma,0}<x_{\Gamma,1}<\ldots<x_{\Gamma,k}=x_\Gamma$ divide $(x_{\Gamma+1},x_\Gamma)$ into equal arcs. Then $\psi_\lambda(f_{\Gamma,i})=\psi_\lambda(f_\Gamma)/J$, and the proof of (3.3) can be carried

out using this and an argument identical to the preceeding one with the functions $\{f_i, i \geqslant 1\}$ replaced by

$$\{f_i, i < \Gamma\} \cup \{f_{i,\Gamma}, 1 \leqslant i \leqslant J\} \cup \{f_i, i > \Gamma\}.$$

4. Construction of g. In this section Ω will be decomposed into disjoint sets $\Omega = \Delta_1 \cup \Delta_2 \cup \ldots$ such that there is a function μ_i with support Δ_i and the distribution of f_i of Theorem 3.1, and such that $E\mu_i^*$ is not too large. Once this is done g is easily constructed.

LEMMA 4.1. *Let α and β be positive numbers and let a, b, and m be positive integers such that $a+b \leqslant m$ and $(a/m)\alpha - (b/m)\beta = 0$. There is an integer N such that if $n \geqslant N$, and A is a set such that $P(A) = 1/nm$ and the sets $T^k A$, $0 \leqslant k < (a+b)n$ are disjoint, then there is a function h with the following properties:*

(i) $P(h = \alpha) = a/m$, $P(h = -\beta) = b/m$, *and* $P(h = 0) = 1-(a+b)/m$,

(ii) $\{h \neq 0\} = \bigcup_{k=0}^{(a+b)n-1} T^k A$, *and*

(iii) $Eh^* \leqslant C(H(h) + E|h|)$,

where C does not depend on α, β, a, b, or m.

Proof. Let $\gamma = n(a+b)-1$ and $B = \bigcup_{k=0}^{\gamma} T^k A$. Let $a_0, a_1, \ldots, a_\gamma$ be numbers, an of which are α and bn of which are $-\beta$ (so $\sum_{k=0}^{\gamma} a_k = 0$) and which satisfy $-\beta \leqslant \sum_{i=0}^{k} a_i \leqslant \alpha$ for each k, $0 \leqslant k \leqslant \gamma$. This can be done by choosing $a_0 = \alpha$ and, if $i > 0$, picking a_i by the rule $a_i = -\beta$ if $\sum_{k=0}^{i-1} a_k \geqslant 0$, otherwise $a_i = \alpha$. Define the function h by

$$h = \sum_{i=0}^{\gamma} a_i I(T^i A).$$

Then $-2\beta \leqslant h + Th + \ldots + T^n h \leqslant 2\alpha$, $n \geqslant 0$, so that if n_0 is a positive integer, $\sup_{n \geqslant n_0} (1/n)\left|\sum_{i=0}^{n-1} T^i h\right| \leqslant 2\max(\alpha, \beta)/n_0$, and thus

$$\{h^* \geqslant 2\max(\alpha, \beta)/n_0\} \cap \tilde{B} \subseteq \bigcup_{i=1}^{n_0-1} T^{-i} B \cap \tilde{B} \subseteq \bigcup_{i=1}^{n_0-1} T^{-i} A,$$

where the $\sim$ denotes complement, a set of probability not exceeding $(n_0-1)P(A) = (n_0-1)/nm$. Therefore

$$(4.1) \qquad Eh^* I(\tilde{B}) \to 0 \quad \text{as} \quad n \to \infty.$$

If $x \in T^k A$, $0 \leqslant k \leqslant \gamma$, let $\varphi(x) = \gamma - k$. Then just as above, we have

$$\Big\{\sup_{k > \varphi(x)} (1/k+1)\Big|\sum_{i=0}^{k} h(T^i(x))\Big| \geqslant 2\max(\alpha, \beta)/n_0\Big\} \cap B \subseteq \bigcup_{i=1}^{n_0-1} T^{-i} A,$$

so that

$$(4.2) \qquad E \sup_{k > \varphi(x)} (1/k+1)\Big|\sum_{i=0}^{k} h(T^i(x))\Big| I(B) \to 0 \quad \text{as} \quad n \to \infty.$$

Now let $\eta = I(B) \max_{0 \leqslant k \leqslant \varphi(x)} (1/k+1) \sum_{i=0}^{k} h(T^i(x))$. An argument very similar to the proof of (2.4) gives that if $\lambda > 0$

$$(4.3) \qquad \int_{\{\eta \geqslant \lambda\}} h\, dm \geqslant \lambda P(\eta \geqslant \lambda).$$

Now $\{\eta \geqslant \lambda\} \supset \{h \geqslant \lambda\}$ and since $\{\eta \geqslant \lambda\} \subset \{h \neq 0\}$, (4.3) implies

$$(4.4) \qquad m\{\eta \geqslant \lambda\} \leqslant \psi_\lambda(h),$$

with equality if equality holds in (4.3). Inequality (4.4), its counterpart for $-h$, and Lemma 3.2 give

$$EI(B) \max_{0 \leqslant k \leqslant \varphi(x)} (1/k+1)\Big|\sum_{i=0}^{k} h(T^i(x))\Big| \leqslant C(H(h) + E|h|),$$

and this together with (4.1) and (4.2) establishes Lemma 4.1.

LEMMA 4.2. *Let $N_1 < N_2 < \ldots$ be integers such that $N_i/N_{i-1} > N_{i-1}$. There exist sets A_i, $i \geqslant 1$, such that $P(A_i) \geqslant 10^{-i}/N_i$ for each i and all the sets $\{T^k A_i, 0 \leqslant k \leqslant N_i - 1, 1 \leqslant i < \infty\}$ are disjoint.*

Proof. Rholin's Theorem (see [6]) says that given $\varepsilon > 0$ and any integer n there is a set A with $P(A) < \varepsilon$, such that $A, TA, \ldots, T^{n-1}A$ are disjoint and $P(\bigcup_{i=0}^{n-1} T^i A) > 1 - \varepsilon$. Using this it is easy to show that for each i there exist sets B_i such that $B_i, TB_i, \ldots, T^{N_i-1}B_i$ are disjoint and $P(B_i) = 2 \cdot 10^{-i}/N_i$, $i \geqslant 0$. Let $C_i = \bigcup_{k=0}^{N_i-1} T^k B_i$.

Now fix $i > 1$. For each point w in B_i let Γ be the collection of those integers k in $[0, N_i - 1]$ (where $[m, n] = \{m, m+1, \ldots, n\}$) which satisfy $T^k w \in C_1$. Let $n_1 = 0$ if $w \in C_1$, otherwise let $n_1 = \min\{k > 0: T^k w \in B_1$ and $k \leqslant N_i - 1\}$ if such exists. If $n_1 = 0$ define $m_1 = \min\{k \geqslant 0: T^k w \in T^{N_1-1}B_1\}$. If $n_1 > 0$ let $m_1 = n_1 + N_1 - 1$ or $N_i - 1$, whichever is smaller. If $m_1 < N_i - 1$, let n_2 be the first integer k greater than m_1 and not exceeding $N_i - 1$, if it exists, such that $T^k w \in B_1$, and let $m_2 = n_2 + N_1 - 1$ or $N_i - 1$, whichever is smaller. Continue like this until either n_j does not exist or $m_j = N_i - 1$, and let J be the number of distinct n_j found. Then $\Gamma = \bigcup_{i=1}^{J} [n_i, m_i]$, and each of the intervals $[n_i, m_i]$ contains N_1 integers, except perhaps the first and the last, which may contain fewer. Thus $P(C_i) \geqslant N_1 EJ$, and since $EJ - 1 \leqslant P(B_1 \cap C_i) \leqslant EJ$, we have

$$P(B_1 \cap C_i) \leqslant P(C_i)/N_1 = 2\cdot 10^{-i}/N_1.$$

Now if $w \notin C_i$ then $T^k w$ is not in C_i for any $k \in [1, N_1 - 1]$ unless $w \in \bigcup_{i=1}^{N_1-1} T^{-k} B_i$, a set of probability at most

$$(N_1 - 1)P(B_i) \leqslant (N_1/N_i) 2\cdot 10^{-i} \leqslant (1/N_1)\cdot 2\cdot 10^{-i}.$$

Thus if

$$D_i = \{w\colon\ w \in B_1 \text{ and } T_w^k \notin C_i,\ 0 \leqslant i \leqslant N_i - 1\}$$

then

$$\begin{aligned} P(D_i) &\geqslant P(B_1) - P(B_1 \cap C_i) - P\Big(\bigcup_{k=1}^{N_1-1} T^{-k} B_i\Big) \\ &\geqslant P(B_1) - 2\cdot 10^{-i}/N_1 - (1/N_1)\cdot 2\cdot 10^{-i} \\ &= 2\cdot 10^{-1}/N_1 - 4(10^{-i})/N_1, \end{aligned}$$

and so

$$P\Big(\bigcap_{i=2}^{\infty} D_i\Big) \geqslant \Big(2\cdot 10^{-1} - 4\sum_{i=2}^{\infty} 10^{-i}\Big)\Big/ N_1 \geqslant 10^{-1}/N_1.$$

Pick A_1 to be any subset of $\bigcap_{i=2}^{\infty} D_i$ of measure exactly $10^{-1}/N_1$. Then if $w \in A_1$, none of $T^k w$, $0 \leqslant k \leqslant N_1 - 1$, belongs to $\bigcup_{i=2}^{\infty} C_i$. Now construct A_2 in exactly the same manner to be a subset of B_2 such that none of $T^k w$, $0 \leqslant k \leqslant N_2 - 1$ belongs to $\bigcup_{i=3}^{\infty} C_i$, and so on.

LEMMA 4.3. *Let $\varepsilon > 0$ and let n be a positive integer. There exists an integer $N(\varepsilon, n) = N$ and a set A such that $A, TA, \ldots, T^{n-1}A$ are disjoint $P\big(\bigcup_{i=0}^{n-1} T^i A\big) > 1 - \varepsilon$, and $\bigcup_{i=0}^{n-1} T^i A$ does not intersect any of the sets $\bigcup_{i=0}^{N_j-1} T^i A_j$, $j \geqslant N$, where the A_j are as in Lemma 4.2.*

Proof. The proof of this lemma resembles the proof just given. Start with a set $\hat{A}$ such that $T^k \hat{A}$, $0 \leqslant k < n$ are disjoint and

$$P\Big(\bigcup_{i=0}^{n-1} T^i \hat{A}\Big) > 1 - (\varepsilon/2).$$

Then we can pick m large enough so

$$P\Big(\hat{A} \cap \Big\{T^i \hat{A} \notin \bigcup_{i=m}^{\infty} \bigcup_{j=0}^{N_i-1} T^j A_j,\ 0 \leqslant i \leqslant n-1\Big\}\Big) > (1-\varepsilon)/n,$$

so we pick A to be this intersection.

Before completing the construction of g we remark that there are many ways to show the existence of a function $\varphi(x)$ on $(0,1)$ which is positive and approaches 0 as x approaches 0 such that

$$Ef^* \leqslant \varphi\big(P(f \neq 0)\big)\|f\|_\infty. \tag{4.5}$$

Let $f \sim g$ mean that f and g are equidistributed. Then, since Ω is nonatomic, if γ is a function with $P(\gamma \neq 0) = x$ there is a function η equidistributed with γ such that $\{\eta \neq 0\} = A$. Thus if $f = \bar{g} + \sum_{i=1}^{\infty} f_i$ is the decomposition of Theorem 3.1, if we can construct functions μ_i such that $\mu_i \sim f_i$, the sets $\{\mu_i \neq 0\}$ are disjoint, and

$$\sum_{i=1}^{\infty} E\mu_i^* \leqslant C\big(H(f) + E|f|\big),$$

then by letting ν_i be any function such that $\{\nu_i \neq 0\} \subset \{\mu_i \neq 0\}$, $\nu_i I(\mu > 0) \sim (f - f_i) I(f_i > 0)$, $\nu_i I(\mu_i < 0) \sim (f - f_i) I(f_i < 0)$, $i > 1$, and ν_0 satisfy $\{\nu_0 \neq 0\} \subset \{\mu_i = 0, i \geqslant 1\}$ and $\nu_0 \sim \bar{g} I(f_i = 0, i \geqslant 1)$, we have $\sum_{i=0}^{\infty} \nu_i \sim \bar{g}$, $\sum_{i=1}^{\infty} \mu_i + \sum_{i=0}^{\infty} \nu_i \sim f$, and

$$\begin{aligned} E\Big(\sum_{i=1}^{\infty} \mu_i + \sum_{i=0}^{\infty} \nu_i\Big)^* &\leqslant \sum_{i=1}^{\infty} E\mu_i^* + \Big\|\sum_{i=0}^{\infty} \nu_i\Big\|_\infty \\ &= \sum_{i=1}^{\infty} E\mu_i^* + \|\bar{g}\|_\infty \leqslant C\big(H(f) + E|f|\big), \end{aligned}$$

so that $\sum_{i=1}^{\infty} \mu_i + \sum_{i=1}^{\infty} \nu_i$ can be taken for g.

Thus to complete the construction of g we need to construct the functions μ_i, $i \geqslant 1$. Assume W.L.O.G. that f is such that g (and thus f_i) takes on only rational values, and write $f_i = \delta_i + \theta_i$, where $\{\delta_i \neq 0\}$ and $\{\theta_i \neq 0\}$ are disjoint, where δ_i satisfies $\int \delta_i dm = 0$ and $P(\delta_i > 0) = k/m_i$ and $P(\delta_i < 0) = j/m_i$ for integers k, j, and m_i, and where

$$0 < \|\theta_i\|_\infty \varphi\big(P(\theta_i > 0)\big) < E|f|2^{-i},$$

φ is as in (4.5). Note $H(\delta_i) = \big(P(\delta_i \neq 0)/P(f_i \neq 0)\big)H(f_i) \leqslant H(f_i)$.

Next, let $R_1, R_2, \ldots$ be sets satisfying $P(R_i) = P(\delta_i \neq 0)/m_i N(\delta_i)$, N as in Lemma 4.1, and all the sets $T^k R_i$, $0 \leqslant k < m_i N(\delta_i)$, $1 \leqslant i < \infty$, are disjoint. That this can be done follows without much difficulty from the last lemma. In this regard we remark that if $m \leqslant n = mj+r$, $0 \leqslant r < m$, and if $A, TA, \ldots, T^{n-1}A$ are disjoint, and if $B = \bigcup_{i=0}^{j-1} T^{im}A$ then $B, TB, \ldots, T^mB$ are disjoint and $P\big(\bigcup_{i=0}^{n-1} T^iA - \bigcup_{j=0}^{m-1} T^jB\big) = rP(A) \leqslant P\big(\bigcup_{i=0}^{n-1} T^iA\big)$.

Finally, construct a function $t_i \sim \delta_i$ such that

$$\{t_i \neq 0\} = \bigcup_{k=0}^{N(\delta_i)m_i-1} T^iR_i$$

satisfying

$$Et_i^* \leqslant C\big(H(\delta_i)+E|\delta_i|\big) \leqslant C\big(H(f_i)+E|f_i|\big),$$

and construct functions a_i, $i \geqslant 1$, such that $a_i \sim \theta_i$, the a_i have disjoint supports, and $\{a_i \neq 0\} \subset \{t_i = 0,\ i \geqslant 1\}$. Then if we take $\mu_i = a_i + t_i$, we have $\sum_{i=1}^{\infty} \mu_i \sim \sum_{i=1}^{\infty} f_i$, and

$$\begin{aligned} E\Big(\sum_{i=1}^{\infty} \mu_i\Big)^* &\leqslant \sum_{i=1}^{\infty} Ea_1^* + \sum_{i=1}^{\infty} Et_i^* \\ &\leqslant \sum_{i=1}^{\infty} \|\theta_i\|_\infty \varphi\big(P(\theta_i \neq 0)\big) + \sum_{i=1}^{\infty} C\big(H(f_i)+E|f_i|\big) \\ &\leqslant C\big(H(f)+E|f|\big), \end{aligned}$$

completing the construction of g.

To conclude, we remark that if $f \geqslant 0$ and $Ef\log^+ f = \infty$, then given $M > 0$ there is an integer $N(M) = N$ such that if $n \geqslant N$ and if A is a set such that $A, TA, \ldots, T^{n-1}A$ are disjoint and $P\big(\bigcup_{i=0}^{n-1} T^iA\big) = P(f > 0)$, a function h can be constructed such that $h \sim f$, $\{h > 0\} = \bigcup_{i=0}^{n-1} T^iA$, and

$$E \sup_{0 \leqslant k \leqslant \varphi(x)} (1/k+1)\Big(\sum_{i=0}^{k} h(T^ix)\Big) > M,$$

where $\varphi(x) = n-1-k$ on T^kA, $0 \leqslant k \leqslant n-1$, $\varphi(x) = 0$ otherwise. This construction is not hard to make after reading [5], and it can be used, together with Lemma 4.3, to show that given any $f \notin L\log L$ there is a function h equidistributed with f such that $Eh^* = \infty$.

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