STUDIA MATHEMATICA, T. LXXIII. (1982)

Symbolic calculus on weighted group algebras

by

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Abstract. Let G be a locally compact group of polynomial growth and ω a non-constant polynomial weight on G. It is shown that $L^1(G,\omega)$ is a symmetric algebra and that functions with arbitrarily small support operate on hermitian L^2-functions in it.

Let A be a semi-simple, commutative Banach *-algebra. The Gelfand transform $\hat{}$ maps A onto a subalgebra $\hat{A}$ in $C_0(M)$ of all continuous functions vanishing at infinity on the space M of all maximal ideals of A. A function $F\colon R\to C$ with $F(0)=0$ operates on an element $f\in A$ if the range of $\hat{f}$ is real and $F\circ\hat{f}\in\hat{A}$.

In 1959 Helson, Kahane, Katznelson and Rudin [3] showed that if F operates on every $f=f^*$ in $L^1(G)$, where G is an infinite locally compact abelian group, then F is real analytic. On the other hand, under some assumptions on a weight ω on a discrete abelian group G also non-analytic functions operate on $l^1(G,\omega)$ (cf. [2], [6], [7] and [8]).

All these results are obtained by estimating the growth of $\|u(nf)\|$ as $n\to\infty$, where $u(f)=\sum_{k=1}^{\infty}\frac{i^kf^k}{k!}$. In particular,

$$\|u(nf)\| = O\big(e^{n^\gamma}\big) \quad \text{as} \quad n\to\infty \tag{1}$$

for a γ, $0<\gamma<1$, guarantees that functions with arbitrarily small supports operate on f.

J. Dixmier [1] extended the notion of function operating on an element f in a non-commutative Banach *-algebra B by repeating the above definition with A being the maximal commutative Banach subalgebra of B containing f. He proved that, if G is a nilpotent group (or, generally, a group of polynomial growth), then any function F, where $F(0)=0$ and F is differentiable sufficiently many times, operates on every f in the algebra $L^1(G)$, provided that f is an L^2-function with compact support. Later A. Hulanicki [4] obtained a result of this type for nilpotent groups of class 2 and hermitian functions in $L^1(G,\omega)\cap L^2(G)$, where ω is a polynomial (c.f. (2) below) non-constant weight on G.

In this note we investigate the algebra $L^1(G, \omega)$, where G is a locally compact group of polynomial growth and ω a polynomial weight with $\int_G \omega^{-p}(s)\,ds < +\infty$ for some $p > 0$.

Theorem 1 is a generalization of our earlier result [9] and shows that $L^1(G, \omega)$ is symmetric. If G is that of [5], we get an example of a dense symmetric subalgebra in non-symmetric algebra $l^1(G)$.

In Theorem 2 we prove (1) for any hermitian L^2-function f in the algebra $L^1(G, \omega)$. The condition $f \in L^2(G)$ is of course automatically satisfied if G is discrete; thus Theorem 2 implies that some non-analytic classes of functions operate on the algebra $l^1(G, \omega)$.

Let G be a locally compact topological group. A measurable and locally bounded (e.g. bounded on compact sets) function $\omega\colon G \to [1, \infty)$ is called a *weight* if $\omega(us) \leqslant \omega(u)\omega(s)$ and $\omega(s^{-1}) = \omega(s)$ for every $u, s \in G$. The space $L^1(G, \omega)$ of all complex functions f on G for which $f\cdot\omega$ belongs to $L^1(G)$ is a Banach $*$-algebra with the algebra operations inherited from $L^1(G)$ and the norm

$$\|f\|_\omega = \|f\cdot\omega\|_1.$$

It is obvious that the sum and the product of two weights is a weight and the same is true for any positive power of a weight. We say that a weight ω_1 is *dominated by* ω_2 if there is a positive constant M such that $\omega_1 \leqslant M\omega_2$. Two weights ω_1, ω_2 are *equivalent* if ω_1 is dominated by ω_2 and ω_2 is dominated by ω_1. For equivalent weights ω_1, ω_2 the algebras $L^1(G, \omega_1)$ and $L^1(G, \omega_2)$ are isomorphic. We call ω a *polynomial weight* if

$$\text{(2)} \qquad \frac{\omega(us)}{\omega(u)+\omega(s)} \quad \text{is bounded on } G\times G.$$

One can easily verify that the sum of two polynomial weights and any positive power of polynomial weight are polynomial weights. That this is not true for the product we will see in Example 3.

EXAMPLE 1. Let $G_1 \subset G_2 \subset \ldots$ be an increasing sequence of closed subgroups of a locally compact group G such that $\bigcup_{n=1}^{\infty} G_n = G$. Let $\alpha_1 \leqslant \alpha_2 \leqslant \ldots$ be an arbitrary increasing sequence of non-negative numbers. Then the function $\omega\colon G \to [1, \infty)$ defined by

$$\omega = 1 + \sum_{n=1}^{\infty} \alpha_n 1_{G_{n+1}-G_n}$$

is a weight on G if it is locally bounded (for example if any compact set of G is fully contained in some G_n). It satisfies the equality

$$\omega(su) = \max\{\omega(s), \omega(u)\}, \qquad s, u \in G,$$

and therefore it is a polynomial weight.

EXAMPLE 2. Let G be a compactly generated group. Let K be a pre-compact symmetric (e.g. $K^{-1} = K$) open set in G such that $\bigcup_{n=1}^{\infty} K^n = G$. Define $\sigma_K(s) = \inf\{n\colon s \in K^n\}$. Then $\sigma_K(su) \leqslant \sigma_K(s) + \sigma_K(u)$ for all $s, u \in G$ and hence

$$\text{(3)} \qquad \Omega_K(s) = e^{\sigma_K(s)}, \qquad s \in G,$$

$$\text{(4)} \qquad \omega_K(s) = 1 + \sigma_K(s), \qquad s \in G$$

are weights on G. If the group G is non-compact (in this case σ_K is unbounded on G) then only the second weight is polynomial.

PROPOSITION 1. *Let G be a compactly generated group. Let K and σ_K be as in Example 2. Every weight ω on G is dominated by Ω_K^δ for suitable $\delta > 0$. If ω is a polynomial weight then it is dominated by ω_K^δ.*

Proof. Denote $A = \sup_{s\in K} \omega(s) < +\infty$. If $s \in K^n - K^{n-1}$, then $s = s_1 s_2 \ldots s_n$ with $s_k \in K$, $k = 1, 2, \ldots, n$. Hence $\omega(s) \leqslant \prod_{i=1}^{n} \omega(s_i) \leqslant A^n = A^{\sigma_K(s)}$ and it suffices to put $\delta = \log_e A$. If, moreover, ω is a polynomial weight and C is a bound of (2) then for $n = 2^m$ we have $\omega(s) \leqslant A(2C)^m = An^\delta \leqslant A\omega_K^\delta(s)$, where $\delta = 1 + \log_2 C$. For if $2^m < n < 2^{m+1}$ then $s = s_1 s_2$, where $s_1, s_2 \in K^{2^m}$; thus $\omega(s) \leqslant \omega(s_1)\omega(s_2) \leqslant A^2\omega_K^\delta(s_1)\omega_K^\delta(s_2) \leqslant A^2\omega_K^{2\delta}(s)$.

EXAMPLE 3. On the additive group R^2 the functions ω_1, ω_2 given by $\omega_i((x_1, x_2)) = 1 + |x_i|$, $i = 1, 2$, are polynomial weights, but their product $\omega = \omega_1\omega_2$ is not. Indeed,

$$\frac{\omega((t,t))}{\omega((t,0)) + \omega((0,t))} = \frac{1+|t|}{2}, \qquad t \in R$$

is unbounded.

LEMMA 1. *Let ω be a polynomial weight. There exists a constant C such that for every f, g in $L^1(G, \omega)$ we have*

$$\text{(5)} \qquad \|f*g\|_\omega \leqslant C(\|f\|_\omega\|g\|_1 + \|f\|_1\|g\|_\omega).$$

This is a trivial consequence of (2).

COROLLARY 1. *Let ω be a polynomial weight and let $f \in L^1(G, \omega)$. The spectral radii $r_\omega(f)$ and $r(f)$ in algebras $L^1(G, \omega)$ and $L^1(G)$ coincide.*

Proof. By (5) we have

$$\|f^2\|_\omega \leqslant 2C\|f\|_\omega\|f\|_1,$$

$$\|f^4\|_\omega \leqslant 2C\|f^2\|_\omega\|f^2\|_1 \leqslant (2C)^2\|f\|_\omega\|f\|_1^3,$$

and so on. Generally

$$\|f^{2^n}\|_\omega \leqslant (2C)^n\|f\|_\omega\|f\|_1^{2^n-1}.$$

Therefore

$$r_\omega(f)=\lim_{n\to\infty}\|f^{2^n}\|_\omega^{2^{-n}}\leqslant\|f\|_1.$$

Substituting f^n instead of f in the last inequality, we get

$$r_\omega(f)=[r_\omega(f^n)]^{1/n}\leqslant\|f^n\|_1^{1/n},$$

and thus $r_\omega(f)\leqslant r(f)$. The converse inequality is obvious, and so $r_\omega(f)=r(f)$ and the corollary follows.

COROLLARY 2. *Let ω be a polynomial weight. Every continuous homomorphism of $L^1(G,\omega)$ into a commutative C^* algebra can be extended to the whole $L^1(G)$.*

Proof. Let A be a commutative C^* algebra and let $\varphi\colon L^1(G,\omega)\to A$. For $f\in L^1(G,\omega)$ we have

$$\|\varphi(f)\|=\lim_{n\to\infty}\|\varphi(f^n)\|^{1/n}=r_\omega(f)=r(f)\leqslant\|f\|_1.$$

So φ is continuous in the $L^1(G)$ norm.

Remark 1. The results of Corollaries 1 and 2 are not true for every weight. This is shown in the following:

EXAMPLE 4. Let G be the additive group of real numbers and $\omega(t)=e^{|t|}$. The mapping $f\to\hat f(z)=\int_{-\infty}^{+\infty}f(t)e^{-izt}\,dt$ is a continuous linear and multiplicative functional on $L^1(G,\omega)$ for any complex number z from the set

$$D=\{z\in C\colon\ -1\leqslant\operatorname{Im}z\leqslant 1\}.$$

Therefore

$$r_\omega(f)\geqslant\sup_{z\in D}|\hat f(z)|.$$

In particular, for the characteristic function of the interval $(0,1)$ we have $\|f^n\|_1=\|f\|_1^n=1$; thus $r(f)=1$, but

$$\hat f(i)=\int_0^1 e^t\,dt=e-1,$$

and so $r_\omega(f)\geqslant e-1>r(f)$.

LEMMA 2. *Let ω be a polynomial weight on G such that ω^{-1} belongs to $L^p(G)$ for a $0<p<\infty$. Suppose $f\in L^1(G,\omega)\cap L^2(G)$. Then there exists a constant A, depending only on ω and such that*

$$\|f\|_1\leqslant A\|f\|_2^{2/(p+2)}\cdot\|f\|_\omega^{p/(p+2)}.\tag{6}$$

Proof. Let a be an arbitrary positive number and let

$$U_a=\{x\in G\colon\ \omega(x)\leqslant a\}.$$

Then, since $1\leqslant a^p\omega^{-p}(x)$ for $x\in U_a$, we have

$$|U_a|=\int_{U_a}1\,dx\leqslant a^p\int_{U_a}\omega^{-p}(x)\,dx\leqslant Ba^p$$

(here $|U_a|$ denotes the Haar measure of the set U_a). Thus

$$\begin{aligned}\|f\|_1&=\|f\cdot 1_{U_a}\|_1+\|f\cdot 1_{G-U_a}\|_1\leqslant\|f\|_2\cdot\|1_{U_a}\|_2+\|(f\cdot\omega)\cdot(1_{G-U_a}/\omega)\|_1\\&\leqslant\|f\|_2B^{1/2}\cdot a^{p/2}+\|f\cdot\omega\|_1\cdot\|1_{G-U_a}/\omega\|_\infty\\&\leqslant\|f\|_2B^{1/2}\cdot a^{p/2}+\|f\|_\omega\cdot a^{-1}.\end{aligned}$$

For $a=(\|f\|_\omega\|f\|_2^{-1}B^{-1/2})^{2/(p+2)}$ both factors of the right side of the last inequality are equal and so

$$\|f\|_1\leqslant 2(\|f\|_2B^{1/2})^{2/(p+2)}\cdot\|f\|_\omega^{p/(p+2)}.$$

This lemma has many consequences, so at first let us consider when the assumption on ω may be realized.

PROPOSITION 2. *Let G be a compactly generated locally compact group. There exists a polynomial weight ω on G such that ω^{-1} belongs to $L^1(G)$ (or any $L^p(G)$, $1\leqslant p<\infty$) if and only if G is of polynomial growth, e.g. for any compact set $K\subset G$ there exists a polynomial P such that $|K^n|\leqslant P(n)$, $n=1,2,\ldots$*

Proof. Suppose ω^{-1} is integrable on G. Let K be any compact set in G. It is contained in a compact set K_0 such that $K_0^{-1}=K_0$ and $\bigcup_{n=1}^\infty K_0^n=G$. Then by Proposition 1, $\omega\leqslant C\omega_{K_0}^\delta$, and thus

$$|K^n|\cdot(1+n)^{-\delta}\leqslant\int_{K_0^n}(1+n)^{-\delta}\leqslant\int_{K_0^n}\omega_{K_0}^{-\delta}\leqslant C\cdot\int_G\omega^{-1}=C_1.$$

Therefore $|K^n|\leqslant C_1(1+n)^\delta$, and G is of polynomial growth.

Conversely, if for a compact set K in G such that $K^{-1}=K$, $\bigcup_{n=1}^\infty K^n=G$ we have $|K^n|\leqslant Cn^k$, $n=1,2,\ldots$, then ω_K^{-k-2} is integrable on G.

Now we are ready to prove a result which gives a generalization of [9], Theorem 6.

THEOREM 1. *Let G be a locally compact group. If ω is a polynomial weight on G such that ω^{-1} belongs to $L^p(G)$ for a $0<p<\infty$, then the algebra $L^1(G,\omega)$ is symmetric.*

Proof. It suffices to prove $r_\omega(f)\leqslant\lambda(f)$ for any hermitian $f\in L^1(G,\omega)$, where $\lambda(f)$ denotes the norm of the operator T_f acting on $L^2(G)$ by convolution.

Let h be a function on G such that $h\in L^1(G,\omega)\cap L^2(G)$. Then each function f^n*h, $n=1,2,\ldots$, satisfies the assumption of Lemma 2, and so

$$\|f^n*h\|_1\leqslant A\|f^n*h\|_2^{2/(p+2)}\cdot\|f^n*h\|_\omega^{p/(p+2)}.$$

But $\|f^n*h\|_2\leqslant[\lambda(f)]^n\cdot\|h\|_2$ and $\|f^n*h\|_\omega\leqslant\|f^n\|_\omega\|h\|_\omega$; thus

$$\varlimsup_{n\to\infty}\|f^n*h\|_1^{1/n}\leqslant[\lambda(f)]^{2/(p+2)}\cdot[r_\omega(f)]^{p/(p+2)}.$$

By [9] Lemma 4, for a given f there exists a continuous function h with compact support such that $r(f)\leqslant\varlimsup_{n\to\infty}\|f^n*h\|_1$. Therefore by Corollary 1

$$r(f)\leqslant[\lambda(f)]^{2/(p+2)}[r(f)]^{p/(p+2)},$$

and so $r(f)\leqslant\lambda(f)$.

Remark 2. It is shown in [5] that there exists a countable locally finite group G, e.g. every finitely generated subgroup of G is finite and such that $l^1(G)$ is not symmetric. It is easy to construct a weight ω on G as in Example 1 such that ω^{-1} is integrable. Then by Theorem 1 the algebra $l^1(G,\omega)$ is a dense symmetric subalgebra in the non-symmetric algebra $l^1(G)$.

LEMMA 3. *Let $1<\lambda<2$. If $a_1,a_2,\ldots$ is a sequence of non-negative real numbers such that*

(a) $a_{n+m}\leqslant a_na_m$, $n,m=1,2,\ldots,$

(b) $a_{2n}\leqslant na_n^\lambda$, $n=1,2,\ldots$

then $a_n=o(e^{n^\gamma})$ for every $\gamma>\log_2\lambda$.

Proof. From (b) we have $a_{2^k}\leqslant 2^{\lambda^{k-2}+2\lambda^{k-3}+\ldots+(k-1)}a_1^{\lambda^k}\leqslant 2^{\beta\lambda^k}$, $k=0,1,2,\ldots,$ where $\beta=(\lambda-1)^{-2}+\log_2a_1$. Now an arbitrary natural number n is of the form $n=\varepsilon_02^0+\varepsilon_12^1+\ldots+\varepsilon_m2^m$, where ε_k is 0 or 1, $\varepsilon_m=1$ and $\log_2n\leqslant m<1+\log_2n$. Therefore by (a)

$$a_n\leqslant\prod_{k=0}^m a_{2^k}^{\varepsilon_k}\leqslant\prod_{k=0}^m 2^{\beta\lambda^k}\leqslant 2^{m\beta\lambda^m}\leqslant 2^{(1+\log_2n)\beta n^{\gamma_0}},$$

where $\gamma_0=\log_2\lambda$. It is obvious that the left side of the last inequality is $o(e^{n^\gamma})$ for every $\gamma>\gamma_0$.

LEMMA 4. *Let ω be as in Theorem 1. For $f\in L^1(G,\omega)$ let us define*

$$u(f)=\sum_{k=1}^\infty\frac{i^kf^k}{k!}.$$

If f is hermitian and belongs to $L^1(G,\omega)\cap L^2(G)$, then

(7) $$\|f^n\|_\omega=[\lambda(f)]^no(e^{n^\gamma}),$$

(8) $$\|u(nf)\|_\omega=o(e^{n^\gamma})$$

for every $\gamma>\log_2\big((2p+2)/(p+2)\big)$.

Proof of (7). Ignoring the trivial case of $f=0$, we may assume $\lambda(f)=1$. We will show that the sequence $a_n=D\cdot\|f^n\|_\omega$, $n=1,2,\ldots$ satisfies conditions (a) and (b) of Lemma 3 for $\lambda=(2p+2)/(p+2)$ and a suitable constant $D\geqslant1$.

We have $\|f^{n+m}\|_\omega\leqslant\|f^n\|_\omega\|f^m\|_\omega$; thus condition (a) is satisfied. Now by (5) and (6)

$$\|f^{2n}\|_\omega\leqslant 2C\|f^n\|_1\cdot\|f^n\|_\omega\leqslant 2CA\|f^n\|_2^{2/(p+2)}\|f^n\|_\omega^\lambda.$$

Since $\lambda(f)=1$, we have $\|f^n\|_2\leqslant\|f\|_2$ for each n; thus choosing D such that $D^{\lambda-1}\geqslant 2CA\|f\|_2^{2/(p+2)}$ we get (b).

Proof of (8). First observe that

(9) $$\|u(f)\|_2\leqslant\|f\|_2.$$

Indeed, let v be the analytic function defined by

$$v(z)=\frac{e^{iz}-1}{z}=\sum_{k=1}^\infty\frac{i^kz^{k-1}}{k!},\qquad z\in C.$$

Then we have $u(f)=v(T_f)(f)$, where $v(T_f)$ is the operator on $L^2(G)$ obtained by the action of v on T_f. Therefore

$$\begin{aligned}\|u(f)\|_2&=\|v(T_f)(f)\|_2\leqslant\|v(T_f)\|\|f\|_2\\&\leqslant\sup_{z\in\mathrm{Sp}\,T_f}|v(z)|\|f\|_2\leqslant\sup_{t\in R}|v(t)|\|f\|_2\leqslant\|f\|_2.\end{aligned}$$

To prove (8) we will apply Lemma 3 with the sequence $a_n=D\big(\|u(nf)\|_\omega+1\big)$ for a suitable constant $D\geqslant1$. We have

$$\begin{aligned}\big\|u\big((n+m)f\big)\big\|_\omega+1&=\|u(nf)*u(mf)+u(nf)+u(mf)\|_\omega+1\\&\leqslant\big(\|u(nf)\|_\omega+1\big)\big(\|u(mf)\|_\omega+1\big);\end{aligned}$$

thus (a) is satisfied. To prove (b), let us consider two cases: First, let $\|u(nf)\|_\omega\leqslant1$. Then

$$\|u(2nf)\|_\omega+1\leqslant\big(\|u(nf)\|_\omega+1\big)^2\leqslant4\leqslant4\big(\|u(nf)\|_\omega+1\big)^\lambda,$$

and so we have (b) for $D^{\lambda-1}\geqslant4$. Second, let $\|u(nf)\|_\omega\geqslant1$. Then by (5), (6) and (9)

$$\begin{aligned}\|u(2nf)\|_\omega+1&\leqslant\big(2C\|u(nf)\|_1+2\big)\|u(nf)\|_\omega+1\\&\leqslant\big(2CA\|nf\|_2^{2/(p+2)}\|u(nf)\|_\omega^{p/(p+2)}+2\big)\|u(nf)\|_\omega+1\\&\leqslant n\big(2CA\|f\|_2^{2/(p+2)}+3\big)\|u(nf)\|_\omega^\lambda,\end{aligned}$$

and it suffices to put $D^{\lambda-1}\geqslant 2CA\|f\|_2^{2/(p+2)}+3$.

THEOREM 2. *Let G be a locally compact group. If ω is a polynomial weight on G such that ω^{-1} belongs to $L^p(G)$ for a $0<p<\infty$, then functions* *with arbitrarily small support operate on each hermitian L^2-function in the algebra $L^1(G, \omega)$.*

Proof. This is an immediate consequence of (8).

References

[1] J. Dixmier, *Opérateurs de rang fini dans les représentations unitaires*, Inst. Hautes Études Sci. Publ. Math. 6 (1960), 13–25.

[2] Y. Domar, *Harmonic analysis based on certain commutative Banach algebras*, Acta Math. 96 (1956), 1–66.

[3] H. Helson, J.-P. Kahane, Y. Katznelson, W. Rudin, *The functions which operate on Fourier transforms*, ibid. 102 (1959), 135–157.

[4] A. Hulanicki, *Growth of the L^1 norm of the convolution powers of functions on nilpotent groups of class* 2, Instituto Nazionale di Alta Matematica, Symposia Mathematica 22 (1977), 439–447.

[5] — *Invariant subsets of non-synthesis. Leptin algebras and non-symmetry*, to appear.

[6] M. Y. Katznelson, *Sur le calcul symbolique dans quelques algèbres de Banach*, Ann. Sc. École Norm. Sup. 63 (1959), 83–123.

[7] N. Leblanc, *Calcul symbolique dans certains algèbres de séries de Fourier à poids*, C. R. Acad. Sci. Paris 266 (1968), 339–341.

[8] N. B. Levina, *Functions that operate on a group algebra with weight* (in Russian), Teor. Funkciĭ Funkcional. Anal. i Priložen. 25 (1976), 82–94.

[9] T. Pytlik, *On the spectral radius of elements in group algebras*, Bull. Acad. Polon. Sci. 21 (1973), 899–902.

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Received October 1, 1980 (1642)