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On the spectra of holomorphic function algebras

by

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Abstract. Let (M, p) be a Riemann domain where $M = N \times \Omega$ for Ω a Stein domain and N the set of natural numbers. Let A be a closed subalgebra of $\mathcal{O}(M)$ such that $\mathcal{O}(\Omega)\circ p \subseteq A$. Let R denote the equivalence relation defined on M by A. The spectrum $\Delta(A)$ with the compactly generated Gelfand topology kG is homeomorphic to the quotient space M/R with the quotient topology Q. Also, under certain conditions, the Gelfand topology G agrees with kG on $\Delta(A)$.

Let (M, p) be a general Riemann domain and let A be a closed subalgebra of $\mathcal{O}(M)$ containing the coordinate functions. Let M^* be the spectrum of the stable algebra B generated by A. Let $\hat{B}$ be the Gelfand transform of B and let A^* be the image of A under the map taking B to $\hat{B}$. The algebras A^* and A are topologically isomorphic. Furthermore, A is more properly viewed as the algebra A^* on M^*.

It is known that if M is A-convex, then $\Delta(A)$ is an analytic space such that $\hat{A} = \mathcal{O}(\Delta(A))$. In general M may not be A-convex even though $\Delta(A)$ is an analytic space with $\hat{A} \subseteq \mathcal{O}(\Delta(A))$. Under certain hypotheses, M^* is A^*-convex whenever $\Delta(A)$ is an analytic space such that $\hat{A} \subseteq \mathcal{O}(\Delta(A))$.

1. Introduction. A *Riemann domain* is a pair (M, p), where M is a topological space, and $p\colon M \to C^N$ is a local homeomorphism. For $x \in M$ and ε sufficiently small, we denote by $\Delta(x, \varepsilon)$ the neighborhood of x which is mapped homeomorphically by p onto the polydisc $\Delta(p(x), \varepsilon)$. A complex-valued function f defined on the Riemann domain (M, p) is said to be *holomorphic* if, for all $\Delta(x, \varepsilon) \subseteq M$, $f\circ p^{-1}$ is holomorphic on $\Delta(p(x), \varepsilon)$ ([8], pp. 43–44). We denote by $\mathcal{O}(M)$ the set of all holomorphic functions on M. If $p = (p_1, \ldots, p_N)$, $a = (a_1, \ldots, a_N)$, and $|a| = \sum_{i=1}^{N} a_i$, we may define the differential operator D^a on $\mathcal{O}(M)$ by

$$D^a f = \frac{\partial^{|a|} f}{\partial p_1^{a_1} \ldots \partial p_N^{a_N}}$$

for all f in $\mathcal{O}(M)$ ([8], p. 47).

Let X and Y be two topological spaces and let $f\colon X \to Y$ be a continuous map. We say that f is a *proper map* if for every compact set K of Y, $f^{-1}(K)$ is a compact subset of X.

Let A be a uniform algebra on a complex manifold M. Let $\Delta(A)$ be the set of nonzero continuous homomorphisms from A to C. We call $\Delta(A)$ the *spectrum* of A. We let $\hat{A} = \{\hat{f}\colon f \in A\}$ where

$\hat{f}(\varphi) = \varphi(f)$ for all $\varphi \in \Delta(A)$. We call $\hat{A}$ the *Gelfand transform* of A. We let e: $M \to \Delta(A)$ be the evaluation map given by $e(x)(f) = f(x)$, for all f in A and all x in M. The Gelfand topology G on $\Delta(A)$ is the weakest topology on $\Delta(A)$ such that $\hat{f}$ is continuous for every f in A.

Let (X, T) be a Hausdorff space. We generate a new topology, called the *compactly generated topology* kT, by letting $U \in kT$ if, and only if, for any T-compact set K in X, there is a T-open set V such that $U \cap K = V \cap K$. If $kT = T$, then we say (X, T) is a *k-space*. We note some facts about compactly generated topologies: the compact sets of (X, T) and (X, kT) are exactly the same, and any first countable or locally compact topological space is a k-space ([6], p. 248). We let kG be the compactly generated Gelfand topology on $\Delta(A)$.

The following theorem is due to Rossi ([15], p. 144).

THEOREM 1. *Let M be an analytic space, and A a uniformly closed algebra of holomorphic functions on M such that M is A-convex. Then $(\Delta(A),\ kG)$ can be given the structure of a Stein space so that $\hat{A} = \mathcal{O}(\Delta)$.*

Cartan has proved the following two theorems. (We note that Cartan defines a proper equivalence relation to be an equivalence relation such that X/R is locally compact and the quotient map is proper.)

THEOREM 2. *If f is a proper, holomorphic mapping of an analytic space X into an analytic space Y, and if R denotes the equivalence relation defined by f, then the quotient space X/R is an analytic space whose holomorphic functions are given by $\{g \in C(X/R)\colon g \circ \pi \in \mathcal{O}(X)\}$* ([4], p. 5).

THEOREM 3. *Let X be an analytic space and let R be the equivalence relation on X defined by a family of analytic mappings. If R is proper, then X/R is an analytic space* ([4], p. 8).

We refer the reader to [8] for the statement of Cartan's Theorems A and B, the Proper Mapping Theorem, and the Direct Image Theorem.

Let Ω be a Stein domain in $\boldsymbol{C}^N$ for some natural number N. Let $M = \boldsymbol{N} \times \Omega$, where $\boldsymbol{N}$ is the set of natural numbers. Let p: $M \to \Omega$ be given by $p(i, x) = x$. We will call the pair (M, p) a *stacked domain*. We note that (M, p) may be viewed as a special example of a Riemann domain. Let $\{z_j\colon 1 \leqslant j \leqslant N\}$ be the set of coordinated functions in $\boldsymbol{C}^N$. Let $p_j = z_j \circ p$, $1 \leqslant j \leqslant N$. We shall refer to the p_j's as the coordinate functions on (M, p).

Let A be a closed subalgebra of $\mathcal{O}(M)$. We say that A is a *nontrivial algebra* if $\mathcal{O}(\Omega)\circ p = \{f \circ p\colon f \in \mathcal{O}(\Omega)\} \subseteq A$. If Ω is polynomially convex, then this is just the requirement that $p_j \in A$, $1 \leqslant j \leqslant N$. Throughout the second and third sections we will assume that (M, p) is a stacked domain and that A is a nontrivial algebra on M. For $f \in \mathcal{O}(M)$ we will sometimes write $f = (f_1, f_2, \ldots)$ where $f_i = f|_{i \times \Omega}$. Thus, we may think of A as a subset of $\prod_{i=1}^{\infty} \mathcal{O}(i \times \Omega)$.

Sections 2 and 3 are concerned with the spectrum $\Delta(A)$ of a nontrivial algebra on a stacked domain. In Section 2 we show $\Delta(A) = M/R$. In Section 3 we investigate the relationships among various topologies commonly associated with $\Delta(A)$.

Let M be any Riemann domain and let A be any closed subalgebra of $\mathcal{O}(M)$. As noted above, if M is A-convex, then $\Delta(A)$ may be given the structure of an analytic space. A natural question arises as to whether the presence of an analytic structure on $\Delta(A)$, such that every function in $\hat{A}$ is holomorphic, necessarily forces M to be A-convex. The following example shows that the answer to this question is, in general, no.

EXAMPLE. Let $M = \boldsymbol{N} \times \Omega$ be a stacked domain. Let $A = \mathcal{O}(\Omega)\circ p = \{f \circ p\colon f \in \mathcal{O}(\Omega)\}$. By the results of Section 2, $\Delta(A) = \Omega$. Clearly, $(\Omega, {}_{\Omega}\mathcal{O})$ is an analytic space. However, M is not A-convex. If K is any compact set in M, then

$$h_A(K) = \{x \in M\colon |f(x)| \leqslant \|f\|_K, \text{ for all } f \in A\} = \boldsymbol{N} \times p(K).$$

Since $\boldsymbol{N} \times p(K)$ is not compact, M is not A-convex.

In examining this example, we note that A is topologically isomorphic to $\mathcal{O}(\Omega)$, and that A does not distinguish between levels of M. Therefore, we may view A and M more "naturally" as $\mathcal{O}(\Omega)$ and Ω. Since Ω is holomorphically convex, it appears that we have been asking the wrong question. We should be concerned with the holomorphic convexity of Ω, instead of the A-convexity of M. We extend this basic idea to stacked domains and nontrivial algebras in Section 4 and to Riemann domains and algebras containing the coordinate functions in Section 5. In both cases, we obtain a new domain M^* and a new algebra A^*, related in a natural way to M and A, such that the analyticity of $\Delta(A)$ forces M^* to be A^*-convex. However, in Section 5 we require the additional hypotheses that the evaluation map from M^* to $\Delta(A)$ is surjective and that the quotient and kG topologies agree on $\Delta(A)$. We showed in Sections 2 and 3 that these hypotheses are always satisfied for nontrivial algebras on stacked domains. Also, in Section 4 we will provide a complete description of the analytic structure of $\Delta(A)$, including the decomposition of $\Delta(A)$ into its irreducible branches.

These results provide a partial converse to a theorem of Rossi [15] and answer partially a question of Cartan [4].

2. The spectrum $\Delta(A)$. We begin by stating the following lemma.

LEMMA 1. *Let X be a locally compact, σ-compact space. Let A be a closed subalgebra of $C(X)$. Let $\Delta(A)$ be the set of all continuous nontrivial homomorphisms of A endowed with the Gelfand topology. Let e: $X \to \Delta(A)$ be the evaluation map. The following are equivalent:*

(1) *e is a proper map.*

(2) *X is A-convex.*

(3) *Given a discrete sequence* $\{x_j\}$ *in* X, *there is a function* $f \in A$ *such that the sequence* $\{f(x_j)\}$ *is unbounded.*

We make several definitions. We define an equivalence relation R on M as follows: xRy if, and only if, $f(x) = f(y)$ for all $f \in A$. We let $M_n = \{1, 2, \ldots, n\} \times \Omega$. We let A_n be the closure (in the topology of uniform convergence on compact subsets of M_n) of the restriction of A to M_n. We let R_n be the equivalence relation defined on M_n by A_n: for x and y in M_n, xR_ny if, and only if, $f(x) = f(y)$ for all $f \in A_n$.

We note that if x and y are in M_n, then xR_ny if, and only if, xRy. In other words, R_n is just the restriction of R to M_n. This follows immediately from the fact that $A|_{M_n}$ is dense in A_n.

We let M_n/R_n be the quotient space of M_n with respect to R_n, Q_n be the quotient topology, and π_n be the quotient map. We let $\Delta(A_n)$ be the spectrum of A_n endowed with the Gelfand topology G_n.

Our next objective is to show that $(M_m/R_n, Q_n)$ and $(\Delta(A_n), G_n)$ are homeomorphic. Because we will use the following lemma at a later point, we separate it from the proof that $(M_n/R_n, Q_n) \cong (\Delta(A_n), G_n)$.

LEMMA 2. *The space* M_n *is* A_n*-convex.*

Proof. We will use Lemma 1 ((3) implies (2)). Let $\{x_j\}$ be a discrete sequence in M_n. We may assume, by taking a subsequence if necessary, that there is a discrete sequence $\{y_j\}$ such that $x_j = (i, y_j)$ for some $i \in \{1, \ldots, n\}$. By Lemma 1, there is a function $f \in \mathcal{O}(\Omega)$ such that $\{f(y_j)\}$ is unbounded. Let $h = f \circ p|_{M_n}$. Clearly, $h \in A_n$ and $\{h(x_j)\} = \{f(y_j)\}$ is unbounded.

PROPOSITION 1. *The spaces* $(M_n/R_n, Q_n)$ *and* $(\Delta(A_n), G_n)$ *are homeomorphic.*

Proof. By Lemma 2 we know that M_n is A_n-convex. Therefore, we may apply a theorem of Rossi ([15], p. 143) to conclude that $\Delta(A_n) = M_n/R_n$, at least as sets. Thus, we may use the maps e_n and π_n interchangeably. To conclude the proof it suffices to show that $e_n\colon M_n \to (\Delta(A_n), G_n)$ is an identification ([6], p. 130). Let U be a set in $\Delta(A_n)$ such that $(e_n)^{-1}(U)$ is open in M_n. We must show that U is a G_n-open set in $\Delta(A_n)$. Let $y \in U$ and let $\{x_1, \ldots, x_k\} = (e_n)^{-1}(y)$. If $k < n$, then let $\{x_{k+1}, \ldots, x_n\} = [(p^{-1} \circ p(x_1)) \cap M_n] \setminus \{x_1, \ldots, x_k\}$. Choose $f \in A_n$ so that $f(x_1) = \ldots = f(x_k) = 0$ and $|f(x_i)| > 1$, $k+1 \leqslant i \leqslant n$. Choose a positive number ε so small that $\Delta(x_i, \varepsilon) \subseteq (e_n)^{-1}(U)$, $1 \leqslant i \leqslant k$ and $\inf\{|f(x)|\colon x \in \Delta(x_i, \varepsilon)\} > 1/2$, $k+1 \leqslant i \leqslant n$. If $k = n$, then let $f \equiv 0$ and choose a positive number ε so small that $\Delta(x_i, \varepsilon) \subseteq (e_n)^{-1}(U)$, $1 \leqslant i \leqslant n$. Then $y \in \{\varphi \in \Delta(A_n)\colon |\varphi(p_j) - y(p_j)| < \varepsilon,\ 1 \leqslant j \leqslant N, \text{ and } |\varphi(f)| < \varepsilon\} \subseteq U$. Hence U is a G_n-open set.

We obtain more information about $(\Delta(A_n), G_n)$ after the following proposition.

PROPOSITION 2. *Let* X *and* Y *be topological spaces such that* X *and* Y *are Hausdorff and* X *is a* k*-space. Let* $f\colon X \to Y$ *be a proper, continuous identification. Then* f *is a closed map.*

Proof. Let F be a closed set in X. Because f is an identification, $f(F)$ will be closed if, and only if, $f^{-1}(f(F))$ is closed in X. Since X is a k-space, $f^{-1}(f(F))$ is closed if, and only if $f^{-1}(f(F)) \cap K$ is closed in K for every compact set K in X. Because X is Hausdorff, $f^{-1}(f(F)) \cap K$ will be closed in K whenever $f^{-1}(f(F)) \cap K$ is compact. But $f^{-1}(f(F)) \cap K$ will be compact if, and only if, each net in $f^{-1}(f(F)) \cap K$ has a cluster point in $f^{-1}(f(F)) \cap K$. Let $\{x_\alpha\}$ be a net in $f^{-1}(f(F)) \cap K$. Since K is compact, $\{x_\alpha\}$ has a cluster point x in K. It suffices to show $x \in f^{-1}(f(F))$. Let $\{x_\beta\}$ be a subnet of $\{x_\alpha\}$ converging to x. Because $\{x_\beta\}$ is a net in $f^{-1}(f(F))$, there is a net $\{y_\beta\}$ in F such that $f(x_\beta) = f(y_\beta)$. Because $\{f^{-1}(f(x_\beta))\}$ is contained in $f^{-1}(f(K))$, we know that $\{f^{-1}(f(y_\beta))\}$ is contained in $f^{-1}(f(K))$ and, in particular, $\{y_\beta\}$ is a subset of $f^{-1}(f(K))$. The properness of f assures us that $f^{-1}(f(K))$ is compact. Therefore, $\{y_\beta\}$ has a cluster point y in $f^{-1}(f(K))$. Let $\{y_\gamma\}$ be a subnet of $\{y_\beta\}$ converging to y. Because $f(y_\gamma) \to f(y)$, $f(x_\gamma) = f(y_\gamma)$, and $f(x_\gamma) \to f(x)$, we have $f(x) = f(y)$. Therefore, $x \in f^{-1}(f(y))$. But F is closed and $\{y_\alpha\}$ is a convergent net in F, so y must be in F. Therefore, $x \in f^{-1}(f(F)) \cap K$.

PROPOSITION 3. *The space* $(M_n/R_n, Q_n)$ *is locally compact.*

Proof. We know π_n is a continuous surjection such that $(\pi_n)^{-1}(x)$ is compact for each $x \in M_n/R_n$. Proposition 2 shows that π_n is closed. Therefore, π_n is a perfect map ([6], p. 235). Because π_n is perfect, $(M_n/R_n, Q_n)$ is locally compact if, and only if, M_n is locally compact ([6], p. 240). It is clear that M_n is locally compact.

PROPOSITION 4. *The quotient space* $(M_n/R_n, Q_n)$ *can be given the structure of an analytic space such that every* $\hat{f} \in \hat{A}_n$ *is holomorphic.*

Proof. By Theorem 3, it suffices to show that $(M_n/R_n, Q_n)$ is locally compact and that the map $\pi_n\colon M_n \to (M_n/R_n, Q_n)$ is proper. These facts were shown in Proposition 3 and Lemma 2.

We conclude this section by showing that $\Delta(A)$ and M/R may be viewed as the same set.

PROPOSITION 5. *As sets,* $\Delta(A) = M/R$.

Proof. Since $A = \varprojlim A_n$, we know that as sets $\Delta(A) = \varinjlim \Delta(A_n)$[2] But $\Delta(A_n) = M_n/R_n$, so $\Delta(A) = \varinjlim M_n/R_n$. Clearly, as sets, $\varinjlim M_n/R_n = M/R$.

3. Topologies on the spectrum $\Delta(A)$. This section will be concerned with various topologies on $\Delta(A)$. We begin with a lemma.

LEMMA 3. *Let* X *and* Y *be topological spaces such that* X *is* σ*-compact and locally compact and* Y *is Hausdorff. If* $f\colon X \to Y$ *is an identification* *and L is a compact subset of Y, then there is a compact subset K of X such that $f(K) = L$.*

Proof. Because X is σ-compact and locally compact, we may write $X = \bigcup_{n=1}^{\infty} K_n$, where K_n is compact and $K_n \subseteq \operatorname{Int} K_{n+1}$. Since f is a continuous surjection, we may write $Y = \bigcup_{n=1}^{\infty} L_n$, where $L_n = f(K_n)$ is a compact subset of Y.

If $L \subseteq f(K_n)$ for some n, let $K = f^{-1}(L) \cap K_n$. Because L is compact and Y is Hausdorff, we know that L and (therefore) $f^{-1}(L)$ are closed. But K_n is compact, so $K = f^{-1}(L) \cap K_n$ is a compact subset of X. It is clear that $f(K) \subseteq L$. To show $L \subseteq f(K)$, we let $y \in L$ and choose $x \in K_n$ such that $f(x) = y$. Clearly, $x \in K_n \cap f^{-1}(L) = K$. Therefore, $f(K) = L$.

If $L \nsubseteq f(K_n)$ for every n, then we may choose a $y_n \in L \setminus f(K_n)$ for each n. Let $S = \{y_1, y_2, \ldots\}$. By choosing a subsequence, if necessary, we may assume $y_n \neq y_m$ whenever $n \neq m$. Since L is compact, $\{y_n\}$ has a cluster point. If S is closed, this cluster point must be some $y_m \in S$. Replacing S by $S \setminus \{y_m\}$, if necessary, we may assume that S is not closed. Because f is an identification, $f^{-1}(S)$ is not closed. Let $x_0 \in \overline{f^{-1}(S)} \setminus f^{-1}(S)$ be a cluster point of $f^{-1}(S)$ not contained in $f^{-1}(S)$. Then there is a net $\{x_a\}$ contained in $f^{-1}(S)$ which converges to x_0. Further, $x_a \neq x_0$ for every a, since $x_0 \notin f^{-1}(S)$. Because $X = \bigcup_{n=1}^{\infty} K_n$ and $K_n \subseteq \operatorname{Int} K_{n+1}$, there is some k such that $x_0 \in \operatorname{Int} K_k$. By dropping to a subnet, if necessary, we may assume that $x_a \in \operatorname{Int} K_k$ for all a. The continuity of f implies that $\{f(x_a)\}$ converges to $y_0 = f(x_0) \in f(K_k)$. Since $x_a \in f^{-1}(S)$, $f(x_a) = y_m$ for some m. But there are only a finite number of y_m contained in $f(K_k)$. Hence, $f(x_a) \equiv y_0$ for a sufficiently large. This implies that $x_0 \in f^{-1}(y_0) = f^{-1}(f(x_a)) \subseteq f^{-1}(S)$. This contradicts the fact that $x_0 \notin f^{-1}(S)$. Therefore, $L \subseteq f(K_n)$ for some n.

The next proposition is a list of statements that are equivalent to the statement that M_n is A_n-convex. In Lemma 2 we showed that M_n is A_n-convex; therefore, these statements are always true. However, since the proof involves showing the equivalence of these statements, we state the proposition in this form. We note that one particular result of this proposition is that $(\Delta(A), kG)$ and $(M/R, Q)$ are homeomorphic.

Before stating the proposition, we establish some notation. Since Ω is a Stein domain, we may write $\Omega = \bigcup_{i=1}^{\infty} L_n$, where $L_n \subseteq \operatorname{Int} L_{n+1}$, $h_{\mathcal{O}(\Omega)}(L_n) = L_n$, and each L_n is a compact subset of Ω. Let $K_n = \{1, 2, \ldots \ldots, n\} \times L_n$,

$$K_n^* = K_n \cup \{y \in M \colon yRx \text{ for some } x \in K_n\},$$

and let

$$V_{ij} = \{x \in \Omega \colon f_i(x) = f_j(x) \text{ for all } f \in A\}.$$

PROPOSITION 6. *The following are equivalent to M_n being A_n-convex.*

(1) *If $p > 0$ and $x \in \{n+p\} \times \left(\Omega \setminus \bigcup_{j=1}^{n} V_{n+p,j}\right)$, then there exists a function $f \in A$ such that $f(x) = 1$ and $f|_{M_n} \equiv 0$.*

(2) *For all n, $h_A(K_n) = K_n^*$.*

(3) *If $\sigma(K) = \{\varphi \in \Delta(A) \colon |\varphi(f)| \leqslant \|f\|_{K_n}$ for all $f \in A\}$, then $e(K_n) = \sigma(K_n)$, for all n.*

(4) *For each n, there exists a compact $L_n^* \subseteq M$ such that $e(L_n^*) = \sigma(K_n)$.*

(5) *The spaces $(\Delta(A), kG)$ and $(M/R, Q)$ are homeomorphic.*

Proof. We will prove this proposition in a number of steps.

Step 1. (1) implies (2). If $x \in h_A(K_n) \setminus K_n^*$, then $|f(x)| \leqslant \|f\|_{K_n}$ for all $f \in A$. By (1) this is not possible. Hence $K_n^* \subseteq h_A(K_n)$.

Step 2. (2) implies (1). Let $x \in \{n+p\} \times \left(\Omega \setminus \bigcup_{j=1}^{n} V_{n+p,j}\right)$. Since $x \notin h_A(K_n)$, there is a $g \in A$ such that $|g(x)| > \|g\|_{K_n}$. We note that $\|g\|_{K_n} \geqslant \|g\|_{j \times L_n}$, $1 \leqslant j \leqslant n$. Let $g_j = g - (g|_{j \times L_n} \circ p)$. Then $g_j(x) \neq 0$ and $g_j|_{j \times L_n} \equiv 0$. By the identity theorem ([8], p. 6), $g_j|_{j \times \Omega} \equiv 0$. We may assume, by multiplying g_j by $1/g_j(x)$ if necessary, that $g_j(x) = 1$. Let $f = \prod_{j=1}^{n} g_j$. Then $f(x) = 1$ and $f|_{M_n} \equiv 0$.

Step 3. (2) is equivalent to (3). We know that $\sigma(K_n) = e(K_n)$ if and only if $e^{-1}(\sigma(K_n)) = e^{-1}(e(K))$. But, $e^{-1}(\sigma(K_n)) = h_A(K_n)$ and $e^{-1}(e(K_n)) = K_n^*$.

Step 4. (3) implies (4). Let $L_n^* = K_n$.

Step 5. (4) implies (5). We know that $kG \subseteq Q$ because $e\colon M \to (\Delta(A), kG)$ is continuous. Let $U \in Q$. Then $e^{-1}(U)$ is open in M. To show that $U \in kG$, we must show that $U \cap H$ is relatively G-open for every kG-compact set H in $\Delta(A)$. Since $\Delta(A) = \bigcup_{n=1}^{\infty} \sigma(K_n)$, every compact H in $\Delta(A)$ is contained in $\sigma(K_n)$ for some n. Thus, it suffices to show that $U \cap \sigma(K_n)$ is relatively G-open for each n. By (4), it suffices to show that if L is a compact set in M, then $U \cap e(L)$ is relatively G-open. Let $\varphi_0 \in U \cap e(L)$. We must find $f_1, \ldots, f_t$ in A and an $\varepsilon > 0$ such that

$$\varphi_0 \in \{\varphi \in e(L) \colon |\varphi(f_i) - \varphi_0(f_i)| < \varepsilon,\ 1 \leqslant i \leqslant t\} \subseteq U \cap e(L).$$

Let $e^{-1}(\varphi_0) \cap L = \{x_1, \ldots, x_s\}$ and let $p^{-1}\big(p(e^{-1}(\varphi_0))\big) \cap L = \{x_1, \ldots, x_s, \ldots \ldots, x_l\}$. Choose $f_1, \ldots, f_{l-s}$ such that $f_i(x_{s+i}) = 1$ and $f_i(x_j) = 0$, $1 \leqslant j \leqslant s$, $1 \leqslant i \leqslant l-s$. Choose $\varepsilon > 0$ so small that $\varepsilon < 1/4$, $\|f_i\|_{\Delta(x_j, \varepsilon)} < 1/4$, $1 \leqslant j \leqslant s$, $1 \leqslant i \leqslant l-s$, $\Delta(x_i, \varepsilon) \subseteq e^{-1}(U)$, $1 \leqslant i \leqslant s$, and

$$\inf\{|f_i(x)| \colon x \in \Delta(x_{s+1}, \varepsilon),\ 1 \leqslant i \leqslant l-s\} > 3/4.$$

Then

$$\{x \in L\colon |p_i(x)-p_i(x_1)| < \varepsilon,\ 1 \leqslant i \leqslant N \text{ and } |f_j(x)-f_j(x_1)| < \varepsilon,\ 1 \leqslant j \leqslant l-s\} \subseteq L \cap e^{-1}(U).$$

This implies that

$$\varphi_0 \in \{\varphi \in e(L)\colon |\varphi(p_i)-\varphi_0(p_i)| < \varepsilon,\ 1 \leqslant i \leqslant N, \text{ and } |\varphi(f_j)-\varphi_0(f_j)| < \varepsilon,\ 1 \leqslant j \leqslant l-s\} \subseteq U \cap e(L).$$

Step 6. (5) implies (4). Since the Q and kG topologies agree on $\Delta(A)$, $\sigma(K_n)$ will be Q-compact for each n. The map $e\colon M \to (\Delta(A), Q)$ is an identification map. By Lemma 3, there is a compact L_n^* such that $e(L) = \sigma(K_n)$.

Step 7. M_n is A_n-convex implies (3). Because M_n is A_n-convex, $\Delta(A_n) = M_n/R_n$. Now,

$$\Delta(A_n) = \bigcup_{j=1}^{\infty} \sigma(\{1, \ldots, n\} \times L_j)$$

so $\sigma(K_n) = \sigma(\{1, \ldots, n\} \times L_n) \subseteq M_n/R_n$. Because L_n is $\mathcal{O}(\Omega)$-convex, $\sigma(K_n) \cap (M_n/R_n) \subseteq e(K_n)$. Clearly, $e(K_n) \subseteq \sigma(K_n)$. Therefore, $e(K_n) = \sigma(K_n)$.

Step 8. (4) implies M_n is A_n-convex. In order to show M_n is A_n-convex for every n it is sufficient to show that for any j, $h_{A_n}(\{1, \ldots, n\} \times L_j)$ is a compact subset of M_n. There is a compact set H_j in M such that $e(H_j) = \sigma(\{1, \ldots, n\} \times L_j)$. Therefore, $e^{-1}(e(H_j)) \cap M_n = h_A(\{1, \ldots, n\} \times L_j) \cap M_n = h_{A_n}(\{1, \ldots, n\} \times L_j)$ is a compact subset of M_n.

At this point we are ready to consider the relationship between the G and kG topologies on $\Delta(A)$. We have not been successful in showing that $(\Delta(A), G)$ and $(\Delta(A), kG)$ are always homeomorphic. However, we have been able to show this for a large number of cases. In order to describe these we need to make some definitions.

DEFINITION. We say that A is *maximal with respect to* R, if the following condition holds: if $f \in \mathcal{O}(M)$ and $f_i|_{V_{ij}} = f_j|_{V_{ij}}$ for all i and j, then $f \in A$. In other words, A is maximal with respect to R if $A = \{f \in \mathcal{O}(M)\colon f$ respects the equivalence relation R defined by $A\}$.

DEFINITION. Let $\{W_\lambda\colon \lambda \in \Lambda\}$ be a collection of subvarieties of Ω. Suppose that given any $x \in \Omega$, there is an $\varepsilon > 0$ such that $\overline{\Delta(x, \varepsilon)} \cap W_\lambda = \emptyset$ for all but a finite number of $\lambda \in \Lambda$. In this case, we say the collection $\{W_\lambda\colon \lambda \in \Lambda\}$ is *locally finite*.

DEFINITION. Let $\{W_\lambda\colon \lambda \in \Lambda\}$ be a locally finite collection of subvarieties. Suppose that, for each pair (i, j) of natural numbers, there is a collection $\mathcal{I}_{ij}$ of subsets I of Λ such that $V_{ij} = \bigcup_{I \in \mathcal{I}_{ij}} \bigcap_{\lambda \in I} W_\lambda$. In this case, we say that the V_{ij} are *globally given by a locally finite collection of subvarieties of* Ω.

PROPOSITION 7. *Suppose that A is maximal with respect to R. Suppose further that the V_{ij} are globally given by a locally finite collection of subvarieties of Ω. Then $(\Delta(A), G) \cong (\Delta(A), kG)$.*

Proof. It suffices to show that given an open set $U \subseteq M$ such that $e^{-1}(e(U)) = U$ and an $x_1 \in U$, there exist a finite number of functions $f_1, \ldots, f_s$ in A and an $\varepsilon > 0$ such that $x_1 \in \{x \in M\colon |f_i(x)-f_i(x_1)| < \varepsilon,$ $1 \leqslant i \leqslant s\} \subseteq U$. We shall assume $x_1 \in 1 \times \Omega$. We let $x_0 = p(x_1)$, $x_j = (j, x_0)$ and $S = \{i \in \mathbf{N} \setminus \{1\}\colon x_i R x_1\}$. We must consider two cases.

Case 1. $S = \emptyset$. This case occurs whenever x_1 is not identified with any other point of M. Let $V_1 = \bigcup_{j \neq 1} V_{1j}$, then

$$V_1 = \bigcup_{j \neq 1} \Big[\bigcup_{I \in \mathcal{I}_{ij}} \Big(\bigcap_{\lambda \in I} W_\lambda\Big)\Big].$$

Since the collection $\{W_\lambda\colon \lambda \in \Lambda\}$ is locally finite, V_1 must be a subvariety.

If $V_1 = \emptyset$, then $V_{1j} = \emptyset$ for all $j \neq 1$. Therefore, the function $f = (0, 1, \ldots, 1, \ldots)$ is an element of A. Since $x_1 \in U$ and U is an open set, we can choose an $\varepsilon < 1/4$ so that $\Delta(x_1, \varepsilon) \subseteq U$. For this ε and f we have $x_1 \in \{x \in M\colon |p_i(x)-p_i(x_1)| < \varepsilon,\ 1 \leqslant i \leqslant N, \text{ and } |f(x)| < \varepsilon\} \subseteq U$.

We now suppose that $V_1 \neq \emptyset$. If $x_0 \in V_1$, then $x_0 \in V_{1j}$ for some j, $x_1 R x_j$ and $j \in S$. But $S = \emptyset$ so $x_0 \notin V_1$. Therefore, we may choose $g \in \mathcal{I}(V_1)$ such that $g(x_0) \neq 0$ ([8], p. 245). We also choose $\varepsilon > 0$ such that $\varepsilon < 1/4$, $\Delta(x_1, \varepsilon) \subseteq U$, and $g(x) \neq 0$ for all $x \in \Delta(x_0, 2\varepsilon)$. If necessary, multiply g by a sufficiently large constant such that $\inf\{|g(x)|\colon x \in \Delta(x_0, \varepsilon)\} > 1$. Let $f = (0, g, \ldots, g, \ldots)$. By the maximality of A and the fact that $g \in \mathcal{I}(V_1) = \mathcal{I}(\bigcup_{j \neq 1} V_{1j})$ we know that f is an element of A. For this ε and f we have $x_1 \in \{x \in M\colon |p_i(x)-p_i(x_1)| < \varepsilon,\ 1 \leqslant i \leqslant N$ and $|f(x)| < \varepsilon\} \subseteq U$.

Case 2. $S \neq \emptyset$. The collection $\{W_\lambda\colon \lambda \in \Lambda\}$ is locally finite, so we may choose an $\varepsilon' > 0$ such that $W_\lambda \cap \overline{\Delta(x_0, \varepsilon')} = \emptyset$ for all but a finite number of $\lambda \in \Lambda$. If $W_\lambda \cap \overline{\Delta(x_0, \varepsilon')} \neq \emptyset$ and $x_0 \notin W_\lambda$, then we may find $\varepsilon_\lambda < \varepsilon'$ such that $W_\lambda \cap \Delta(x_0, \varepsilon_\lambda) = \emptyset$. We let

$$\varepsilon'' = \tfrac{1}{2}\min\{\varepsilon', \varepsilon_\lambda\colon x_0 \notin W_\lambda,\ W_\lambda \cap \overline{\Delta(x_0, \varepsilon')} \neq \emptyset\}.$$

Thus, if $W_\lambda \cap \overline{\Delta(x_0, \varepsilon'')} \neq \emptyset$, then $x_0 \in W_\lambda$. We let $I_0 = \{\lambda \in \Lambda\colon W_\lambda \cap \overline{\Delta(x_0, \varepsilon'')} \neq \emptyset\}$ and let $I_0^* = \Lambda \setminus I_0$. We now choose $\varepsilon''' < \varepsilon''$ so that $\Delta(x_1, \varepsilon''') \subseteq U$.

If $V_{ij} \cap \overline{\Delta(x_0, \varepsilon''')} \neq \emptyset$, then

$$V_{ij} \supseteq \bigcap_{\lambda \in I_0} W_\lambda;$$

hence, $x_0 \in V_{ij}$. This implies that $x_i R x_j$. Let $S^* = S \cup \{1\}$. If $i \in S^*$ and $j \notin S^*$, then $V_{ij} \cap \overline{\Delta(x_0, \varepsilon''')} = \emptyset$. Otherwise, x_i would be identified with x_j, and x_1 and x_j would necessarily have to be identified. Let

$$Y = \bigcup_{\substack{i \in S^* \\ j \notin S^*}} V_{ij}.$$

Then $Y \cap \overline{\Delta(x_0, \varepsilon''')} = \emptyset$. Further, Y is a subvariety, since it is a locally finite union of subvarieties. Choose $g \in \mathscr{I}(Y)$ such that $g(x_0) \neq 0$ ([8], p. 245). Choose $\varepsilon < \varepsilon'''$ such that $\varepsilon < 1/4$ and $g(x) \neq 0$ for all $x \in \overline{\Delta(x_0, 2\varepsilon)}$. Throughout the rest of this section, we let $\Delta = \Delta(x_0, \varepsilon)$ and $\bar{\Delta} = \overline{\Delta(x_0, \varepsilon)}$. We multiply g by an appropriate constant, if necessary, so that $\inf\{|g(x)|: x \in \bar{\Delta}\} > 1$. Let $f = (f_j)$ where

$$f_j = \begin{cases} 0 & \text{if} \quad j \in S^*, \\ g & \text{if} \quad j \notin S^*. \end{cases}$$

If i and j are both contained in S^* or $\boldsymbol{N} \setminus S^*$, then it is clear that $f_i|_{V_{ij}} = f_j|_{V_{ij}}$. If $i \in S^*$ and $j \notin S^*$, then $V_{ij} \subseteq Y$. Therefore, $f_i|_{V_{ij}} = g|_{V_{ij}} = f_j|_{V_{ij}} = 0$. Since $f_i|_{V_{ij}} = f_j|_{V_{ij}}$ for all i and j, the maximality of A mplies that $f \in A$. We now have

$$\{x \in M: |p_i(x) - p_i(x_1)| < \varepsilon, \ 1 \leqslant i \leqslant N, \text{ and } |f(x)| < \varepsilon\} \subseteq S^* \times \Omega.$$

We establish more notation. For $I \subseteq I_0$ let $I' = I_0 \setminus I$. For $I \subseteq I_0$ let

$$W_I = \Big(\bigcup_{\lambda \in I_0^*} W_\lambda\Big) \cup \Big(\bigcup_{\lambda \in I} W_\lambda\Big) \cup Y$$

and let $W^I = \bigcap_{\lambda \in I} W_\lambda$. Let $W = \bigcup_{\lambda \in \Lambda} W_\lambda$.

Because Ω is a Stein domain, there are functions g_i^I, $I \subseteq I_0$, $1 \leqslant i \leqslant n_I$, and h_i, $1 \leqslant i \leqslant r$, such that

$$W_I = \bigcap_{i=}^{n_I} Z(g_i^I)$$

and

$$W = \bigcap_{i=1}^{r} Z(h_i)$$

([7], p. 94). We let $g_I = (g_1^I, \ldots, g_{n_I}^I): \Omega \to \boldsymbol{C}^{n_I}$ and $|g_I| = \sup\{|g_i^I(x)|: 1 \leqslant i \leqslant n_I\}$. Similarly, we let $h = (h_1, \ldots, h_r): \Omega \to \boldsymbol{C}^r$ and $|h(x)| = \sup\{|h_i(x)|: 1 \leqslant i \leqslant r\}$. Let $I \subseteq I_0$. For $k \in S$ we define constants $c(I, k)$ by

$$c(I, k) = \begin{cases} 0 & \text{if} \quad W^{I'} \subseteq V_{1k}, \\ 1 & \text{if} \quad W^{I'} \nsubseteq V_{1k}. \end{cases}$$

For $k \notin S$ we let $c(I, k) = 0$. Finally, we let $X_k = \{x \in \bar{\Delta} \cap W: |c(I, k) g_I(x)| = 0, \ I \subseteq I_0\}$.

We claim that $X_k \subseteq V_{1k}$ for all $k \in S^*$. If $k = 1$, then $V_{1k} = \Omega$ and the claim is clear. Suppose that $k \neq 1$ and let $x \in X_k$. We let $I(x) = \{\lambda \in I_0: x \notin W_\lambda\}$ and $I'(x) = I_0 \setminus I(x)$. If $I'(x) = I_0$, then

$$x \in \bigcap_{\lambda \in I_0} W_\lambda \subseteq V_{1k}.$$

Suppose now that $I'(x)$ is strictly contained in I_0. Then $I(x) \neq \emptyset$ and $x \notin W_{I(x)}$. Consequently, $|g_{I(x)}(x)| \neq 0$. If $c(I(x), k) = 1$, then $|c(I(x), k) g_{I(x)}(x)| \neq 0$. This would imply $x \notin X_k$. Therefore, $c(I(x), k) = 0$. But this implies $W^{I'(x)} \subseteq V_{1k}$. By the definition of $I'(x)$ we have $x \in W^{I'(x)}$. Thus, $X_k \subseteq V_{1k}$ for all $k \in S^*$.

Now let j and k be elements of S^* such that $j \neq k$. We claim that if $W^{I'} \subseteq V_{jk}$, then $c(I, j) = c(I, k)$. If $j = 1$ or $k = 1$, then this is clear by the definition of $c(I, k)$. So we may suppose j and $k \neq 1$. If $c(I, k) = 0$, then $W^{I'} \subseteq V_{1k}$ and $W^{I'} \subseteq V_{1k} \cap V_{jk}$. This implies that $W^{I'} \subseteq V_{1j}$; reversing the roles of j and k finishes the argument.

Now let j and k be elements of S^* such that $j < k$. We claim that $c(I, k) g_I|_{V_{jk}} = c(I, k) g_I|_{V_{jk}}$ for all $I \subseteq I_0$. Let $x \in V_{jk}$. If $x \in W_I$, then $g_I(x) = 0$ and the result is clear. Thus, we may assume $x \notin W_I$. Then $g_I(x) \neq 0$. If $W^{I'} \subseteq V_{jk}$, then $c(I, j) = c(I, k)$ and $c(I, j) g_I(x) = c(I, k) g_I(x)$. The only other possibility is that $x \notin W_I$, $x \in V_{jk}$ and $W^{I'} \nsubseteq V_{jk}$. We show that this cannot happen. Since $x \notin W_I$ and $x \in V_{jk}$, we know that $x \in W^J$ for some $J \subseteq I'$. We suppose that J is maximal in the sense that J is the largest subset of I' such that $x \in W^J$. If $W^{I'} \nsubseteq V_{jk}$, then $W^J \nsubseteq V_{jk}$ ($J \subseteq I'$ implies $W^{I'} \subseteq W^J$). But

$$V_{jk} = \bigcup_{L \in \mathscr{I}_{jk}} \bigcap_{\lambda \in L} W_\lambda,$$

so if $W^J \nsubseteq V_{jk}$, then $J \notin \mathscr{I}_{jk}$. Since $x \in V_{jk}$, there is some λ, $\lambda \notin J$ such that $x \in W_\lambda$. If $\lambda \in I'$, then $x \in W^{J \cup \{\lambda\}}$ and $J \cup \{\lambda\} \subseteq I'$. The maximality of J implies that this cannot happen. Therefore, $\lambda \notin I'$. Thus, it must be the case that $\lambda \in I \cup I_0^*$. But this implies that $x \in W_I$, a contradiction to our assumption that $x \notin W_I$.

Now let $j \in S^*$ and $k \notin S^*$. We claim that $c(I, j) g_I|_{V_{jk}} = c(I, k) g_I|_{V_{jk}}$. If $j \in S^*$ and $k \notin S^*$, then $V_{jk} \subseteq Y \subseteq W_I$. Therefore, $g_I|_{V_{jk}} \equiv 0$.

We define additional constants by induction. Let $j = \inf S$. Since $X_j \subseteq V_{1j}$, it is clear that $j \times X_j \subseteq U$. Because $\bar{\Delta}$ is a compact set, there is a δ_j such that

$$j \times X_j^* = j \times \{x \in \bar{\Delta} \cap W: |c(I, j) g_I(x)| \leqslant \delta_j, \ I \subseteq I_0\} \subseteq U.$$

Let $\delta(I, j) = \delta_j$ for all $I \subseteq I_0$. Now let $l \in S$. Suppose that for all $k < l$,

$k \in S$, and for all $I \subseteq I_0$, $\delta(I, k)$ has been defined so that

$$k \times X_k^* = k \times \{x \in \bar{\Delta} \cap W\colon |c(I, k) g_I(x)| \leqslant \delta(I, k),\ I \subseteq I_0\} \subseteq U.$$

We also suppose that if k and m are distinct elements of S less than l, then $\delta(I, k) = \delta(I, m)$ whenever $W^{I'} \subseteq V_{km}$. If $k \in S$, $k < l$, and $W^{I'} \subseteq V_{kl}$, we let $\delta(I, l) = \delta(I, k)$. Let k and m be elements of S such that $k \neq m$, $k < l$ and $m < l$. If $W^{I'} \subseteq V_{kl}$ and $W^{I'} \subseteq V_{ml}$, then $W^{I'} \subseteq V_{kl} \cap V_{ml} \subseteq V_{km}$. Hence, $\delta(I, m) = \delta(I, k)$, so no ambiguity is present in this definition.

We let

$$\mathscr{I}^l = \{I \subseteq I_0\colon \text{there is a } k < l,\ k \in S \text{ and } W^{I'} \subseteq V_{kl}\}.$$

Let

$$X_l' = \{x \in \bar{\Delta} \cap W\colon |c(I, l) g_I(x)| \leqslant \delta(I, l) \text{ for } I \in \mathscr{I}^l \text{ and } |c(I, l) g_I(x)| = 0 \text{ for } I \subseteq I_0,\ I \notin \mathscr{I}^l\}.$$

We claim that $l \times X_l' \subseteq U$. Let (l, x) be an element of $l \times X_l'$, let $J' = \{\lambda \in \Lambda\colon x \in W_\lambda\}$, and let $J = \{\lambda \in I_0\colon x \notin W_\lambda\}$. Now, $x \in X_l'$ implies that $x \in \bar{\Delta} \cap W$. Therefore, $J' \subseteq I_0$ and

$$x \notin \Big(\bigcup_{\lambda \in I_0^*} W_\lambda\Big) \cup \Big(\bigcup_{\lambda \in I_0 \setminus J'} W_\lambda\Big) \cup Y = W_J.$$

We consider two possibilities: either $J' \subseteq \mathscr{I}^l$ or $J' \notin \mathscr{I}^l$.

If $J' \notin \mathscr{I}^l$, then $|c(I, l) g_I(x)| = 0$. If $c(I, l) = 0$, then $W^{J'} \subseteq V_{1l}$; hence, $(l, x) \in U$. If $c(I, l) = 1$, then $|g_I(x)| = 0$. Hence, $x \in W_J$. This is a contradiction.

If $J' \in \mathscr{I}^l$, then there is a $k \in S$ such that $k < l$, $W^{J'} \subseteq V_{kl}$ and $(k, x) R (l, x)$. We will show that $(k, x) \in U$. This will imply that $(l, x) \in U$. It suffices to show that $(k, x) \in k \times X_k^*$, since, by our assumption, $k \times X_k^* \subseteq U$. If $I \subseteq I_0$, then either $I \subseteq J$ or $I \nsubseteq J$. If $I \nsubseteq J$, then there is a $\lambda \in I \setminus J$. Now, $\lambda \in J'$, so $x \in W_\lambda \subseteq W_I$. Therefore, $g_I(x) = 0$ and $|c(I, k) g_I(x)| = 0 \leqslant \delta(I, k)$. If $I \subseteq J$, then $J' \subseteq I'$ and $W^{I'} \subseteq W^{J'} \subseteq V_{kl}$. Thus, $I \in \mathscr{I}^l$ and $\delta(I, k) = \delta(I, l)$. But $(l, x) \in l \times X_l'$, so $|c(I, k) g_I(x)| = |c(I, l) g_I(x)| \leqslant \delta(I, l) = \delta(I, k)$. For all $I \subseteq I_0$, we have shown that $|c(I, k) g_I(x)| \leqslant \delta(I, k)$. Therefore, $(k, x) \in k \times X_k^*$.

We may now find a δ_l such that

$$l \times \{x \in \bar{\Delta} \cap W\colon |c(I, l) g_I(x)| \leqslant \delta(I, l) \text{ for } I \subseteq \mathscr{I}^l \text{ and } |c(I, l) g_I(x)| \leqslant \delta_l \text{ for } I \subseteq I_0,\ I \notin \mathscr{I}^l\} \subseteq U.$$

We define $\delta(I, l) = \delta_l$ for every $I \subseteq I_0$ such that $I \notin \mathscr{I}^l$. By our choices of $\delta(I, l)$ we have

$$l \times X_l^* = l \times \{x \in \bar{\Delta} \cap W\colon |c(I, l) g_I(x)| \leqslant \delta(I, l),\ I \subseteq I_0\} \subseteq U.$$

We define some more functions. For $I \subseteq I_0$, we define $G_I \in \mathcal{O}(M)^{n_I}$ by $G_I(k, x) = G(I, k)(x)$ where

$$G(I, k)(x) = \begin{cases} \dfrac{\varepsilon c(I, k)}{\delta(I, k)} g_I(x) & \text{if } k \in S, \\ 0 & \text{if } k \notin S. \end{cases}$$

We claim that $G_I \in A^{n_I}$. This will be the case if $G(I, k)|_{V_{jk}} = G(I, j)|_{V_{jk}}$ for all j and k. If $j \notin S$ and $k \notin S$, then this is clear. If $j \in S^*$ and $k \notin S^*$, then $V_{jk} \subseteq Y \subseteq W_I$. Therefore, $g_I|_{V_{jk}} \equiv 0$. Finally, we assume that j and k are elements of S^* and that $j < k$. We have shown above that if $x \in V_{jk}$, then $c(I, j) g_I(x) = c(I, k) g_I(x)$ for all $I \subseteq I_0$. If $c(I, j) g_I(x) \neq 0$, then we must show $\delta(I, j) = \delta(I, k)$. We do this by showing $W^{I'} \subseteq V_{jk}$. Since $c(I, j) g_I(x) = c(I, k) g_I(x) \neq 0$, $c(I, j) = c(I, k) = 1$. Therefore, $W^{I'} \nsubseteq V_{1j}$ and $W^{I'} \nsubseteq V_{1k}$. Let $J' = \{\lambda\colon x \in W_\lambda\}$. Because $x \notin W_I$, we have $J' \subseteq I'$ and $W^{I'} \subseteq W^{J'}$. Now,

$$V_{jk} = \bigcup_{L \in \mathscr{I}_{jk}} W^L,$$

so there is some L such that $x \in W^L = \bigcap_{\lambda \in L} W_\lambda$. Further, $L \subseteq J' \subseteq I'$, because, if $x \in W_\lambda$, then $\lambda \in J'$. Therefore, $W^{I'} \subseteq W^{J'} \subseteq W^L \subseteq V_{jk}$ and $\delta(I, j) = \delta(I, k)$.

Since $j \times X_j^* \subseteq U$ for all $j \in S$, we know that

$$\{(j, x)\colon |p_i(j, x) - p_i(1, x_0)| \leqslant \varepsilon,\ 1 \leqslant i \leqslant N,\ |f(j, x)| \leqslant \varepsilon,\ |G_I(j, x)| \leqslant \varepsilon,\ I \subseteq I_0, \text{ and } |h(x)| = 0\} \subseteq U.$$

For $j \in S$ choose η_j such that

$$j \times \{x \in \bar{\Delta}\colon |G_I(j, x)| < \varepsilon,\ I \subseteq I_0,\ |h(x)| < \eta_j\} \subseteq U.$$

We define a function $H \in \mathcal{O}(M)^r$ by

$$H(k, x) = \begin{cases} \dfrac{\varepsilon}{\eta_k} h(x) & \text{if } k \in S, \\ 0 & \text{if } k \notin S. \end{cases}$$

We claim that $H \in A^r$. This is clear, since $V_{kl} \subseteq W$ for every k and l and $H \in \mathscr{I}(W)^r$.

Let

$$U' = \{(j, x) \in M\colon |p_i(j, x) - p_i(j, x_0)| < \varepsilon,\ 1 \leqslant i \leqslant N,\ |f(j, x)| < \varepsilon,\ |G_I(j, x)| < \varepsilon,\ I \subseteq I_0,\ |H(j, x)| < \varepsilon\}.$$

We know that $U' \subseteq U$ and that all components of the vector-valued functions are functions in A. Therefore, $(\Delta(A), kG) \cong (\Delta(A), G)$.

4. Stacked domains and nontrivial algebras. We begin by making several definitions. Let $i(1) = 1$ and for $j > 1$ let $i(j) = \inf\{k: 1 \leqslant k \leqslant j$ and $V_{kj} = \Omega\}$. Let $S = \{i(j): j \in N\}$, let $M^* = S \times \Omega$, let $\pi^*: M \to M^*$ be given by $\pi^*(j, x) = (i(j), x)$, let $p^* = p \circ (\pi^*)^{-1}$, and let $A^* = \{f \circ (\pi^*)^{-1}: f \in A\}$. Clearly, (M^*, p^*) is a Riemann domain, and A^* is a nontrivial algebra on M^*.

We will be working with M^* and A^* for the rest of this section. This is equivalent to dealing with M and A under the assumption that $V_{ij} \neq \Omega$ for any i and j, $i \neq j$. In order to simplify notation, we will make this assumption throughout the rest of this section. We will also use (X, T) to refer to the homeomorphic spaces $(\Delta(A), kG)$ and $(M/R, Q)$.

We will say that (X, T) *has an analytic structure* if there is a structure sheaf ${}_X\mathcal{O}$ such that $((X, T), {}_X\mathcal{O})$ is an analytic space, and such that $\hat{A}$ is contained in the algebra $\Gamma(X, {}_X\mathcal{O})$ of global sections of ${}_X\mathcal{O}$. The ultimate goal of this section is to show that if (X, T) has an analytic structure (and $V_{ij} \neq \Omega$ for any distinct i and j), then M is A-convex. We will do this by first showing that the map $\pi: M \to (X, T)$ is proper and then appealing to Lemma 1.

We note some facts about Fréchet algebras and their spectra. If A is any Fréchet algebra and K is a subset of $\Delta(A)$, then K is compact in the G topology if and only if K is compact in the kG topology. For any uniform algebra A, $(\Delta(A), G)$ is A-convex. Therefore, for any uniform algebra A, $(\Delta(A), kG)$ is A-convex.

For the rest of this section, we will assume that there is a sheaf $\mathcal{A}$ on X such that $((X, T), \mathcal{A})$ is a Stein analytic space and that $\hat{A} = \Gamma(X, \mathcal{A})$. The following proposition will show that we can make this assumption without any strengthening of our current hypothesis.

PROPOSITION 8. *Suppose that A is a uniform algebra. If $((\Delta(A), kG), {}_{\Delta(A)}\mathcal{O})$ is an analytic space and $\hat{A} \subseteq \Gamma(\Delta(A), {}_{\Delta(A)}\mathcal{O})$, then there is a sheaf $\mathcal{A}$ on $\Delta(A)$ such that $((\Delta(A), kG), \mathcal{A})$ is a Stein analytic space and $\hat{A} = \Gamma(\Delta(A), \mathcal{A})$.*

Proof. Since A is a uniform algebra, $(\Delta(A), kG)$ is $\hat{A}$-convex. By Theorem 1, we know there is a sheaf $\mathcal{A}$ on $\Delta(\hat{A})$ such that $(\Delta(\hat{A}), \mathcal{A})$ is a Stein space and $\hat{\hat{A}} = \Gamma(\Delta(\hat{A}), \mathcal{A})$. But A is a uniform algebra, so $\hat{A} = \hat{\hat{A}}$ and $\Delta(\hat{A}) = \Delta(\hat{\hat{A}})$. Therefore, we may view $((\Delta(A), kG), \mathcal{A})$ as a Stein space such that $\hat{A} = \Gamma(\Delta(A), \mathcal{A})$.

We establish some notation. Let $\Omega_j = j \times \Omega$, $X_j = \pi(\Omega_j)$, $\pi_j = \pi|_{\Omega_j}$, and T_j be the relative topology on X_j inherited from T. We will show that each X_j is an irreducible branch of $((X, T), \mathcal{A})$. Then, using the local finiteness of the irreducible branches of an analytic space, we will show that π is a proper map.

PROPOSITION 9. *The map $\pi_j: \Omega_j \to (X_j, T_j)$ is a homeomorphism.*

Proof. The map π_j is clearly injective, surjective, and continuous. We need only show it is open. Let $U \subseteq \Omega_j$ be open. Then $\pi_j(U) = V \cap X_j$ where $V = \pi(N \times U)$. Since $N \times U$ is open and $N \times U = \pi^{-1}(\pi(N \times U))$, we know $V \in T$. Therefore, $\pi_j(U) \in T_j$.

PROPOSITION 10. *The set X_j is closed in (X, T) and the map π_j: $\Omega_j \to (X, T)$ is proper.*

Proof. Since π is an identification, it suffices to show that $\pi^{-1}(X_j)$ is closed. This follows immediately from the fact that $\pi^{-1}(X_j) = \big(\bigcup_{i \neq j} i \times V_{ij}\big) \cup \Omega_j$.

Let K be a compact set in X. Since X_j is closed in (X, T), $K \cap X_j$ is compact. By Proposition 9, $(\pi_j)^{-1}(K \cap X_j)$ is compact. Therefore, $(\pi_j)^{-1}(K) = (\pi_j)^{-1}(K \cap X_j)$ is compact and π_j: $\Omega_j \to (X, T)$ is a proper map.

In order to proceed, it is necessary to show that the G topology and the kG topology agree on $\Delta(A)$. To do this we must first prove the following proposition.

PROPOSITION 11. *The analytic space $((X, T), \mathcal{A})$ is of dimension less than or equal to N.*

Proof. The space $((X, T), \mathcal{A})$ is of dimension less than or equal to N if the dimension of each of its irreducible branches is less than or equal to N. The dimension of an irreducible branch is determined by the dimension of the set of its regular points. Let $\mathcal{R}(X)$ be the regular points of $((X, T), \mathcal{A})$. Since $(\mathcal{R}(X), \mathcal{A}|_{\mathcal{R}(X)})$ is a complex manifold, it suffices to show that the topological dimension of X is less than or equal to $2N$.

We must first show that (X, T) is a separable metric space in order that the topological dimension of (X, T) may be defined. Because (X, T) is the continuous image of a separable space, it must be separable. Because (X, T) is the spectrum of a Fréchet algebra, it is Hausdorff and it is the union of a countable number of compact sets. Because $((X, T), \mathcal{A})$ is an analytic space, (X, T) is locally metrizable and locally compact. Any locally compact, Hausdorff space that is the union of a countable number of compact sets is paracompact ([9], p. 79). A locally metrizable Hausdorff space is metrizable if and only if it is paracompact ([9], p. 81). Therefore, (X, T) is metrizable.

A separable metric space has topological dimension less than or equal to $2N$ if it is the countable union of closed sets, each of topological dimension less than or equal to $2N$. Since (X_j, T_j) is homeomorphic to Ω_j, the topological dimension of (X_j, T_j) is $2N$. Further, each X_j is a closed subset of X since $\pi^{-1}(X_j) = \Omega_j \cup \big(\bigcup_{i \in N} i \times V_{ij}\big)$.

PROPOSITION 12. *The G and kG topologies agree on $\Delta(A)$.*

Proof. By the previous proposition, $((\Delta(A), kG), \mathcal{A})$ is a Stein space of dimension less than or equal to N. Therefore, the kG topology must be the weakest topology such that every f in $\Gamma(\Delta(A), \mathscr{A})$ is continuous ([8], p. 222). But this is exactly the definition of the G topology on $\Delta(A)$. Therefore, the two topologies must agree.

PROPOSITION 13. *Each point of (X, T) has a neighborhood basis $\{U_n\}$ of open sets such that each U_n is $\hat{A}$-convex.*

Proof. This is a well-known theorem.

PROPOSITION 14. *The maps $\pi\colon (M, {}_M\mathscr{O}) \to ((X, T), \mathscr{A})$ and $\pi_j\colon (\Omega_j, {}_{\Omega_j}\mathscr{O}) \to ((X, T), \mathscr{A})$ are holomorphic. Further, X_j is a subvariety of X, and $((X_j, T_j), \mathscr{A}|_{X_j})$ is a Stein analytic space.*

Proof. We first show that $\pi\colon (M, {}_M\mathscr{O}) \to ((X, T), \mathscr{A})$ is holomorphic. Let $x \in M$, $y = \pi(x)$, and $\underline{h} \in \mathscr{A}_y$. We must show $\underline{h \circ \pi} \in {}_M\mathscr{O}_x$. Choose a neighborhood W of y and $h \in \Gamma(W, \mathscr{A})$ such that W is $\hat{A}$-convex and (h, W) is a representative of the germ $\underline{h}$. Because $((X, T), \mathscr{A})$ is Stein and W is $\hat{A}$-convex, we may approximate h on compact subsets of W by global sections of $\mathscr{A}$ ([8], p. 214). Choose $\hat{h}_m \in \hat{A}$ such that $\{\hat{h}_m\}$ converges uniformly to h on compact subsets of W. If we let $h_m = \hat{h}_m \circ \pi$, then $h_m \in A$. Let K be a compact subset of $\pi^{-1}(W)$. Then $\pi(K)$ is a compact subset of W. The uniform convergence of $\{\hat{h}_m\}$ on $\pi(K)$ forces the uniform convergence of $\{h_m\}$ on K. Each h_m is holomorphic on $\pi^{-1}(W)$, so $f = \lim h_m$ is also a holomorphic function on $\pi^{-1}(W)$. If $\underline{f}$ is the germ of f at x, then $\underline{f} = \underline{h \circ \pi} \in {}_M\mathscr{O}_x$.

To see that $\pi_j\colon (\Omega_j, {}_{\Omega_j}\mathscr{O}) \to ((X, T), \mathscr{A})$ is holomorphic, we note that π_j is just the composition of $\pi\colon (M, {}_M\mathscr{O}) \to ((X, T), \mathscr{A})$ with the embedding of $(\Omega_j, {}_{\Omega_j}\mathscr{O})$ is $(M, {}_M\mathscr{O})$.

We showed in Proposition 10 that $\pi_j\colon \Omega_j \to (X, T)$ is a proper map. Thus, by the Proper Mapping Theorem, X_j is a subvariety of $((X, T), \mathscr{A})$.

Since X_j is a subvariety of $((X, T), \mathscr{A})$ and $((X, T), \mathscr{A})$ is a Stein space, it follows that $((X_j, T_j), \mathscr{A}|_{X_j})$ is also a Stein space ([8], p. 210).

At this point we prove a general proposition about mappings of analytic spaces.

PROPOSITION 15. *Let $(X, {}_X\mathscr{O})$ and $(Y, {}_Y\mathscr{O})$ be analytic spaces such that $(Y, {}_Y\mathscr{O})$ is a Stein space. Let $f\colon (X, {}_X\mathscr{O}) \to (Y, {}_Y\mathscr{O})$ be a holomorphic homeomorphism (onto). Suppose further that the induced map $f^*\colon \Gamma(Y, {}_Y\mathscr{O}) \to \Gamma(X, {}_X\mathscr{O})$ given by $f^*(h) = h \circ f$ is surjective. Then f is an isomorphism of analytic spaces.*

Proof. Let $\mathscr{F}$ be the direct image sheaf on Y generated by the presheaf $\{\mathscr{F}(U)\}$ where $\mathscr{F}(U) = \{g \in C(U)\colon g \circ f \in \Gamma(f^{-1}(U), {}_X\mathscr{O})\}$. Since f is one-to-one, the fibers of f are discrete and ${}_X\mathscr{O}$ is a coherent sheaf of ${}_X\mathscr{O}$-modules. Applying the Direct Image Theorem we conclude that $\mathscr{F}$ is a coherent sheaf of ${}_Y\mathscr{O}$-modules. We note that $\mathscr{F}$ and ${}_Y\mathscr{O}$ have the same global sections; $\Gamma(Y, \mathscr{F}) \subseteq \Gamma(Y, {}_Y\mathscr{O})$ because f^* is surjective, and $\Gamma(Y, {}_Y\mathscr{O}) \subseteq \Gamma(Y, \mathscr{F})$ because f is a mapping of analytic spaces. Now $\mathscr{F}$ is a coherent sheaf on Y and $(Y, {}_Y\mathscr{O})$ is a Stein space; hence, by Cartan's Theorem A, F_y is generated by $\Gamma(Y, {}_Y\mathscr{O})$ for all $y \in Y$. Since the global sections $\Gamma(Y, {}_Y\mathscr{O})$ and $\Gamma(Y, \mathscr{F})$ are equal, we may conclude that the sheaves, $\mathscr{F}$ and ${}_Y\mathscr{O}$, are equal. Let $\underline{h} \in {}_X\mathscr{O}_x$. In order to show that f is an isomorphism of analytic spaces, it suffices to show that $\underline{h \circ f^{-1}} \in {}_Y\mathscr{O}_{f(x)}$. Clearly, $\underline{h \circ f^{-1}} \in \mathscr{F}_{f(x)}$ by the definition of the direct image sheaf. Since $\mathscr{F}_{f(x)} = {}_Y\mathscr{O}_{f(x)}$ we have $\underline{h \circ f^{-1}} \in {}_Y\mathscr{O}_{f(x)}$.

We are now ready to show that $((X_j, T_j), \mathscr{A}|_{X_j})$ is a complex manifold and that X_j is an irreducible branch of $((X, T), \mathscr{A})$.

PROPOSITION 16. *The analytic space $((X_j, T_j), \mathscr{A}|_{X_j})$ is a connected complex manifold of dimension N.*

Proof. We have shown in Proposition 14 that $((X_j, T_j), \mathscr{A}|_{X_j})$ is a Stein space and that

$$\pi_j\colon (\Omega_j, {}_{\Omega_j}\mathscr{O}) \to ((X_j, T_j), \mathscr{A}|_{X_j})$$

is a mapping of analytic spaces. We have shown in Proposition 9 that $\pi_j\colon \Omega_j^1 \to (X_j, T_j)$ is a homeomorphism. We observe that the induced mapping $(\pi_j)^*\colon \Gamma(X_j, \mathscr{A}|_{X_j}) \to \Gamma(\Omega_j, {}_{\Omega_j}\mathscr{O})$ is surjective. Thus, by Proposition 15, $\pi_j\colon (\Omega_j, {}_{\Omega_j}\mathscr{O}) \to ((X_j, T_j), \mathscr{A}|_{x_j})$ is an isomorphism of analytic spaces. Since $(\Omega_j, {}_{\Omega_j}\mathscr{O})$ is a complex manifold of dimension N, it must also be the case that $((X_j, T_j), \mathscr{A}|_{X_j})$ is a complex manifold of dimension N.

PROPOSITION 17. *For all j, X_j is an irreducible branch of $((X, T), \mathscr{A})$.*

Proof. It suffices to show that X_j is a maximal irreducible subvariety of X. From Proposition 14 we know that X_j is a subvariety of X. From Proposition 16 we know that $((X_j, T_j), \mathscr{A}|_{X_j})$ is a complex manifold. Thus, X_j must be an irreducible subvariety of $((X, T), \mathscr{A})$. If Y is a subvariety containing X_j, then $\bigcup_{i=1}^{\infty} (X_i \cap Y)$ is an irreducible decomposition of Y (unless, of course, $Y = X_j$). Therefore, X_j is a maximal irreducible subvariety of X.

The following two propositions will establish the properness of π and the A-convexity of M.

PROPOSITION 18. *If $x \in X$, then there is an open set $U \in T$ such that $x \in U$ and $\overline{\pi^{-1}(U)}$ is a compact set in M.*

Proof. Choose an open neighborhood D about x such that $D \cap X_j$ is empty for all but a finite number of X_j's. We can find such a neighborhood because the family $\{X_i\}$ of irreducible branches of X is locally finite ([8], p. 155). The inverse image of D is an open set contained in a finite number of levels of M, say, $\pi^{-1}(D) \subseteq \bigcup_{k=1}^{n} (i_k \times \Omega)$. Since $x \in D$, $\pi^{-1}(x)$ must consist of a finite number of points. Denote these by $j_1 \times y, \ldots, j_m \times y$. Choose $\varepsilon > 0$ so small that $j_l \times \Delta(y, \varepsilon) \subseteq \pi^{-1}(D)$, $1 \leqslant l \leqslant m$. Let

$$D^* = \bigcup_{k=1}^{n} i_k \times \Delta(y, \varepsilon)$$

and let $D^{**} = D^* \cap \pi^{-1}(D)$. Since $\overline{D^*}$ is a compact subset of M and $\overline{D^{**}}$ is a closed subset of $\overline{D^*}$, we know that $\overline{D^{**}}$ is a compact subset of M. We want to show that $\pi^{-1}(\pi(D^{**})) = D^{**}$. One inclusion is obvious. To see the other one, let $x_0 \in \pi^{-1}(\pi(D^{**}))$. Because $p(\pi^{-1}(\pi(D^{**}))) \subseteq \Delta(y, \varepsilon)$, $p(x_0) \in \Delta(y, \varepsilon)$. Further, $x_0 \in \pi^{-1}(D)$, so $x_0 \in i_k \times \Delta(y, \varepsilon)$ for some k, $1 \leqslant k \leqslant n$. Thus, $x_0 \in D^*$ and $x_0 \in \pi^{-1}(D)$. This implies that $x_0 \in D^{**} = D^* \cap \cap \pi^{-1}(D)$. We let $U = \pi(D^{**})$. Since $\pi^{-1}(U) = \pi^{-1}(\pi(D^{**})) = D^{**}$ is an open set, U is an open set. Furthermore, $\pi^{-1}(U) = D^{**}$, so $\overline{\pi^{-1}(U)} = \overline{D^{**}}$ is a compact set in M.

Proposition 19. *The map π: $M \to (X, T)$ is a proper mapping and M is A-convex.*

Proof. Let K be a compact set in (X, T). We must show that $\pi^{-1}(K)$ is compact. By Proposition 18, for $x \in K$, we may find an open set $U(x)$ about x such that $\pi^{-1}(U(x))$ has compact closure. Since K is compact and $\{U(x)\colon x \in K\}$ is an open cover of K, we may find a finite subcover $\{U(x_1), \ldots, U(x_n)\}$. Thus,

$$\pi^{-1}(K) \subseteq \bigcup_{i=1}^{n} \pi^{-1}(U(x_i)) \subseteq \bigcup_{i=1}^{n} \overline{\pi^{-1}(U(x_i))}.$$

Because $\bigcup_{i=1}^{n} \overline{\pi^{-1}(U(x_i))}$ is compact and $\pi^{-1}(K)$ is closed, we know that $\pi^{-1}(K)$ is compact. We conclude that M is A-convex by applying Lemma 1.

5. Riemann domains and algebras containing the coordinate functions. Throughout this section we assume that (M, p) is a Riemann domain spread over C^N and that A is a closed subalgebra of $\mathcal{O}(M)$ containing the coordinate functions p_j, $1 \leqslant j \leqslant N$.

We let (X, T) be the spectrum of A endowed with the kG topology. We assume that there is a sheaf $\mathcal{O}$ on X such that $((X, T), \mathcal{O})$ is an analytic space and $\hat{A} \subseteq \Gamma(X, \mathcal{O})$. By Proposition 8, we may assume that there is a sheaf $\mathscr{A}$ on X such that $((X, T), \mathscr{A})$ is a Stein analytic space and $\hat{A} = \Gamma(X, \mathscr{A})$.

The example in the beginning of this chapter indicates that these assumptions do not imply that M is A-convex. As in Section 4, we will find a new domain M^* and a new algebra A^* closedly related to M and A. After adding the hypotheses that the evaluation map from M^* to $\Delta(A)$ is surjective and that the kG and quotient topologies agree on $\Delta(A)$, we will be able to show that M^* is A^*-convex. In order to describe M^* and A^* we must make several definitions and prove some propositions.

Definition. Let B be a closed subalgebra of $\mathcal{O}(M)$. If B is closed under differentiation, then we say B is a *stable algebra*. If A is a subalgebra of $\mathcal{O}(M)$ and B is the closure of the algebra generated by $\{D^a f\colon f \in A\}$, then we say that B is the *stable algebra generated by* A.

We will need the following lemma in the proof of Proposition 20.

Lemma 4. *Let X be a Stein manifold and let A be a closed subalgebra of $\mathcal{O}(x)$ such that A separates points of X, A provides local coordinates for X, and X is A-convex. Then $A = \mathcal{O}(X)$.*

Proof. The proof may be found in Cartan's Séminaire ([3], Théorème 4, p. 9–10).

The following proposition is essentially due to Bishop [1].

Proposition 20. *Let (M, p) be a Riemann domain. Let B be a stable algebra on M. Then the spectrum of B endowed with the Gelfand topology, $(\Delta(B), G)$, can be given the structure of a complex manifold in such a way that*

(1) *the evaluation map e_B: $M \to \Delta(B)$ is a complex analytic mapping and has open image in $\Delta(B)$,*

(2) *the functions $\hat{p}_i \in \hat{B}$ are the global local coordinates of $\Delta(B)$,*

(3) *$\Delta(B)$ is a Stein manifold and every component of $\Delta(B)$ intersects $e_B(M)$,*

(4) *$\hat{B} = \mathcal{O}(\Delta(B))$, and*

(5) *the G, kG, and manifold topologies all agree on $\Delta(B)$.*

Proof. The proof of (1), (2), and $\hat{B} \subseteq \mathcal{O}(\Delta(B))$ may be found in the proof of Corollary 1 to Theorem 2 in Bishop's paper on holomorphic completions [1].

Since $\Delta(B)$ is the spectrum of the uniform algebra B, we know that $\hat{B}$ separates points of $\Delta(B)$, and that $\Delta(B)$ is $\hat{B}$-convex. By (2), we know that the $\hat{p}_i$ are local coordinates for $\Delta(B)$. Therefore, the containment of $\hat{B}$ in $\mathcal{O}(\Delta(B))$ implies that $\Delta(B)$ is a Stein manifold. Since $\hat{B} \subseteq \mathcal{O}(\Delta(B))$ and $\Delta(B)$ is a Stein manifold, we may apply Lemma 4 to conclude that $\hat{B} = \mathcal{O}(\Delta(B))$. Because $\hat{B} = \mathcal{O}(\Delta(B))$ and B is a uniform algebra, we know that every component of $\Delta(B)$ intersects $e_B(M)$. Thus, we have (3) and (4).

To show (5), we note that the G and manifold topologies agree. Since the manifold topology is locally compact, the G and kG topologies agree.

For simplicity in notation, we let $q_i = \hat{p}_i$, $1 \leqslant i \leqslant N$, and let $q = \hat{p}$. We are now ready to define several equivalence relations on M.

Definition. We say that $xR_A y$ if, and only if, $f(x) = f(y)$ for all $f \in A$. We say that $xR_B y$ if, and only if, $f(x) = f(y)$ for all $f \in B$. Let

$p(x, \varepsilon)$ be the restriction of p to $\Delta(x, \varepsilon)$. Then $p(x, \varepsilon)\colon \Delta(x, \varepsilon) \to \Delta(p(x), \varepsilon)$ is a homeomorphism. We define R_1 by saying that xR_1y if, and only if, there is an $\varepsilon > 0$ such that

$$p(x, \varepsilon)^{-1}(z) R_A p(y, \varepsilon)^{-1}(z) \quad \text{for all } z \in \Delta(p(x), \varepsilon).$$

We note $R_1 \subseteq R_A$ and $R_B \subseteq R_A$.

Proposition 21. *The relations R_1 and R_B are the same.*

Proof. We first show that $R_1 \subseteq R_B$. If xR_1y, then there is an $\varepsilon > 0$ and neighborhoods $\Delta(x, \varepsilon)$ and $\Delta(y, \varepsilon)$ which are "totally identified" by A. Thus.

$$f|_{\Delta(x,\varepsilon)} \circ p(x, \varepsilon)^{-1} \equiv f|_{\Delta(y,\varepsilon)} \circ p(y, \varepsilon)^{-1} \quad \text{for all } f \in A.$$

Therefore, if $D^a f$ is any derivative of f, then

$$(D^a f)|_{\Delta(x,\varepsilon)} \circ p(x, \varepsilon)^{-1} \equiv (D^a f)|_{\Delta(y,\varepsilon)} \circ p(y, \varepsilon)^{-1} \quad \text{for all } f \in A.$$

Therefore the algebra of functions generated by $\{D^a f\colon f \in A\}$ "totally identifies" $\Delta(x, \varepsilon)$ and $\Delta(y, \varepsilon)$. Clearly, taking limits preserves the total identification of $\Delta(x, \varepsilon)$ and $\Delta(y, \varepsilon)$. Therefore, xR_By.

For the other inclusion, we will show that if $x \operatorname{non} R_1 y$, then $x \operatorname{non} R_B y$. We know that $R_B \subseteq R_A$, so we assume $x \operatorname{non} R_1 y$ and $xR_A y$. Choose an $\varepsilon > 0$ so that $\Delta(x, \varepsilon)$ and $\Delta(y, \varepsilon)$ are homeomorphic to $\Delta(p(x), \varepsilon)$. Since $x \operatorname{non} R_1 y$, there is a sequence $\{z_n\} \subseteq \Delta(p(x), \varepsilon)$ such that $\{z_n\}$ converges to $p(x)$ and $p(x, \varepsilon)^{-1}(z_n) \operatorname{non} R_A p(y, \varepsilon)^{-1}(z_n)$. Therefore,

$$p(x, \varepsilon)^{-1}(z_n) \operatorname{non} R_B p(y, \varepsilon)^{-1}(z_n).$$

This implies

$$e_B(p(x, \varepsilon)^{-1}(z_n)) \neq e_B(p(y, \varepsilon)^{-1}(z_n)),$$

where e_B is the evaluation map, $e_B\colon M \to \Delta(B)$. Since B is a stable algebra we know there exists a $\delta > 0$ and a neighborhood U of $e_B(x)$ such that $q\colon U \to \Delta(p(x), \delta)$ is a homeomorphism. We assume that $e_B(x) = e_B(y)$ and obtain a contradiction. Since $z_n \in \Delta(p(x), \delta)$ for some n, we know that $(q|_U)^{-1}(z_n) \in U$. Therefore, $e_B(p(x, \delta)^{-1}(z_n))$ and $e_B(p(y, \delta)^{-1}(z_n))$ are both mapped to z_n by $q|_U$. Since $e_B(p(x, \delta)^{-1}(z_n))$ and $e_B(p(y, \delta)^{-1}(z_n))$ are distinct points in $\Delta(B)$, we have a contradiction to the fact that $q|_B$ is a homeomorphism. Thus, it must be the case that $e_B(x) \neq e_B(y)$. This implies that there is an $f \in B$ such that $f(x) \neq f(y)$. Therefore, $x \operatorname{non} R_B y$.

Proposition 22. *Let Q_B be the quotient topology on M/R_B and let T_1 be the manifold topology of $\Delta(B)$ restricted to $e_B(M)$. Then $(M/R_B, Q_B)$ is homeomorphic to $(e_B(M), T_1)$.*

Proof. Since R_B is precisely the equivalence relation determined by the map e_B, we may view M/R_B and $e_B(M)$ as the same set. A set U is Q_B-open if, and only if, for every $x \in e_B^{-1}(U)$, there is a $\delta = \delta(x) > 0$ such that $p(x, \delta)\colon \Delta(x, \delta) \to \Delta(p(x), \delta)$ is a homeomorphism and $\Delta(x, \delta) \subseteq e_B^{-1}(U)$. But this is true if, and only if, for every $e_B(x) \in U$ there is a $\delta = \delta(x) > 0$ such that $q\colon e_B(\Delta(x, \delta)) \to \Delta(p(x), \delta)$ is a homeomorphism and $e_B(\Delta(x, \delta)) \subseteq U$. This can occur if, and only if, U is an open set in the manifold topology. Therefore, the topologies agree.

We let $\tilde{A}$ be the image of A under the Gelfand transform which takes B to $\hat{B}$. We let $e_{\tilde{A}}\colon \Delta(B) \to X$ be the evaluation map given by $e_{\tilde{A}}(x)(f) = \tilde{f}(x)$.

We are now able to state precisely the hypotheses that we will be assuming for the rest of this section. The first hypothesis is that the map $e_{\tilde{A}}\colon \Delta(B) \to X$ is surjective. The second is that the quotient topology determined by $e_{\tilde{A}}$ and the topology T agree on X.

We have obtained the domain M^* and the algebra A^*, namely $\Delta(B)$ and $\tilde{A}$. The rest of this section will be devoted to showing that $\Delta(B)$ is $\tilde{A}$-convex by showing that $e_{\tilde{A}}$ is a proper map.

At this point we are ready to define some equivalence relations on $\Delta(B)$.

Definition. Let $xR_{\tilde{A}}y$ if, and only if, $\tilde{f}(x) = \tilde{f}(y)$ for all $\tilde{f}$ in $\tilde{A}$. Let $xR_{\hat{B}}y$ if, and only if, $\hat{f}(x) = \hat{f}(y)$ for all $\hat{f} \in \hat{B}$. Let $q(x, \varepsilon)$ be the restriction of q to $\Delta(x, \varepsilon)$. Let xR_2y if, and only if, there is an $\varepsilon > 0$ such that $q(x, \varepsilon)^{-1}(z) R_{\tilde{A}} q(y, \varepsilon)^{-1}(z)$ for all $z \in \Delta(q(x), \varepsilon)$.

Proposition 23. *The relation $R_{\hat{B}}$ is the identity relation. The algebra $\hat{B} = \mathcal{O}(\Delta(B))$ is the stable algebra on $\Delta(B)$ generated by $\tilde{A}$. The relation R_2 is the identity relation.*

Proof. Since $\hat{B} = \mathcal{O}(\Delta(B))$ and $\Delta(B)$ is a Stein manifold, we have separation of points of $\Delta(B)$ by functions in $\hat{B}$. Therefore, $R_{\hat{B}}$ is the identity relation.

Bishop has shown that $\widehat{D^a f} = D^a \hat{f}$ for every $f \in B$ [1]. Therefore, $\hat{B}$ is the stable algebra generated by $\tilde{A}$.

If $R_2 \subseteq R_{\hat{B}}$, then R_2 must be the identity relation. The proof that $R_2 \subseteq R_{\hat{B}}$ follows exactly the proof that $R_1 \subseteq R_B$ in Proposition 21.

In Section 4 we showed that π was proper by using the fact that each $X_j = \pi(\Omega_j)$ is an irreducible branch of X. In our more general situation, we are unable to determine the irreducible branches of X. However, this information is unnecessary, since we will be able to use the local finiteness of germs of irreducible subvarieties to show that $e_{\tilde{A}}$ is a proper map.

Proposition 24. *For every ε such that $\Delta(x, \varepsilon) \leqslant (\Delta(B), q)$ the map*

$$e_{\tilde{A}}|_{\Delta(x,\varepsilon)}\colon \Delta(x, \varepsilon) \to (e_{\tilde{A}}(\Delta(x, \varepsilon)), T)$$

is a homeomorphism.

Proof. It is clear that $e_{\tilde{A}}|_{\Delta(x,\varepsilon)}$ is one-to-one, onto, and continuous. It remains to show that it is an open map. If $y \in \Delta(x, \varepsilon)$ and $\Delta(y, \delta) \subseteq \Delta(x, \varepsilon)$, then

$$e_{\tilde{A}}|_{\Delta(x,\varepsilon)}(\Delta(y, \varepsilon)) = e_{\tilde{A}}(\Delta(x, \varepsilon)) \cap \{\varphi \in \Delta(A): |\varphi(q_i) - e_{\tilde{A}}(y)(q_i)| < \varepsilon, 1 \leqslant i \leqslant N\}.$$

Therefore, $e_{\tilde{A}}|_{\Delta(x,\varepsilon)}(\Delta(y, \varepsilon))$ is open in the relative topology of $e_{\tilde{A}}(\Delta(x, \varepsilon))$ and $e_{\tilde{A}}|_{\Delta(x,\varepsilon)}$ is an open map.

At this point we recall Propositions 11, 12, and 13 from Section 4. We note that their statements and proofs carry over exactly to our current situation. The following proposition and its proof are similar to Proposition 14 and its proof. Because the statement is somewhat different from Proposition 14, we list it here as a new proposition.

PROPOSITION 25. *The map*

$$e_{\tilde{A}}: (\Delta(B), {}_{\Delta(B)}\mathcal{O}) \to ((X, T), \mathscr{A})$$

is holomorphic.

PROPOSITION 26. *For any $\Delta(x, \varepsilon) \subseteq \Delta(B)$ there is an open set $D(x, \varepsilon) \subseteq X$ such that if $W(x, \varepsilon) = e_{\tilde{A}}(\Delta(x, \varepsilon))$, then $W(x, \varepsilon)$ is closed in $D(x, \varepsilon)$.*

Proof. This follows from the fact that $W(x, \varepsilon)$ is a locally compact subspace of the locally compact space (X, T) ([6], p. 239).

PROPOSITION 27. *For any $\Delta(x, \varepsilon) \subseteq \Delta(B)$, the map*

$$e_{\tilde{A}}|_{\Delta(x,\varepsilon)}: (\Delta(x, \varepsilon), {}_{\Delta(B)}\mathcal{O}|_{\Delta(x,\varepsilon)}) \to (D(x, \varepsilon), \mathscr{A}|_{D(x,\varepsilon)})$$

is holomorphic.

Proof. We know that

$$e_{\tilde{A}}|_{\Delta(x,\varepsilon)}: (\Delta(x, \varepsilon), {}_{\Delta(B)}\mathcal{O}|_{\Delta(x,\varepsilon)}) \to ((X, T), \mathscr{A})$$

is holomorphic and that $e_{\tilde{A}}(\Delta(x, \varepsilon)) = W(x, \varepsilon) \subseteq D(x, \varepsilon)$; hence, the result follows.

PROPOSITION 28. *For any $\Delta(x, \varepsilon) \subseteq \Delta(B)$ the map*

$$e_{\tilde{A}}|_{\Delta(x,\varepsilon)}: \Delta(x, \varepsilon) \to D(x, \varepsilon)$$

is a proper map. Further, the set $W(x, \varepsilon)$ is an irreducible subvariety of $(D(x, \varepsilon), \mathscr{A}|_{D(x,\varepsilon)})$.

Proof. Let L be a compact subset of $D(x, \varepsilon)$. Since $W(x, \varepsilon)$ is a closed subset of $D(x, \varepsilon)$, $W(x, \varepsilon) \cap L$ is a compact subset of $W(x, \varepsilon)$. Since $e_{\tilde{A}}|_{\Delta(x,\varepsilon)}$ is a homeomorphism and $(e_{\tilde{A}}|_{\Delta(x,\varepsilon)})^{-1}(L) = (e_{\tilde{A}}|_{\Delta(x,\varepsilon)})^{-1}(W(x, \varepsilon) \cap L)$, $(e_{\tilde{A}}|_{\Delta(x,\varepsilon)})^{-1}(L)$ is a compact subset of $\Delta(x, \varepsilon)$. Therefore, $e_{\tilde{A}}|_{\Delta(x,\varepsilon)}: \Delta(x, \varepsilon) \to D(x, \varepsilon)$ is a proper map.

By the Proper Mapping Theorem, $W(x, \varepsilon)$ is a subvariety of $D(x, \varepsilon)$. Suppose now that $W(x, \varepsilon)$ is not irreducible. Let $W(x, \varepsilon) = V_1 \cup \ldots \cup V_n$ be the decomposition of $W(x, \varepsilon)$ into irreducible branches. Because $e_{\tilde{A}}|_{\Delta(x,\varepsilon)}$ is holomorphic and the inverse image of a subvariety in $W(x, \varepsilon)$ must be a subvariety in $\Delta(x, \varepsilon)$, we know that $(e_{\tilde{A}}|_{\Delta(x,\varepsilon)})^{-1}(V_1) \cup \ldots \cup (e_{\tilde{A}}|_{\Delta(x,\varepsilon)})^{-1}(V_n)$ is a decomposition of $\Delta(x, \varepsilon)$. Since manifolds are necessarily irreducible and $\Delta(x, \varepsilon)$ is a manifold, this is not possible. Therefore, $W(x, \varepsilon)$ is an irreducible subvariety of $D(x, \varepsilon)$.

The last two propositions of this section correspond to Propositions 18 and 19 in Section 4.

PROPOSITION 29. *If $x \in X$, then there is an open set $U \in T$ such that $x \in U$ and $\overline{(e_{\tilde{A}})^{-1}(U)}$ is a compact set in $\Delta(B)$.*

Proof. Let $x \in X$ and suppose that $(e_{\tilde{A}})^{-1}(x) = \{x_1, x_2, \ldots\}$. Let $U_i = \Delta(x_i, \varepsilon_i)$ where the ε_i are sufficiently small that $x_j \notin \Delta(x_i, \varepsilon_i)$ for any distinct i and j. We may choose such ε_i's because $q^{-1}(q(x_i))$ must be a discrete set in $(\Delta(B), q)$ for any i (otherwise q would not be a local homeomorphism). Let $W_i = e_{\tilde{A}}(U_i)$. By Propositions 26 and 28, we may find an open set D_i in (X, T) such that W_i is an irreducible subvariety of D_i. Let $\underline{W_{i_x}}$ be the germ of W_i at x. Then $\underline{W_{i_x}}$ is the germ of an irreducible subvariety at x.

We claim that if $i \neq j$, then $\underline{W_{i_x}} \neq \underline{W_{j_x}}$. If it were the case that $\underline{W_{i_x}} = \underline{W_{j_x}}$, then there is an open neighborhood D of x such that $D \cap W_i = D \cap W_j$. This implies $(e_{\tilde{A}})^{-1}(D) \cap U_i$ and $(e_{\tilde{A}})^{-1}(D) \cap U_j$ must be "totally identified" by $\tilde{A}$. Hence, x_i and x_j are in the same equivalence class of R_2. This contradicts Proposition 23 which states that R_2 is the identity relation.

Therefore, we know that $\bigcup_{i=1}^{\infty} \underline{W_{i_x}}$ is the decomposition of $\underline{X_x}$ into germs of irreducible subvarieties at x. Since $((X, T), \mathscr{A})$ is an analytic space, there can be at most a finite number of such germs, $\underline{W_{1_x}}, \ldots, \underline{W_{n_x}}$. Let U be an open neighborhood of x such that $U \cap X = (U \cap W_1) \cup \ldots \cup (U \cap W_n)$. Then $(e_{\tilde{A}})^{-1}(U) \subseteq \bigcup_{i=1}^{n} U_i$ and so must be relatively compact.

PROPOSITION 30. *The map $e_{\tilde{A}}: \Delta(B) \to (X, T)$ is a proper map. Further, $\Delta(B)$ is $\tilde{A}$-convex.*

Proof. The proof is exactly the proof of Proposition 19.

It should be noted that there is a correspondence between the results of this section and those of Section 4. It turns out that the domain M^* and the algebra A^* in Section 4 are, in fact, the spectrum $\Delta(B)$ of the stable algebra B generated by A and the image $\tilde{A}$ of A under the Gelfand transform. The facts that $R_1 = R_B$ and that R_2 is the identity relation correspond to the fact that all of the "total collapsing" of levels of the stacked domain occurs in passing from M to M^*.

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On meromorphic functions with values in locally convex spaces

by

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Abstract. Meromorphic functions on a complex manifold with values in a sequentially complete locally convex space are investigated. It is shown that each such meromorphic function on a Stein manifold can be written in the form f/σ, where f is a vector-valued holomorphic function and σ is a complex-valued holomorphic function. This result is applied to the extending and lifting problem of vector-valued meromorphic functions. We also investigate Cousin's First and Second Problem for vector-valued meromorphic functions.

Meromorphic functions on an open set in C with values in Banach spaces have been investigated by several authors ([6], [13]). The aim of this paper is to study meromorphic functions on a complex manifold with values in a sequentially complete locally convex space.

In § 1 we prove that the pole set of each vector-valued meromorphic function either is empty or is an analytic set of codimension 1. Section 2 is devoted to proving that each meromorphic function on a Stein manifold with values in a sequentially complete locally convex space can be represented in the form f/σ, where f is a vector-valued holomorphic function and σ is a complex-valued holomorphic function. An application of this result to the extending and lifting problem of vector-valued meromorphic functions is given in § 3. In § 4 we investigate Cousin's First Problem for meromorphic functions with values in a Fréchet space and Cousin's Second Problem in a commutative Banach algebra with unit element.

Notations and definitions. Given a locally convex space L. By $\mathcal{U}(L)$ we denote the set of all balanced convex neighbourhoods of zero in L. For each $U \in \mathcal{U}(L)$ let $L(U)$ denote the completion of $L/p(U)^{-1}(0)$ equipped with the norm $p(U)$, where $p(U)$ is the Minkowski functional of U, and let $\pi(U)$ denote the canonical map from L into $L(U)$. If $U, V \in \mathcal{U}(L)$ and $V \subset U$, then $\omega(V, U)$ denotes the canonical map from $L(V)$ into $L(U)$. The strongly dual space of L is denoted by L'.

Let X be a complex manifold and let L be a sequentially complete locally convex space. By $\mathcal{O}^L$ we denote the sheaf of germs of holomorphic functions on X with values in L. We write $\mathcal{O} = \mathcal{O}^C$. Given a connected