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On meromorphic functions with values in locally convex spaces

by

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Abstract. Meromorphic functions on a complex manifold with values in a sequentially complete locally convex space are investigated. It is shown that each such meromorphic function on a Stein manifold can be written in the form f/σ, where f is a vector-valued holomorphic function and σ is a complex-valued holomorphic function. This result is applied to the extending and lifting problem of vector-valued meromorphic functions. We also investigate Cousin's First and Second Problem for vector-valued meromorphic functions.

Meromorphic functions on an open set in C with values in Banach spaces have been investigated by several authors ([6], [13]). The aim of this paper is to study meromorphic functions on a complex manifold with values in a sequentially complete locally convex space.

In § 1 we prove that the pole set of each vector-valued meromorphic function either is empty or is an analytic set of codimension 1. Section 2 is devoted to proving that each meromorphic function on a Stein manifold with values in a sequentially complete locally convex space can be represented in the form f/σ, where f is a vector-valued holomorphic function and σ is a complex-valued holomorphic function. An application of this result to the extending and lifting problem of vector-valued meromorphic functions is given in § 3. In § 4 we investigate Cousin's First Problem for meromorphic functions with values in a Fréchet space and Cousin's Second Problem in a commutative Banach algebra with unit element.

Notations and definitions. Given a locally convex space L. By $\mathcal{U}(L)$ we denote the set of all balanced convex neighbourhoods of zero in L. For each $U \in \mathcal{U}(L)$ let $L(U)$ denote the completion of $L/p(U)^{-1}(0)$ equipped with the norm $p(U)$, where $p(U)$ is the Minkowski functional of U, and let $\pi(U)$ denote the canonical map from L into $L(U)$. If $U, V \in \mathcal{U}(L)$ and $V \subset U$, then $\omega(V, U)$ denotes the canonical map from $L(V)$ into $L(U)$. The strongly dual space of L is denoted by L'.

Let X be a complex manifold and let L be a sequentially complete locally convex space. By $\mathcal{O}^L$ we denote the sheaf of germs of holomorphic functions on X with values in L. We write $\mathcal{O} = \mathcal{O}^C$. Given a connected open set U in X, consider the set

$$\tilde{\mathscr{M}}(U, L) = \{(f, \sigma) \in \mathcal{O}(U, L) \times \mathcal{O}(U) \setminus \{0\}\}$$

where $\mathcal{O}(U, L) = H^0(U, \mathcal{O}^L)$ and $\mathcal{O}(U) = \mathcal{O}(U, C)$. We define on $\tilde{\mathscr{M}}(U, L)$ a relation $R(U, L)$ by

$$(f, \sigma) R(U, L)(g, \beta) \quad \text{iff} \quad \beta f = \sigma g.$$

Since U is connected, $R(U, L)$ is an equivalent relation on $\tilde{\mathscr{M}}(U, L)$ and, moreover, if $(f, \sigma)R(U, L)(g, \beta)$, then $(f|V, \sigma|V)R(V, L)(g|V, \beta|V)$ for all connected open sets V in U. Hence the formula

$$U \mapsto \tilde{\mathscr{M}}(U, L)/R(U, L),$$

where U is connected and open in X, defines a sheaf $\mathscr{M}^L$ over X. The sheaf $\mathscr{M}^L$ is said to be the *sheaf of germs of meromorphic functions on X with values in L*. An element $\mathfrak{m} \in H^0(U, \mathscr{M}^L)$ is called a *meromorphic function on U with values in L.*

Finally we recall that a function f on an open set Ω in a locally convex space L with values in a sequentially complete locally convex space F is said to be *holomorphic* iff f is continuous and $f|\Omega \cap L_0$ is holomorphic for all finite dimensional subspaces L_0 of L.

§ 1. The pole set of a vector-valued meromorphic function. Let X be a complex manifold and let L be a sequentially complete locally convex space. Let $\mathfrak{m} \in H^0(X, \mathscr{M}^L)$. Put

$$P(\mathfrak{m}) = \{z \in X: \mathfrak{m}_z \notin \mathcal{O}_z^L\}.$$

Then $P(\mathfrak{m})$ is called the *pole set of* $\mathfrak{m}$. Let us prove the following

THEOREM 1.1. *Let X be a complex manifold and let L be a sequentially complete locally convex space. Let $\mathfrak{m} \in H^0(X, \mathscr{M}^L)$. Then for each $z \in P(\mathfrak{m})$ there exist a neighbourhood U of z in X and elements $f \in \mathcal{O}(U, L)$, $\sigma \in \mathcal{O}(U)$, such that*

$$\mathfrak{m}|U = f/\sigma \quad \textit{and} \quad P(\mathfrak{m}) \cap U = \{z \in U: \sigma(z) = 0\}.$$

The following is an immediate consequence of Theorem 1.1.

COROLLARY 1.1. *Let X be a complex manifold and let L be a sequentially complete locally convex space. Let $\mathfrak{m} \in H^0(X, \mathscr{M}^L) \setminus \mathcal{O}(X, L)$. Then $P(\mathfrak{m})$ is an analytic set in X of codimension 1.*

The proof of Theorem 1.1 is based on the following

LEMMA 1.1. *Let $z_0 \in X$ and let $\beta \in \mathcal{O}_{z_0}$ be an irreducible element. Let $f \in \mathcal{O}_{z_0}^L$ such that $u'f|V(\beta) = 0$ for all $u' \in L'$, where $V(\beta)$ denotes the germ of the zero set of a representative of β at z_0 and L is a sequentially complete locally convex space. Then there exists a unique element $g \in \mathcal{O}_{z_0}^L$ such that $f = \beta g$.*

Proof. Let G be a Stein neighbourhood of z_0 in X and let $\tilde{f}$ and $\tilde{\beta}$ be holomorphic functions on G such that $\tilde{f}_{z_0} = f$ and $\tilde{\beta}_{z_0} = \beta$. Applying Theorem A [5] to the coherent sheaf $\mathscr{S}$ over G generated by $\{u'\tilde{f}: u' \in L'\}$, we find a finite set $\{u_j'\tilde{f}\}_{j=1}^m$ and a neighbourhood G_0 of z_0 in G such that $\{u_j'\tilde{f}|G_0\}$ generates $H^0(G_0, \mathscr{S})$ as a $\mathcal{O}(G_0)$-module. Since β is irreducible and since $u_j'f|V(\beta) = 0$, it follows that there exists a unique element $a_j \in \mathcal{O}_{z_0}$ such that $u_j'f = a_j\beta$. Hence there exist a connected neighbourhood G_1 of z_0 in G_0 and holomorphic functions $\tilde{a}_j$, $j = 1, 2, \ldots, m$ on G_1 such that $u_j'\tilde{f} = \tilde{a}_j\tilde{\beta}$ on G_1. Whence $u'\tilde{f}|G_1 = g(\ , u')\tilde{\beta}|G_1$ for all $u' \in L'$. Let us observe that g is linear in variable u' and holomorphic in variable z.

Let $U \in \mathscr{U}(L)$. Since

$$\sup\{|g(z, u')|: z \in K, u' \in U^0\} < \infty$$

where U^0 denotes the polar of U, for all compact sets K in $G_1 \setminus V(\tilde{\beta})$ and, since g is G-holomorphic on $G_1 \times L(U)'$, it follows that $g|G_1 \times L(U)'$ is holomorphic ([10]). Hence, it is easy to see that the map $\tilde{g}_U: G_1 \to L(U)'$ induced by g is holomorphic. Since $\tilde{g}_U(G_1 \setminus V(\tilde{\beta})) \subset L(U)$ and $G_1 \subset \overline{G_1 \setminus V(\beta)}$, we infer that $\tilde{g}_U(G_1) \subset L(U)$. Obviously $\pi(U)\tilde{f} = \tilde{\beta}\tilde{g}_U\pi(U)$ and $\tilde{g}_U = \omega(V, U)\tilde{g}_V$ for all $V, U \in \mathscr{U}(L)$ and $V \subset U$. Thus there exists a unique holomorphic function $\tilde{g}$ on G_1 with values in the completion $\hat{L}$ of L such that $\tilde{f} = \tilde{\beta}\tilde{g}$. Since L is sequentially complete and since $G_1 \setminus V(\tilde{\beta})$ is dense in G_1, it follows that $\tilde{g}(G_1) \subset L$ and hence the lemma is proved.

LEMMA 1.2. *Let $\mathfrak{m} \in H^0(X, \mathscr{M}^L)$ and let $z_0 \in X$. If $(u'\mathfrak{m})_{z_0} \in \mathcal{O}_{z_0}^L$ for all $u' \in L'$, then $\mathfrak{m}_{z_0} \in \mathcal{O}_{r_0}^L$.*

Proof. Select a neighbourhood G of z_0 in X such that $\mathfrak{m}|_G = f/\sigma$, where $f \in \mathcal{O}(G, L)$ and $\sigma \in \mathcal{O}(G)$. Since $(u'\mathfrak{m})_{z_0} \in \mathcal{O}_{z_0}^L$, we have $uf'|V(\sigma)_{z_0} = 0$ for $u' \in L'$. On the other hand, since $\mathcal{O}_{z_0}$ is a unique factorization domain, by Lemma 1.1 it follows that $f_{z_0} = \sigma g$ for some $g \in \mathcal{O}_{z_0}^L$. Hence $\mathfrak{m}_{z_0} \in \mathcal{O}_{z_0}^L$.

Proof of Theorem 1.1. Let $z_0 \in P(\mathfrak{m})$ and let W be a neighbourhood of z_0 in X such that

$$\mathfrak{m}|_W = g/\beta, \quad \text{where} \quad g \in \mathcal{O}(W, L) \text{ and } \beta \in \mathcal{O}(W) \setminus \{0\}.$$

Consider the coherent sheaf $\mathscr{S}$ over W generated by $\{u'g: u' \in L'\}$. By Theorem A [5] we can assume that W is a neighbourhood V of z_0 such that $\mathscr{S}$ is generated by $\{u_j'g\}_{j=1}^s$. Let us prove that

$$P(\mathfrak{m}) = \bigcup_{j=1}^{s} Pu_j'\mathfrak{m}. \tag{1.1}$$

Obviously $\bigcup_{j=1}^{s} Pu_j'\mathfrak{m} \subseteq P(\mathfrak{m})$. Let $z \notin \bigcup_{j=1}^{s} Pu_j'\mathfrak{m}$ and let $u' \in L'$. Since

$$(u'\mathfrak{m})_z = (u'g/\beta)_z = \sum a_j(u_j'g/\beta)_z + \sum (u_j'g/\beta)_z \sum a_{ij}(u_i'g)_z + \ldots,$$

where

$$(u'g)_z = \sum a_j(u'_j g)_z + \sum a_{ij}(u'_i g)_z(u'_j g)_z + \ldots, \quad a_j, a_{ij}, \ldots \in \mathcal{O}_z,$$

it follows that $z \notin P(u'\mathfrak{m})$. Thus by Lemma 1.2 we get $z \notin P(\mathfrak{m})$ and hence (1.1) is proved.

Let $\beta_{z_0} = \beta_1^{n_1} \ldots \beta_q^{n_q}$, where $\beta_1, \ldots, \beta_q$ are irreducible elements in $\mathcal{O}_{z_0}$. Since $\operatorname{codim} P(\mathfrak{m})_{z_0} = 1$, there exists a subset $I \subset \{1, \ldots, q\}$ such that $P(\mathfrak{m})_{z_0} = \bigcup_{j \in I} V(\beta_j)$. On the other hand, since $g_{z_0} | V_i \setminus \bigcup_{j \neq i} V_i \cap V_j = 0$ and since $V_i \setminus \bigcup_{j \neq i} V_i \cap V_j$ is dense in V_i ([5]) for all $i \notin I$, by Lemma 1.1 it follows that g_{z_0} is written in the form $g_{z_0} = \prod_{j \in I} \beta_j^{n_j} f$ for some element $f \in \mathcal{O}_z^L$. Hence setting $\sigma = \prod_{j \in I} \beta_j^{n_j}$ we get the relations

$$\mathfrak{m}_{z_0} = f/\sigma \quad \text{and} \quad P(\mathfrak{m})_{z_0} = V(\sigma).$$

The theorem is proved.

§ 2. The representative of vector-valued meromorphic functions. In this section we prove the following

THEOREM 2.1. *Let X be a Stein manifold and let L be a sequentially complete locally convex space. Let $\mathfrak{m} \in H^0(X, \mathcal{M}^L)$. Then there exist $f \in \mathcal{O}(X, L)$ and $\sigma \in \mathcal{O}(X)$ such that $\mathfrak{m} = f/\sigma$.*

We need the following

LEMMA 2.1. *Let X be a complex manifold and let L be a sequentially complete locally convex space. Let $\mathfrak{m} \in H^0(X, \mathcal{M}^L)$. Then there exists a divisor d on X such that $d_z\mathfrak{m}_z \in \mathcal{O}_z^L$ for all $z \in X$.*

Proof. Let $z \in X$. From the proof of Theorem 1.1 it follows that $\mathfrak{m}_z$ can be represented in the form h_z/σ_z, where $h_z \in \mathcal{O}_z^L$ and $\sigma_z \in \mathcal{O}_z$ such that

(2.1) $\quad h_z \neq 0$ on every irreducible branch of $P(\mathfrak{m})_z$ and $P(\mathfrak{m})_z = V(\sigma_z)$.

It is easy to see that if g_z/β_z is another representative of $\mathfrak{m}_z$ satisfying condition (2.1), then

$$\sigma_z/\beta_z \in \mathcal{O}_z^* = \{\gamma \in O_z\colon \gamma(z) \neq 0\}.$$

Thus the family $\{\sigma_z\colon z \in X\}$ defines a divisor d on X such that $d_z\mathfrak{m}_z \in \mathcal{O}_z^L$ for all $z \in X$.

Proof of Theorem 2.1. Let $\mathcal{M}^* = \mathcal{M}^C \setminus \{0\}$. By $\mathcal{D}$ we denote the sheaf of germs of divisors on X. Let τ denote the canonical map from $\mathcal{M}^*$ onto $\mathcal{D}$. Consider the exact sequences

(2.2) $$0 \to Z \to \mathcal{O} \xrightarrow[\exp]{} \mathcal{O}^* \to 1,$$

(2.3) $$1 \to \mathcal{O}^* \to \mathcal{M}^* \xrightarrow{\tau} \mathcal{D} \to 1$$

of sheaves of abelian groups over X, where $\exp \sigma = e^{2\pi i \sigma}$. Let $\alpha\colon H^1(X, \mathcal{O}^*) \to H^2(X, Z)$ and $\beta\colon H^0(X, \mathcal{D}) \to H^1(X, \mathcal{O}^*)$ be canonical maps constructed from the exact sequences (2.2) and (2.3), respectively. Since $H^1(X, \mathcal{O}) = 0$, α is injective. Let $c = \alpha\beta$ and let d be a divisor on X such that $d_z\mathfrak{m}_z \in \mathcal{O}_z^L$ for all $z \in X$. Select a positive divisor d^+ on X such that $c(d^+) = c(d^{-1})$. Since $\alpha\beta(dd^+) = 0$ and $\operatorname{Ker} \alpha = 0$, we have $\beta(dd^+) = 0$. Hence, by the exactness of the sequence

$$\ldots \to H^0(X, \mathcal{M}^*) \xrightarrow{\tau} H^0(X, \mathcal{D}) \to H^1(X, \mathcal{O}^*) \to \ldots,$$

$dd^+ = \tau\tilde{\mathfrak{m}}$ for some $\tilde{\mathfrak{m}} \in H^0(X, \mathcal{M}^*)$. Since X is Stein, $\tilde{\mathfrak{m}} = \beta/\sigma$, where $\beta, \sigma \in \mathcal{O}(X)$ ([5]). On the other hand, since $\tilde{\mathfrak{m}}_z\mathfrak{m}_z = d_z^+ d_z\mathfrak{m}_z \in \mathcal{O}_z^L$ for $z \in X$, we infer that $\tilde{\mathfrak{m}}\mathfrak{m} \in \mathcal{O}(X, L)$. Hence $\mathfrak{m} = \tilde{\mathfrak{m}}\mathfrak{m}/\tilde{\mathfrak{m}} = \sigma\tilde{\mathfrak{m}}\mathfrak{m}/\beta$, $\sigma\tilde{\mathfrak{m}}\mathfrak{m} \in \mathcal{O}(X, L)$, $\beta \in \mathcal{O}(X)$. The theorem is proved.

§ 3. Extending and lifting vector-valued meromorphic functions. The aim of this section is to apply Theorem 2.1 to the extension and selection of meromorphic functions with values in Fréchet spaces.

THEOREM 3.1. *Let S be a submanifold of a Stein manifold X and let F be a Fréchet space. Then the restriction map $R_F\colon H^0(X, \mathcal{M}^F) \to H^0(S, \mathcal{M}^F)$ is surjective.*

Proof. Let $\mathfrak{m} \in H^0(S, \mathcal{M}^F)$. By Theorem 2.1, $\mathfrak{m}$ can be represented in the form $\mathfrak{m} = f/\sigma$, where $f \in \mathcal{O}(S, F)$, $\sigma \in \mathcal{O}(S)$. By a theorem of Bungart ([3]) there exist $\tilde{f} \in \mathcal{O}(X, F)$ and $\tilde{\sigma} \in \mathcal{O}(X)$ such that $\tilde{f}|S = f$ and $\tilde{\sigma}|S = \sigma$. Hence setting $\tilde{\mathfrak{m}} = \tilde{f}/\tilde{\sigma}$ we get a meromorphic extension of $\mathfrak{m}$ on X.

THEOREM 3.2. *Let X be a Stein manifold and let E and F be Banach spaces. Let $J\colon X \times E \to F$ be a holomorphic map such that $J(z, \ )\colon E \to F$ is surjective and linear for all $z \in X$. Then the map $\hat{J}\colon H^0(X, \mathcal{M}^E) \to H^0(X, \mathcal{M}^F)$ induced by J is surjective.*

Proof. Let $\mathfrak{m} \in H^0(X, \mathcal{M}^F)$. Select a representative f/σ of $\mathfrak{m}$, where $f \in \mathcal{O}(X, F)$, $\sigma \in \mathcal{O}(X)$. By a result of Leiterer ([7]) there exists a holomorphic function $\tilde{f}\colon X \to E$ such that $J(z, \tilde{f}(z)) = f(z)$ for all $z \in X$. Hence setting $\tilde{\mathfrak{m}} = \tilde{f}/\sigma$ we get the required selection of $\mathfrak{m}$.

THEOREM 3.3. *Let X be a complex manifold having a countable topology and let J be a continuous linear map from a Fréchet space E onto a Fréchet space F. Then the map $\hat{J}\colon H^0(X, \mathcal{M}^E) \to H^0(X, \mathcal{M}^F)$ induced by J is surjective.*

We need the following:

Let $M = (c_j^n)_{j,n=1}^{\infty}$ be a matrix of positive numbers. For any Fréchet space F by $l(M, F)$ we denote the Fréchet space of sequences $\{v_n\}$ in F such that

$$p_j^U(\{v_n\}) = \sum_{n=1}^{\infty} c_j^n p(U) v_n < \infty \quad \text{for all } U \in \mathscr{U}(F) \text{ and } j \geqslant 1.$$

LEMMA 3.1. *Let J be a continuous linear map from a Fréchet space E onto a Fréchet space F. Then the map $\hat{J}$: $l(M, E) \to l(M, F)$ induced by J is surjective.*

Proof. Let $\{U_n\}_{n=1}^{\infty}$ be a decreasing basis of balanced convex neighbourhood of zero in E. Since J is open, $\{JU_n\}$ forms a basis of neighbourhoods of zero in F. Thus, for each n, J induces a continuous linear map J_n from $E_n = E(U_n)$ onto $F_n = F(U_n)$. Since E_n and F_n are Banach spaces, it is easy to see that $\hat{J}_n$: $l(M, E_n) \to l(M, F_n)$ is surjective for all $n \geqslant 1$. On the other hand, since every canonical map from $\operatorname{Ker} J_{n+1} \to \operatorname{Ker} J_n$ has a dense image, it follows that every canonical map from $l(M, \operatorname{Ker} J_{n+1})$ into $l(M, \operatorname{Ker} J_n)$ also has a dense image. Hence, by a result of Palamodov ([12]), $\hat{J}$ is surjective.

Proof of Theorem 3.3. Let $\mathfrak{m} \in H^0(X, \mathscr{M}^F)$. Select a countable open cover $\{U_j\}$ of X such that $\mathfrak{m} | U_j = f_j/\sigma_j$, where $f_j \in \mathscr{O}(U_j, F)$ and $\sigma_j \in \mathscr{O}(U_j)$ for all j. Applying a theorem of Bishop ([2]) to the sequence $\{f_j\}$, we get a sequence $\{v_n\} \subset F$ and a sequence $\{P_n\}$ of one-dimensional continuous linear mutually annihilating projections of F such that

$$P_n v_n = v_n \quad \text{for} \quad n \geqslant 1,$$

$$(v_n) \in l(M, F), \tag{3.1}$$

$$f_j(z) = \sum_{n=1}^{\infty} P_n f_j(z) = \sum_{n=1}^{\infty} f_j^n(z) v_n \quad \text{for all } z \in U_j \text{ and for all } j$$

where $M = (c_j^n)$, $c_j^n = \sup\{|f_j^n(z)|: z \in V_j\} < \infty$ and $\{V_j\}$ is an open cover of X such that $V_j \Subset U_j$ for $j \geqslant 1$. Since $P_n P_m = 0$ for $n \neq m$, by (3.1) we have

$$\sigma_i f_j^n = \sigma_j f_i^n \quad \text{on} \quad U_i \cap U_j \quad \text{for} \quad i, j \geqslant 1. \tag{3.2}$$

Applying Lemma 3.1 to the sequence $\{v_n\}$, we find an element $(u_n) \in l(M, E)$ such that $Ju_n = v_n$ for all n. Thus, by (3.1), the formula

$$\tilde{f}_j(z) = \sum_{n=1}^{\infty} f_j^n(z) u_n \quad \text{for} \quad z \in V_j$$

defines a holomorphic function $\tilde{f}_j$: $V_j \to E$ such that $J\tilde{f}_j = f_j$. By (3.2) we have $\sigma_i \tilde{f}_j = \beta_j \tilde{f}_i$ on $V_i \cap V_j$ for all $i, j \geqslant 1$. Hence setting $\tilde{\mathfrak{m}} | V_j = \tilde{f}_j/\sigma_j$ we get an $\tilde{\mathfrak{m}} \in H^0(X, \mathscr{M}^E)$ such that $\hat{J}(\tilde{\mathfrak{m}}) = \mathfrak{m}$. The theorem is proved.

§ 4. Cousin's Problems for vector-valued meromorphic functions.

1. *Cousin's First Problem.* Let X be a complex manifold and let L be a sequentially complete locally convex space. Let $\{U_\alpha: \alpha \in \Lambda\}$ be an open cover of X and let $\mathfrak{m}_\alpha$ be meromorphic functions on U_α with values in L such that

$$\mathfrak{m}_\alpha - \mathfrak{m}_\beta \in H^0(U_\alpha \cap U_\beta, \mathscr{O}^L) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{4.1}$$

Does there exist an element $\mathfrak{m} \in H^0(X, \mathscr{M}^L)$ such that

$$\mathfrak{m} - \mathfrak{m}_\alpha \in H^0(U_\alpha, \mathscr{O}^L) \quad \text{for all } \alpha \in \Lambda? \tag{4.2}$$

A family $D = \{\mathfrak{m}_\alpha \in H^0(U_\alpha, \mathscr{M}^L)\}$ satisfying (4.1) is called a *Cousin's First Data on X with values in L*. A meromorphic function $\mathfrak{m} \in H^0(X, \mathscr{M}^L)$ satisfying (4.2) is said to be a *solution of D*.

Considering the exact sequence

$$0 \to \mathscr{O}^L \to \mathscr{M}^L \to \mathscr{M}^L/\mathscr{O}^L \to 0,$$

it follows that if $H^1(X, \mathscr{O}^L) = 0$, then every Cousin's First Data on X with values in L has a solution.

Let us prove the following

THEOREM 4.1. (i) *If X is a complex manifold having a countable topology and if $H^1(X, \mathscr{O}) = 0$, then every Cousin's First Data on X with values in a Fréchet space has a solution.*

(ii) *If X is a complex manifold having a non-constant holomorphic function and if F is a Fréchet space which does not admit a continuous norm, then Cousin's First Problem does not have a solution for some data with values in F'.*

Proof. (i) Consider the Dolbeaut complex on X

$$0 \to \mathscr{O} \to \Omega^0 \xrightarrow{\bar{\partial}} \Omega^1 \to \dots$$

where for each $q \geqslant 0$ and for each sequentially complete locally convex space L we denote by Ω_L^q the sheaf of germs of C^∞-forms of bidegree $(0, q)$ on X with values in L. We write $\Omega^q = \Omega_{\mathbf{C}}^q$ for all $q \geqslant 0$. Let us note that the sequence

$$0 \to \mathscr{O}^F \to \Omega_F^0 \xrightarrow{\bar{\partial}_F} \Omega_F^1 \to \dots \tag{4.3}$$

is exact, where F is a Fréchet space. Since $H^1(X, \mathscr{O}) = 0$, the sequence

$$0 \to H^0(X, \mathscr{O}) \to H^0(X, \Omega^0) \to \operatorname{Ker} \bar{\partial} \to 0$$

is exact. Hence it follows that the sequence

$$0 \to H^0(X, \mathcal{O}^F) \to H^0(X, \Omega_F^0) \xrightarrow{\bar\partial_F} H^0(X, \Omega_F^1)$$

is exact. Combining this with the exactness of the sequence (4.3), we infer that $H^1(X, \mathcal{O}^F) = 0$. Hence statement (i) is proved.

(ii) Let F be a Fréchet space which does not admit a continuous norm. By a theorem of Bessaga and Pełczyński ([1]) there exists a complemented subspace E of F which is isomorphic to C^∞. Without loss of generality we can assume that $C^\infty \subset F$ and X is connected.

Let f be a non-constant holomorphic function on X. Put $a = \sup\{|f(z)|: z \in X\}$. Since X is connected, $|f(z)| < a$ for all $z \in X$. Thus, considering the function $(a - f(z))^{-1}$, we can assume that f is not bounded on X. Take a sequence $\{z_n\} \subset X$ such that $|f(z_n)| \to \infty$ and $f(z_i) \neq f(z_j)$ for $i \neq j$. Select an $\varphi \in \mathcal{O}(C)$ such that $\varphi^{-1}(0) = \{f(z_n)\}$. Put $V = (\varphi f)^{-1}(0) = \bigcup_{j=1} V_j$, where $V_j = f^{-1}(f(z_j))$. Since $\{f(z_j)\}$ is discrete and since f is continuous, there exists an open cover $\{U_j\}_{j=0}^\infty$ of X such that

$$V_j \subset U_j \quad \text{for} \quad j \geqslant 1,$$

$$U_j \cap U_j = \emptyset \quad \text{for} \quad i \neq j, i, j \geqslant 1,$$

and

$$V \cap U_0 = \emptyset.$$

Put $\mathfrak{m}_j = e_j/\varphi(f)$ for $j \geqslant 1$ and $\mathfrak{m}_0 = 0$, where $e_0 = 0$, $e_j = (\overbrace{0, \ldots, 1}^{j}) \in C^{\infty\prime}$. Let us show that the data $\{\mathfrak{m}_j \in H^0(U_j, \mathcal{M}^{F})\}$ does not have a solution.

For a contradiction, there exists a meromorphic function $\mathfrak{m}$ on X with values in $C^{\infty\prime}$ such that

$$h_j = \mathfrak{m} - \mathfrak{m}_j \in H^0(U_j, \mathcal{O}^{C^{\infty\prime}}) \quad \text{for all } j \geqslant 0.$$

Since $h_i - h_j = (e_i - e_j)/\varphi(f)$ on $U_i \cap U_j$, we have

$$\varphi(f) h_i - e_i = \varphi(f) h_j - e_j \quad \text{on} \quad U_i \cap U_j \quad \text{for all } i, j \geqslant 0.$$

Thus the formula $h(z) = \varphi(fz) h_j(z) - e_j$ on U_j defines a holomorphic function on X with values in $C^{\infty\prime}$ such that

(4.4) $$h(z_j) = e_j \quad \text{for all } j \geqslant 1.$$

On the other hand, by the connectedness of X and by the relation $C^{\infty\prime} \cong \oplus Ce_j$ it is easy to see that $h(X) \subset Ce_1 \oplus \ldots \oplus Ce_n$ for some n. This contradicts (4.4). Hence (ii) is proved.

Corollary 4.1. *Let X be a complex manifold having a non-constant holomorphic function and let F be a non-zero Fréchet space. If $H^1(X, \mathcal{O}^F) = 0$, then $\mathcal{O}(X, F)$ cannot be complemented in $C^\infty(X, F)$.*

Proof. For a contradiction, by an argument as in ([9]) we have

$$H^1(X, \mathcal{O}^{F \hat{\otimes} C^{\infty\prime}}) = 0.$$

Since $F \neq 0$, it follows that $H^1(X, \mathcal{O}^{C^{\infty\prime}}) = 0$. This is impossible by Theorem 4.1. The corollary is proved.

Remark 4.1. Let us note that, when X is Stein and $F = C$, Corollary 4.1 has been established by Palamodov ([11]).

Remark 4.2. In [8] we have proved that if X is a locally irreducible complex space having a countable topology and if $\mathcal{O}(X, F)$ is complemented in $C^\infty(R(X), F)$ for some non-zero Fréchet space F, where $R(X)$ denotes the regular part of X, then $\mathcal{O}(X) \cong C^m$ for some $m \leqslant \infty$.

2. *Cousins's Second Problem.* All algebras in this section are assumed to be commutative with a unit element. Let B be a Banach algebra and let $G(B)$ denote the Banach–Lie group of invertible elements in B. By $G_e(B)$ we denote the component of $G(B)$ containing a unit element. Put $S(B) = B \setminus G(B)$. Let X be a complex manifold. By $\mathcal{M}^{S(B)}$ we denote the sheaf over X given by the formula

$$U \mapsto \{\mathfrak{m} \in H^0(U, \mathcal{M}^B): \mathfrak{m}(z) \in S(B) \text{ for all } z \notin P(\mathfrak{m})\},$$

where U is a connected open subset of X.

Let $\{U_\alpha: \alpha \in A\}$ be an open cover of X. Suppose that, for every $\alpha \in A$, we are given an element

$$\mathfrak{m}_\alpha \in H^0(U_\alpha, \mathcal{M}^B) \setminus H^0(U_\alpha, \mathcal{M}^{S(B)})$$

such that

(4.5) $$\mathfrak{m}_\alpha = \mathfrak{m}_\beta f_{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta \text{ for } \alpha, \beta \in A,$$

where $f_{\alpha\beta} \in H^0(U_\alpha \cap U_\beta, \mathcal{O}^{G_e(B)})$. Does there exist an $\mathfrak{m} \in H^0(X, \mathcal{M}^B)$ such that

(4.6) $$\mathfrak{m} | U_\alpha = \mathfrak{m}_\alpha f_\alpha \quad \text{for all } \alpha \in A,$$

where $f_\alpha \in H^0(U_\alpha, \mathcal{O}^{G_e(B)})$? Let us note that if (4.6) holds, then (4.5) holds also with $f_{\alpha\beta} = f_\beta f_\alpha^{-1}$. A family

$$D = \{\mathfrak{m}_\alpha \in H^0(U_\alpha, \mathcal{M}^B) \setminus H^0(U_\alpha, \mathcal{M}^{S(B)})\}$$

satisfying (4.5) is called a *Cousin Second Data on X with values in B*. A meromorphic function on X with values in B satisfying (4.6) is said to be a *solution of D*. Consider the exponent homomorphism $\exp_B$: $B \to G_e(B)$, $b \mapsto \sum (2\pi i b)^n/n!$. Let us note that $\exp_B$ is an analytic cover. Hence the sequence

(4.7) $$0 \to \tilde{N}^B \to \mathcal{O}^B \xrightarrow{\exp_B} \mathcal{O}^{G_e(B)} \to e$$

is exact, where $\tilde{N}_z^B = \operatorname{Ker} \exp_B$ for all $z \in X$.

The exact sequence (4.7) induces the exact cohomological sequence

$$(4.8)\qquad \ldots\to H^1(X,\mathcal{O}^B)\to H^1(X,\mathcal{O}^{G_e(B)})\underset{a_B}{\to}H^2(X,\tilde{N}^B)\to\ldots$$

Let us prove the following

THEOREM 4.2. *Let X be a complex manifold having a countable topology and let B be a Banach algebra. If $H^1(X,\mathcal{O})=H^2(X,\tilde{N}^B)=0$, then every Cousin Second Data on X with values in B has a solution.*

Proof. (a) Since if V is a non-empty open subset of a connected open set $U\subset X$ and if $\mathfrak{m}\in H^0(U,\mathcal{M}^B)$, $\mathfrak{m}|V\in H^0(V,\mathcal{M}^{S(B)})$, then $\mathfrak{m}\in H^0(U,\mathcal{M}^{S(B)})$, the formula

$$U\mapsto H^0(U,\mathcal{M}^B)\setminus H^0(U,\mathcal{M}^{S(B)})$$

defines a sheaf $\mathcal{V}^B$ over X. It is easy to check that $\mathcal{V}^B$ is a sheaf of abelian semigroups.

(b) Consider the exact sequence

$$(4.9)\qquad e\to\mathcal{O}^{G_e(B)}\to\mathcal{V}^B\overset{\tau_B}{\to}\mathcal{D}^B\to e,\qquad \mathcal{D}^B=\mathcal{V}^B/\mathcal{O}^{G_e(B)}$$

of sheaves of abelian semigroups over X. Let $d\in H^0(X,\mathcal{D}^B)$. Select a simple cover $\mathcal{U}=\{U_a\}_{a\in A}$ of X and $\mathfrak{m}_a\in H^0(U_a,\mathcal{M}^B)\setminus H^0(U_a,\mathcal{M}^{S(B)})$ such that $\tau_B\mathfrak{m}_a=d|U_a$ for $a\in A$. Since $\mathcal{U}$ is simple, it follows that for $a,\beta\in A$, $U_a\cap U_\beta\neq\emptyset$ there exists a unique element $\theta_{a\beta}\in H^0(U_a\cap U_\beta,\mathcal{O}^{G_e(B)})$ such that $\mathfrak{m}_a=\theta_{a\beta}\mathfrak{m}_\beta$ on $U_a\cap U_\beta$.

Since $\theta_{a\beta}\theta_{\beta\gamma}\theta_{\gamma a}=e$ for $a,\beta,\gamma\in A$, $U_a\cap U_\beta\cap U_\gamma\neq\emptyset$, the formula

$$\beta_B(d)(U_a,U_\beta)=\theta_{a\beta}\quad\text{for}\quad a,\beta\in A,\ U_a\cap U_\beta\neq\emptyset$$

defines an element $\beta_B(d)\in H^1(X,\mathcal{O}^{G_e(B)})$. It is easy to see that $\beta_B(d)$ is independent of the choice of $\{\mathfrak{m}_a,U_a\}$ and thus depends only on d. Let us note that β_B is a homomorphism and

$$(4.10)\qquad \operatorname{Ker}\beta_B=\operatorname{Im}\tau_B.$$

(c) Since $H^1(X,\mathcal{O})=0$, we have $H^1(X,\mathcal{O}^B)=0$. Hence by the exactness of the sequence (4.8) and by the hypothesis $H^2(X,\tilde{N}^B)=0$ it follows that $H^1(X,\mathcal{O}^{G_e(B)})=0$. By (4.10) we infer that τ_B is surjective. This completes the proof of theorem.

COROLLARY 4.1. *Let X be a complex manifold having a countable topology and let B be a Banach algebra which has a locally connected closed boundary. Let $H^1(X,\mathcal{O})=H^2(X,\boldsymbol{Z})=0$. Then every Cousin Second Data on X with values in B has a solution.*

Proof. By Theorem 4.2 it suffices to check that $H^2(X,\tilde{N}^B)=0$. Since B has a locally connected compact boundary and since $\operatorname{Ker}\exp_B\cap R(B)=0$, where $R(B)$ denotes the radical of B, it follows that $\operatorname{Ker}\exp_B=\boldsymbol{Z}^n$ for some n. Hence

$$H^2(X,\tilde{N}^B)=H^2(X,\boldsymbol{Z}^n)=\oplus H^2(X,\boldsymbol{Z})=0.$$

The corollary is proved.

EXAMPLE 4.1. Let Ω be a connected relatively compact open subset of C^n and let $A(\Omega)$ (resp. $H^\infty(\Omega)$) denote the Banach algebra of continuous functions on $\overline{\Omega}$ which are holomorphic on Ω (resp. of bounded and holomorphic functions on Ω). Let $B\in(C(\Omega),H^\infty(\Omega),A(\Omega))$. Since Ω is connected, $\operatorname{Ker}\exp_B=\boldsymbol{Z}$ and thus $H^2(\Omega,\tilde{N}^B)=H^2(\Omega,\boldsymbol{Z})$. Hence if $H^1(\Omega,\mathcal{O})=H^2(\Omega,\boldsymbol{Z})=0$, then every Cousin Second Data on Ω with values in B has a solution.

EXAMPLE 4.2. Let X be a contractible Stein manifold. Then $H^1(X,\mathcal{O}^G)=0$ for every Banach–Lie group G ([4]). Hence every Cousin Second Data on X with values in a Banach algebra has a solution.

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